Bose Hubbard model

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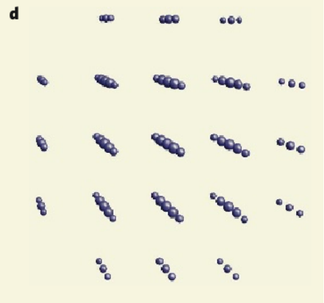
Outline

1. Optical lattices
   1. Laser traps
   2. Standing light waves
   3. periodic potentials
2. Bose hubbard model
   1. Simple model to describe particles in periodic potentials
   2. Hamiltonian: $H_B \rightarrow H_{MF}$
   3. SF - MI phase transition
3. SF - MI phase transition
4. Summary

From NATURE|Vol 453|5 June 2008 .



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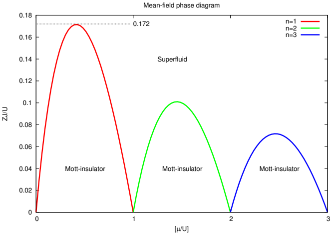
From http://plentifulmarketing.com .



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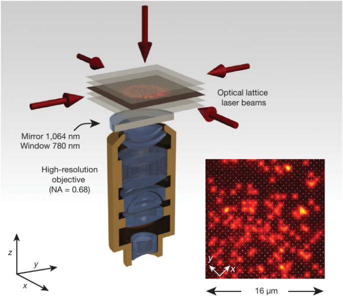
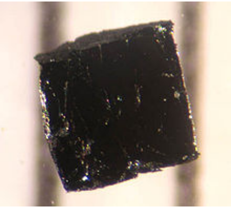
From: NATURE | Vol 467 | 2 September 2010 .



Motivation

Why are we interessted in the Bose Hubbard model?

- High temperature superconductor
- $Bi_2Sr_2Ca_2Cu_3O_{10}(BSCCO)$ (110 K)
- Principle not yet undertsood

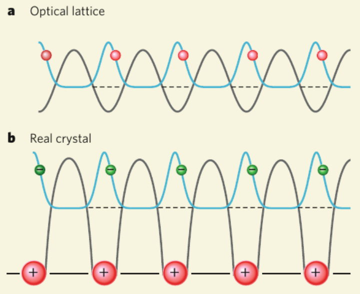


From http://en.wikipedia.org .
Adapted from http://www.profitpath.com .

Simplify

Break the problem down to make it simpler

1. Rebuild the solid state using optical lattices
   1. $d_{lat} \approx \frac{\lambda_{Laser}}{2} \approx 10^{-6}\,\text{m}$
   2. $d_{lat} \approx 10^{-10}\,\text{m}\ \left(\mathring{A}\right)$
2. Investigate superconductivity
3. Explain the superfluid - Mott-Insulator transition



From NATURE|Vol 453|5 June 2008 .

Outline

Particle confinement

1. Focused laser beam →potential well
2. Standing light wave
3. $d = \frac{\lambda_{Laser}}{2}$
4. 2D, 1D and 0D confinement
5. $d_{lat} \approx \frac{\lambda_{Laser}}{2} \approx 10^{-6}\,\text{m}$
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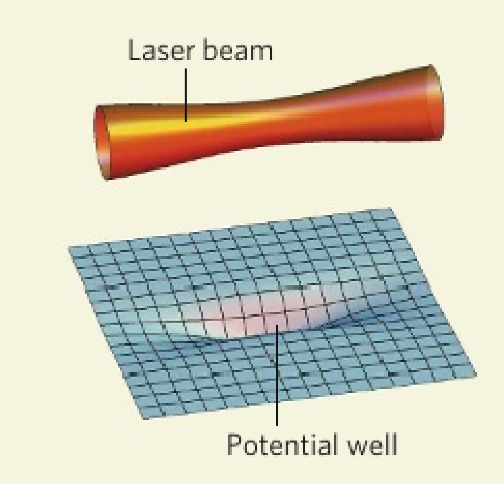

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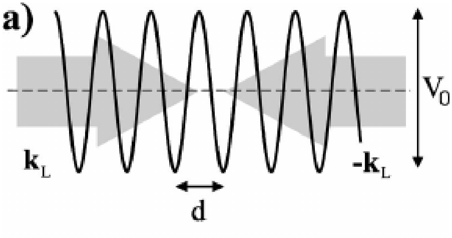

Adapted from arXiv:physics/9902072v1 .

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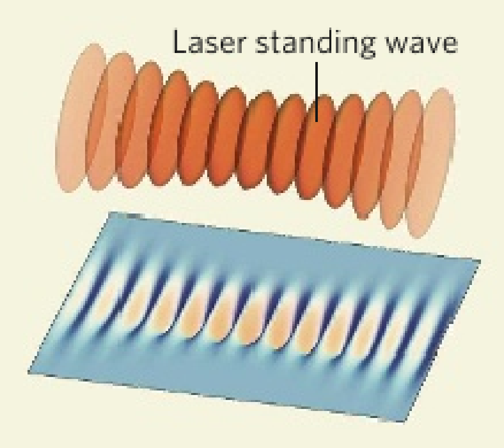

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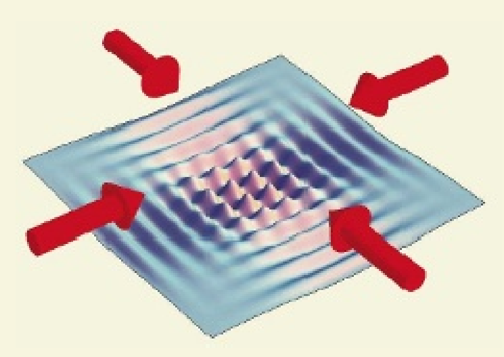

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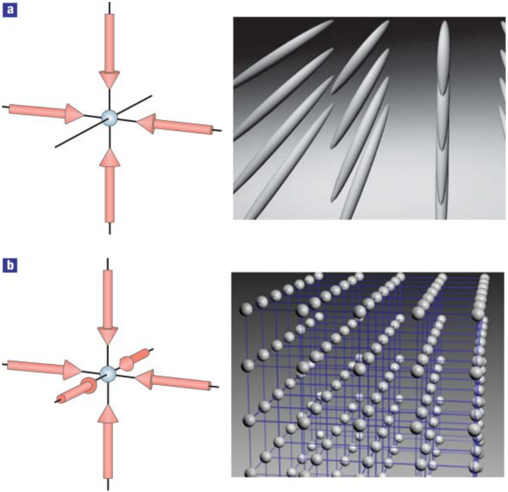
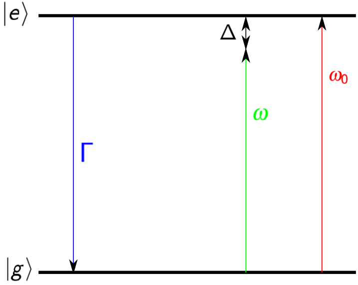


From Nature Physics 1, 23 - 30 (2005) .

Light forces

- Use the principle of the AC Stark shift
- Electric dipole moment is induced in the atom
- Energy shift ΔE



$$\Delta E = U_{dip} = -\frac{1}{2}\langle \boldsymbol{pE}\rangle = -\frac{1}{2}\Re(\alpha(\omega))\left\langle E^2(t)\right\rangle$$

$$I = 2\varepsilon_0 c \cdot \left|\tilde{E}\right|^2$$

$$U_{dip} = -\frac{1}{2\varepsilon_0 c}\Re(\alpha(\omega)) \cdot I$$

Dynamic polarization

$$\omega = \omega_0 + \Delta$$

Potential

$$U_{dip}(\boldsymbol{r}) = \frac{3\pi c^2}{2\omega_0^3}\frac{\Gamma}{\Delta} I(\boldsymbol{r})$$

Black Board Calculation I

Light forces

$$U_{dip} = -\frac{3\pi c^2}{2\omega_0^3}\left(\frac{\Gamma}{\omega_0 - \omega} + \underbrace{\frac{\Gamma}{\omega_0 + \omega}}_{\text{neglect}}\right) I(x)$$

Rotating-wave approximation:

- Resonances at $\pm\omega_0$
- Tune laser near $\omega_0 \rightarrow \Delta \equiv \omega - \omega_0 \rightarrow |\Delta| \ll \omega_0$
- $\Delta \ll \omega + \omega_0$

Proof.

$$U_{dip} = \frac{3\pi c^2}{2\omega_0^3} \cdot \frac{\Gamma}{\Delta} \cdot I(x) \propto \frac{\Gamma}{\Delta} \cdot I(x)$$

□

Detuning

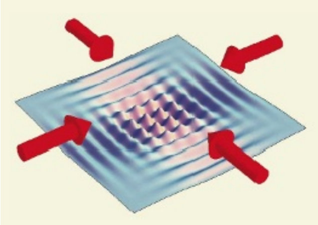
$$U_{dip} = \frac{3\pi c^2}{2\omega_0^3} \cdot \frac{\Gamma}{\Delta} \cdot I(x) \propto \frac{\Gamma}{\Delta} \cdot I(x)$$

$$\Gamma_{sc} = \frac{3\pi c^2}{2\hbar\omega_0^3} \cdot \left(\frac{\Gamma}{\Delta}\right)^2 \cdot I(x) \propto \left(\frac{\Gamma}{\Delta}\right)^2 \cdot I(x)$$

- Below atomic resonance ("red" shifted) $(\omega < \omega_0)$
  - $\Delta < 0 \rightarrow U_{dip} < 0$
  - attractive potential

Experimental issues

- Potential $\propto \frac{I}{\Delta}$
- Scattering rate $\propto \frac{I}{\Delta^2}$



From NATURE|Vol 453|5 June 2008 .

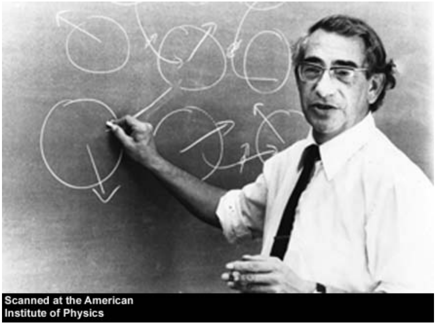
Fact

Use for optical traps (at a certain potential depth)

- *large detunings*
- *high intensities*

Outline

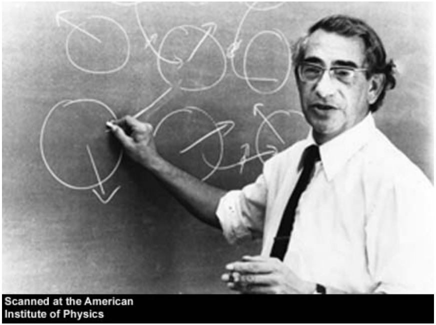
Fermi Hubbard model (FHM)



- Proposed 1963 by John Hubbard
- Describe e^- in transition metals
- Investigate magnetic properties

From: theor.jinr.ru .

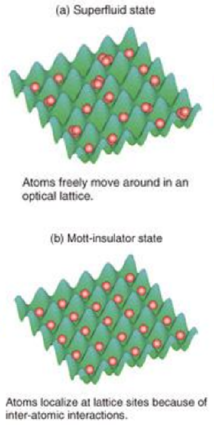
Bose Hubbard model (BHM)



- Cooper pairs (CP) - Josephson tunneling
- He atoms moving on substrates
- Ultracold atoms in optical lattices
- **SF-MI phase transition**
- CP/e^- transition is neglected

From: theor.jinr.ru .

SF and MI Introduction



Adapted from: www.ntt-review.jp .

- U=0: Superfluid phase

$$|\Psi_{SF}\rangle_{(U=0)} = \frac{1}{\sqrt{N!}} \left(\frac{1}{\sqrt{N_L}} \sum_{i=1}^{N_L} \hat{b}_i^\dagger \right)^N |0\rangle$$

  - phase coherence

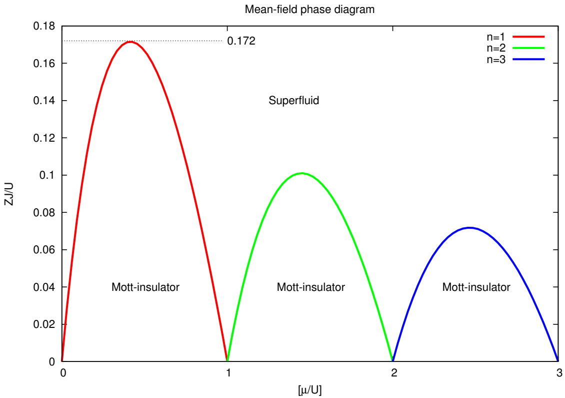
- J=0: Mott insulating phase

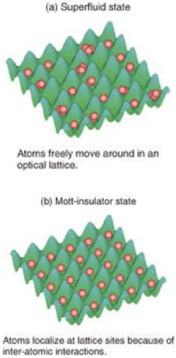
$$|\Psi_{MI}\rangle_{(J=0)} \propto \prod_{i=1}^{N_L} \left(\hat{b}_i^\dagger \right)^n |0\rangle$$

  - no phase coherence
  - perfect correlation in n

Further plan

What we want to do within this part.





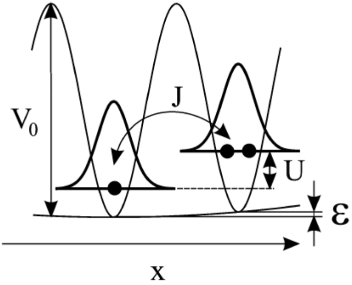
Bose Hubbard Hamiltonian

According to a tight binding model the Hamiltonian is

$$H_B = -J\sum_{\langle ij\rangle}\left(\hat{b}_i^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{b}_i\right) - \mu \sum_i \hat{n}_{bi} + \frac{U}{2}\sum_i \hat{n}_{bi}\left(\hat{n}_{bi}-1\right)$$

$$\left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij}$$

- J: Allows hopping of bosons / Josephson tunneling
- μ: Chemical potential $\equiv$ number of bosons
- $U > 0$: Repulsive interaction



From D. Jaksch, P. Zoller / Annals of Physics 315 (2005) 52–79 .

Mean-field theory

Now: Have a look on a **single** lattice site !

Model the properties of H_B by the best possible sum of single-site hamiltonians:

$$H_{MF} = \sum_i \left(-\mu \hat{n}_{bi} + \frac{U}{2} \hat{n}_{bi} \left(\hat{n}_{bi} - 1 \right) - \Psi_B^* \hat{b}_i - \Psi_B \hat{b}_i^\dagger \right)$$

- Ψ_B: variational parameter (complex)
- μ: Chemical potential
- $U > 0$: Repulsive interaction

Black Board Calculation II

Phase transition

$$H_B = -J\sum_{\langle ij\rangle}\left(\hat{b}_i^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{b}_i\right) - \mu\sum_i \hat{n}_{bi} + \frac{U}{2}\sum_i \hat{n}_{bi}\left(\hat{n}_{bi} - 1\right)$$

Set $J = 0 \rightarrow$ single-site Hamiltonian becomes

$$H_{B.i} = -\mu\hat{n}_{bi} + \frac{U}{2}\hat{n}_{bi}\left(\hat{n}_{bi} - 1\right) = \frac{U}{2}\left[\hat{n}_{bi}\left(\hat{n}_{bi} - 1 - 2\frac{\mu}{U}\right)\right]$$

$$E_0 = \langle n|H_{B.i}|n\rangle = \frac{U}{2}\left[n\left(n - 1 - 2\frac{\mu}{U}\right)\right]$$

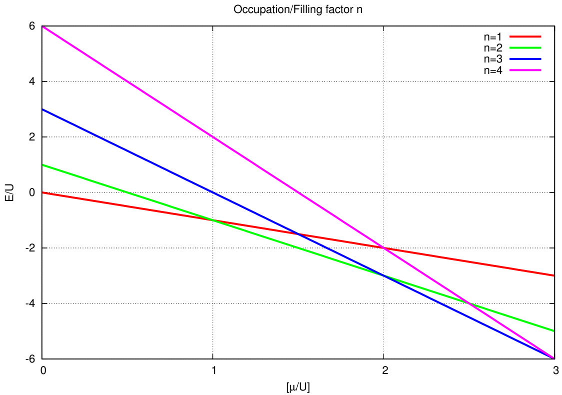
$$n = \text{integer}\left(\frac{\mu}{U} + \frac{1}{2}\right)$$

Phase transition

$$E = \frac{U}{2}\left[n\left(n-1-2\frac{\mu}{U}\right)\right]$$

$$\frac{E}{U} = \frac{1}{2}\left[n\left(n-1-2\frac{\mu}{U}\right)\right]$$

$$n = \begin{cases} 0 & \mu/U < 0 \\ 1 & 0 < \mu/U < 1 \\ 2 & 1 < \mu/U < 2 \\ \vdots & \vdots \\ n & n-1 < \mu/U < n \end{cases}$$

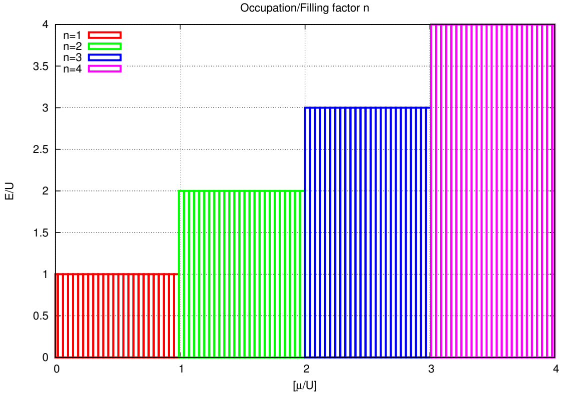


Phase transition

$$n = \begin{cases} 0 & \mu/U < 0 \\ 1 & 0 < \mu/U < 1 \\ 2 & 1 < \mu/U < 2 \\ \vdots & \vdots \\ n & n-1 < \mu/U < n \end{cases}$$

$$n = \frac{\partial E}{\partial \mu} = \text{const.}$$

$$\frac{\partial n}{\partial \mu} = 0$$



Corollary

Mott insulating phase is ***incompressible !***

Phase transition

Expand E_0 in powers of Ψ_B

$$E_0 = E_{00} + q\left|\Psi_B\right|^1 + r\left|\Psi_B\right|^2 + s\left|\Psi_B\right|^3 + \mathcal{O}\left(\left|\Psi_B\right|^4\right)$$

Landau: Symmetry of phase transition (second order) leads to

$$E_0 = E_{00} + r\left|\Psi_B\right|^2 + \mathcal{O}\left(\left|\Psi_B\right|^4\right)$$

Use pertubation theory till second order and identify r

$$r = ZJ \cdot (1 - ZJx_0)$$

$$x_0\left(\mu/U\right) = \frac{n\left(\mu/U\right)+1}{Un\left(\mu/U\right)-\mu} + \frac{n\left(\mu/U\right)}{\mu - u\left(n\left(\mu/U\right)-1\right)}$$
$$= -\frac{\mu+U}{(Un-\mu)\left(U(n-1)-\mu\right)}$$

Phase transition

$$H_{MF.i} = -\mu\hat{n} + \frac{U}{2}\hat{n}(\hat{n}-1) - ZJ\left(\Psi_B^*\hat{b} + \Psi_B\hat{b}^\dagger - \Psi_B^2\right)$$

$$\hat{V} = -JZ\Psi_B\left(\hat{b}+\hat{b}^\dagger\right)$$

$$\begin{aligned} E_n^{(1)} &= \left\langle n|\hat{V}|n\right\rangle \\ &= -JZ\Psi_B\left\langle n|\hat{b}|n\right\rangle - JZ\Psi_B\left\langle n|\hat{b}^\dagger|n\right\rangle \\ &= -JZ\Psi_B\sqrt{n}\langle n|n-1\rangle - JZ\Psi_B\sqrt{n+1}\langle n|n+1\rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Psi_n^{(1)} &= -JZ\Psi_B\sum_{m\neq n}|m\rangle\cdot\frac{\left\langle m|\hat{V}|m\right\rangle}{E_n-E_m} \\ &= -JZ\Psi_B\left(\frac{\sqrt{n}}{E_n-E_{n-1}}|n-1\rangle + \frac{\sqrt{n+1}}{E_n-E_{n+1}}|n+1\rangle\right) \end{aligned}$$

Phase transition

$$E_n^{(2)} = \left\langle n \middle| \hat{V} \middle| \Psi_n^{(1)} \right\rangle$$
$$= J^2 Z^2 \Psi_B^2 \left(\frac{n}{E_n - E_{n-1}} + \frac{n+1}{E_n - E_{n+1}} \right)$$

$$E_n^{(0)} = JZ\Psi_B^2 + \frac{U}{2} n(n-1) - \mu n$$

$$E_n - E_{n-1} = U(n-1) - \mu$$
$$E_n - E_{n+1} = \mu - Un$$

$$E_n^{(2)} = -\frac{U+\mu}{(\mu - Un)(U(n-1)-\mu)} J^2 Z^2 \Psi_B^2$$

$$E_0 = \frac{U}{2} n(n-1) - \mu n + \left(JZ - \frac{U+\mu}{(\mu - Un)(U(n-1)-\mu)} J^2 Z^2 \right) \Psi_B^2$$

Phase transition

$$r = ZJ \cdot \underbrace{(1 - ZJx_0)}_{=0} = 0$$

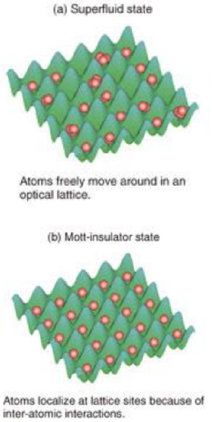
Corollary

$$\frac{ZJ}{U} = \frac{\left(n - \frac{\mu}{U}\right)\left(1 - n + \frac{\mu}{U}\right)}{1 + \frac{\mu}{U}}$$

Critical point ($n = 1$):

$$\frac{ZJ}{U} \approx 0.172 \dots \leftrightarrow \frac{U}{JZ} \approx 5.8$$

Phase transition



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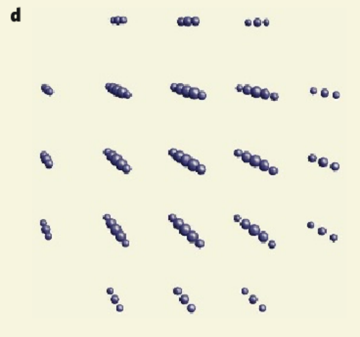
Outline

Quantum Phase transition

1. Optical lattice with $d_{lat} = 426\,\text{nm}$
2. Expand 15 ms (ToF)
3. Take image
4. Coherence/Decoherence
5. $|\Psi(x)\rangle \propto \cos\left(\frac{\pi}{d_{lat}} \cdot x\right)$
6. $|\Psi(k)\rangle \propto \delta\left(k \pm \frac{\pi}{d_{lat}}\right)$



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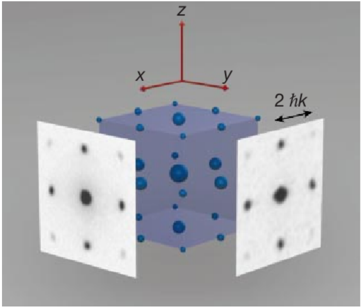

From: NATURE | VOL 415 | 3 JANUARY 2002

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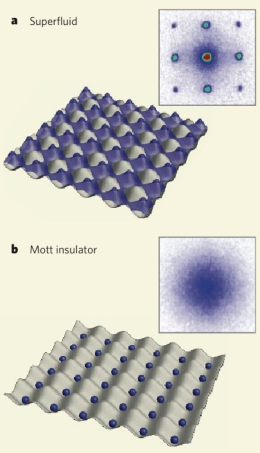

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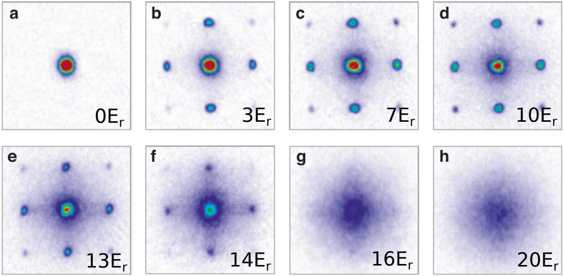
Quantum Phase transition - Experiment I

$U/J = 0$



$U/J = 36$ $U/J = \infty$

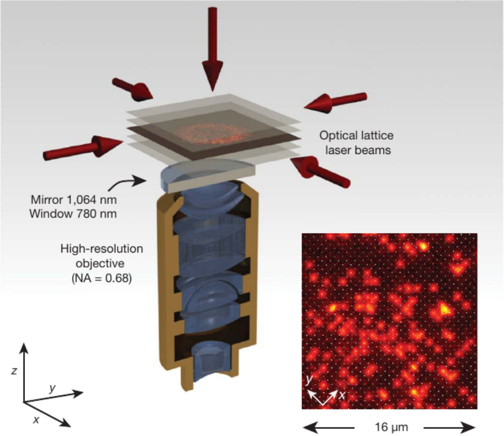
From: NATURE | VOL 415 | 3 JANUARY 2002 .

Example

SF-MI transition (experiment): $U/J = 36|_{Z=6} \rightarrow U/JZ = 6$

SF-MI transition (theory): $U/JZ = 5.8$

Quantum Phase transition - Experiment II



From: NATURE | Vol 467 | 2 September 2010 .

- BEC in 2D optical lattice
- high-resolution fluorescence imaging
- very strong optical particle confinement
- only Mott insulating phases visible
- occupation mod2

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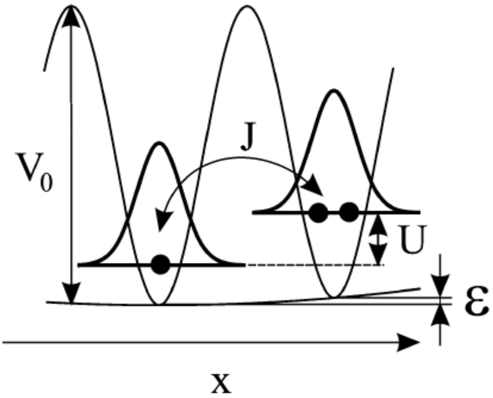

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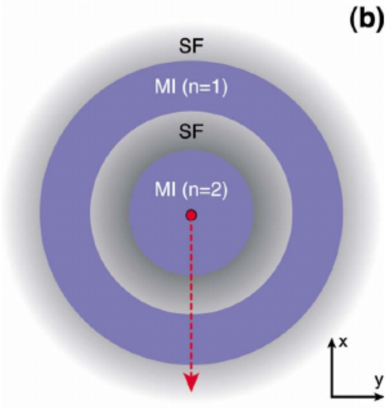

From: Rev.Mod.Phys., Vol.80, No.3, July-Sept 2008 . (2005) 52–79 .

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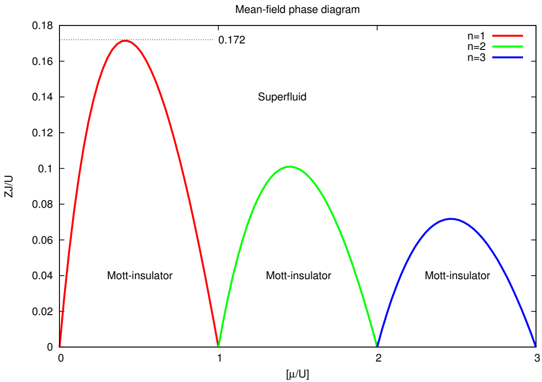

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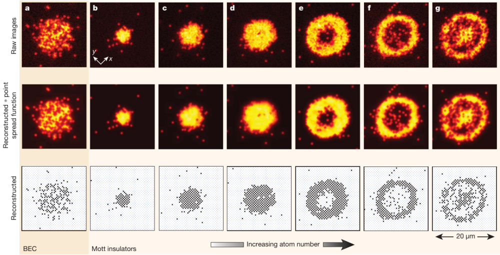

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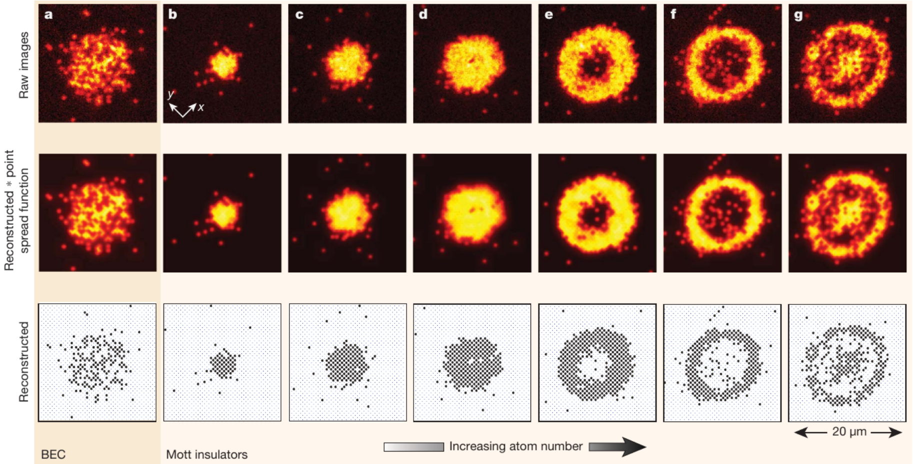
- BEC in 2D optical lattice
- high-resolution fluorescence imaging
- very strong optical particle confinement
- only Mott insulating phases visible
- occupation mod2

Quantum Phase transition - Experiment II



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Outline

Summary

What we have learned today

1. Optical lattice:

$$U_{dip}(\mathbf{r}) = \frac{3\pi c^2}{2\omega_0^3}\frac{\Gamma}{\Delta}I(\mathbf{r})$$

2. Bose Hubbard Hamiltonian

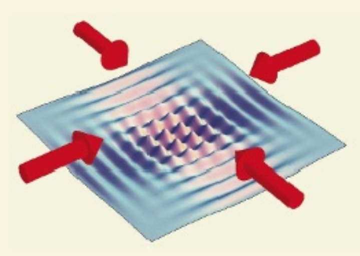
$$H_B = -J\sum_{\langle ij\rangle}\left(\hat{b}_i^\dagger\hat{b}_j + \hat{b}_j^\dagger\hat{b}_i\right) - \mu\sum_i \hat{n}_{bi} + \frac{U}{2}\sum_i \hat{n}_{bi}\left(\hat{n}_{bi}-1\right)$$

$$H_{MF} = \sum_i\left(-\mu\hat{n}_{bi} + \frac{U}{2}\hat{n}_{bi}\left(\hat{n}_{bi}-1\right) - \Psi_B^*\hat{b}_i - \Psi_B\hat{b}_i^\dagger\right)$$

3. Phase diagram given by

$$\frac{ZJ}{U} = \frac{\left(n-\frac{\mu}{U}\right)\left(1-n+\frac{\mu}{U}\right)}{1+\frac{\mu}{U}}$$

4. Today experimentally proofen



From NATURE|Vol 453|5 June 2008 .

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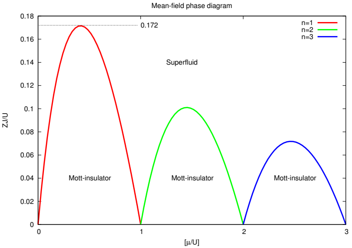

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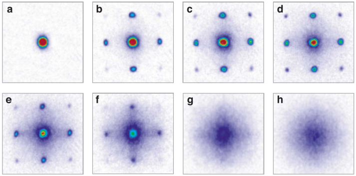

From: NATURE | VOL 415 | 3 JANUARY 2002

Thank you!

Thank you for your attention!

For Further Reading I