Superfluidity in 2 Dimensions: Kosterlitz & Thouless Phase Transition

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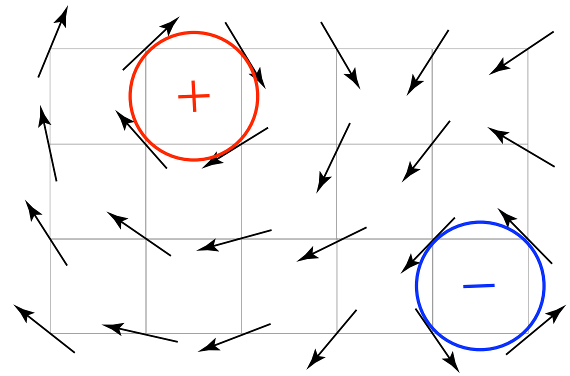
Introduction

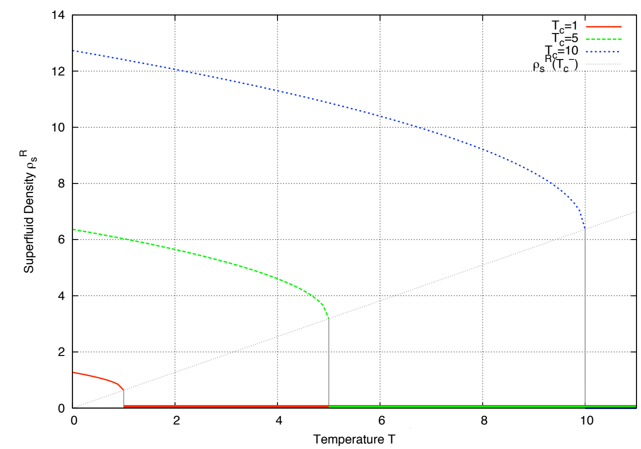
- 1966 – Mermin & Wagner–Theorem
  - No spontaneous long-range order in isotropic 2D lattice at finite temperature
  - No phase transition expected (anti-/ferromagnetism)

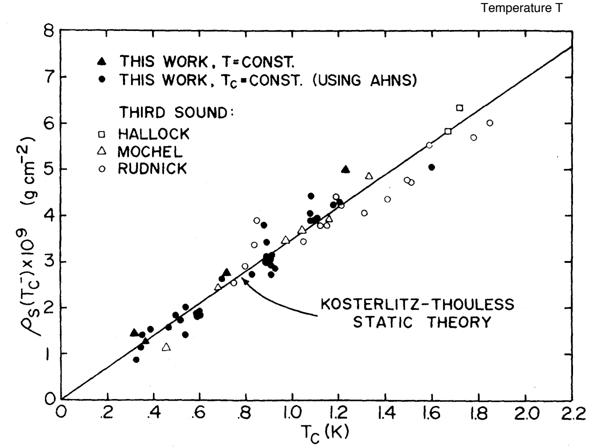
- 1973 – Kosterlitz & Thouless (KT)
  - phase transition in systems in weak external fields
  - Vortex pairs binding ←→ unbinding

Outline

- Introduction
- 2D XY-Model
  - Hamiltonian
  - Thermodynamics
  - Vortices
- Symmetry and Phase Transition
  - Mermin & Wagner Theorem
  - Correlation function
    - Low Temperature
    - High Temperature
  - Superfluid density
- Kosterlitz Thouless Phase Transi
  - Superfluid density
  - Experiment
- Conclusion & Outlook

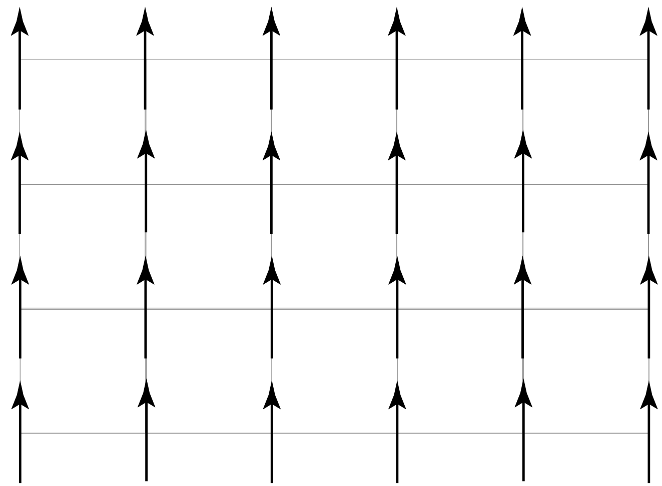


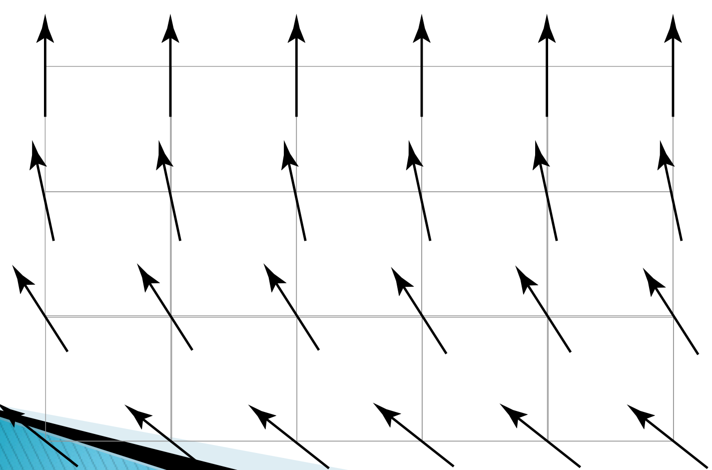


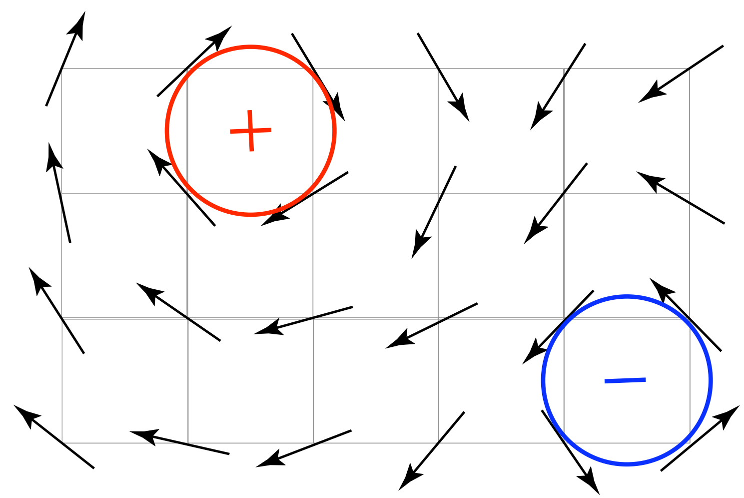


2D XY–Model

- Planar rotors
- Unit length
- 2D square lattice
- Study systems as:
  - films of superfluid helium
  - superconducting materials
  - fluctuating surfaces







2D XY–Model: Hamiltonian

- Hamiltonian:

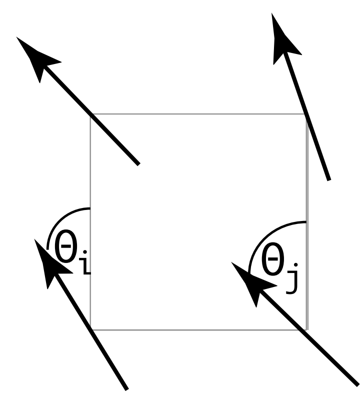
$$H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

- Hamiltonian at low temperature:

With $\cos(\theta_i - \theta_j) \approx 1 - \frac{1}{2}(\theta_i - \theta_j)^2$ and $\theta_i - \theta_j = \partial_x \theta$

$$H = E_0 + \frac{J}{2} \int d\mathbf{r} (\nabla \theta)^2$$

With $E_0 = 2JN$



2D XY-Model: Thermodynamics

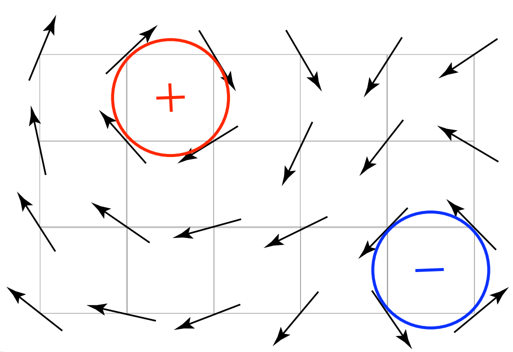
- Partition function at low temperature

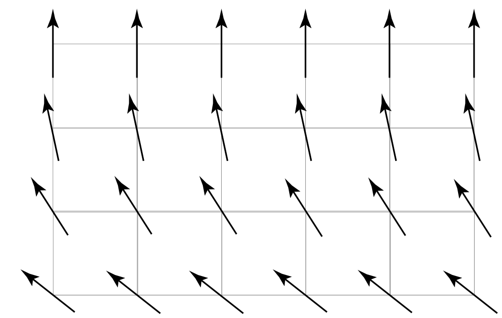
$$Z = e^{-\beta E_0} \int D[\theta] \exp\{-\beta \frac{J}{2} \int d\mathbf{r} (\nabla \theta)^2\}$$

$$Z = e^{-\beta E_0} \sum_{\theta_{vor}} \int D[\theta_{sw}] \exp\{-\beta (H[\theta_{vor}] + \frac{1}{2} \int d\mathbf{r}_1 \int d\mathbf{r}_2 \theta_{sw}(\mathbf{r}_1) \frac{\delta^2 H}{\delta\theta(\mathbf{r}_1)\delta\theta(\mathbf{r}_2)} \theta_{sw}(\mathbf{r}_2))\}$$

Sum: local minima → vortices (left term)

+ fluctuations → spin waves (right term)





2D XY–Model: Thermodynamics

- Extremal condition:

$$\frac{\delta H}{\delta\theta(\mathbf{r})} = 0 \;\Rightarrow\; \nabla^2\theta(\mathbf{r}) = 0$$

- Solutions:

1. Ground State: $\theta(\mathbf{r}) = constant.$
2. Vortices:
   - Centre of vortex encircled: $\oint \nabla\theta(\mathbf{r}) \cdot d\mathbf{l} = 2\pi n.$
   - No vortex centre encircled: $\oint \nabla\theta(\mathbf{r}) \cdot d\mathbf{l} = 0.$

Vortices: Energy

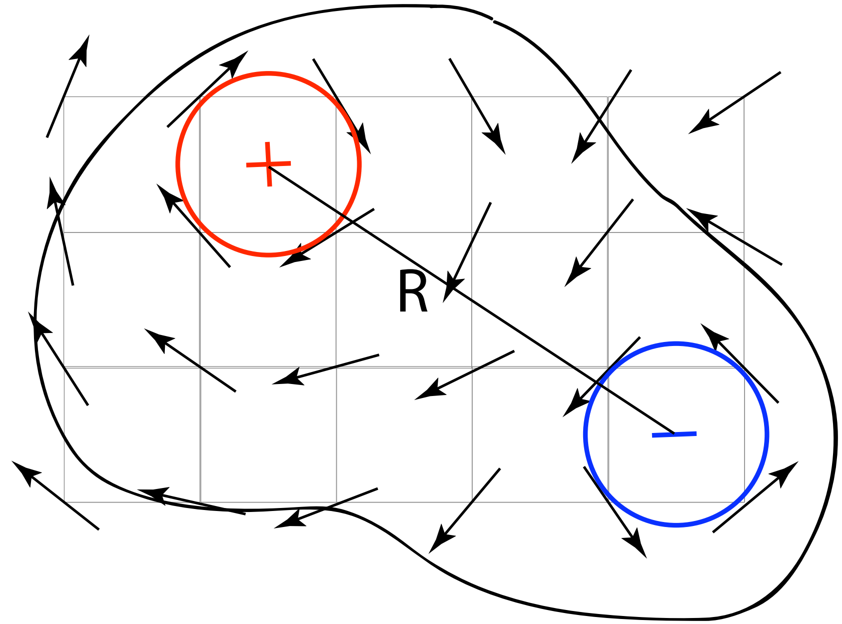
- Calculate energy of a single vortex:
  - With: $$2\pi n = \oint \nabla\theta(\mathbf{r}) \cdot d\mathbf{l} = 2\pi r|\nabla\theta|$$
  - Substitute this into the Hamiltonian:

$$\begin{aligned}
E_{vor} - E_0 &= \frac{J}{2}\int d\mathbf{r}[\nabla\theta(\mathbf{r})]^2 \\
&= \frac{Jn^2}{2}\int_0^{2\pi}\int_a^L r dr \frac{1}{r^2} \\
&= \pi n^2 J \ln(\frac{L}{a}).
\end{aligned}$$

Vortices: Pair Energy

- The energy of a vortex pair is given by:

$$E_{2vor}(R) = 2E_c + E_1 \ln(R/a)$$



Vortices: Free energy F=E−TS

- The free energy of a single vortex:

$$F = E_0 + (\pi J - 2k_B T)\ln(L/a)$$

1. $T < \pi J/2k_B$: Free energy diverges
2. $T > \pi J/2k_B$: Free energy minimized
   → Existence of free vortices lower free energy

- Identify the critical temperature:

$$T_{KT} = \pi J/2k_B$$

Symmetry and Phase Transition

Why is KT phase transition special?

Phase Transition usually linked with symmetry breaking.

Long-range order required.

We will observe no long range order.
But find a phase transition.

Mermin & Wagner Theorem

- In 2D fluctuations at finite temperature
  - No long-range order
  - No usual phase transition
  - No symmetry break possible

→ Mean magnetisation goes to zero at finite temperature
→ Fluctuations destroys long-range order

Correlation function

- Correlation function:
  - Low temperature
  - High temperature

- Find two solutions
  → hint for phase transition

Correlation function: Low temperature

- Look at correlation function :

$$
\begin{aligned}
\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle &= \langle \cos(\theta(\mathbf{r}) - \theta(0))\rangle \\
&= Re[\langle \exp\{i(\theta(\mathbf{r}) - \theta(0))\}\rangle] \\
&= \exp[g(r)]
\end{aligned}
$$

→ Black Board Calculation

Black Board Calculation

Correlation function: Low temperature

- Behaviour of correlation function:

$$\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle \simeq \begin{cases} e^{-\mathrm{const}.T} & \text{for } d>2 \\ \left(\frac{r}{L}\right)^{-\eta} & \text{for } d=2 \\ \exp(-\frac{T}{2Ja}r) & \text{for } d=1. \end{cases}$$

- for $r \to \infty$

1. $d > 2$: correlation survives → ordered phase.
2. $d = 2$: decays with exponent $\eta = T/2\pi J$
   → quasi-long-range order
1. $d = 1$: decays to zero exponentially

Correlation function: High temperature

- Start again with:

$$\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle = \langle \cos(\theta(\mathbf{r}) - \theta(0))\rangle$$
$$= Re[\langle \exp\{i(\theta(\mathbf{r}) - \theta(0))\}\rangle]$$

- Behaviour of correlation function:

$$\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle = e^{-\frac{r}{\xi}}$$

- For $r \rightarrow \infty$

  → correlation decays exponentially

- With: $\xi = L/\ln(2T/J)$

Correlation function: Summary

- Two solutions:
  1. Low temperature
  $$\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle \simeq \left(\frac{r}{L}\right)^{-T/2\pi J}$$
     - power-law decay → Quasi-long-range order
  2. High temperature
  $$\langle \mathbf{S}(\mathbf{r})\mathbf{S}(0)\rangle = e^{-\frac{r}{\xi}}$$
     - Exponential decay → No long-range order
- CONCLUSION: Implies there must be a phase transition

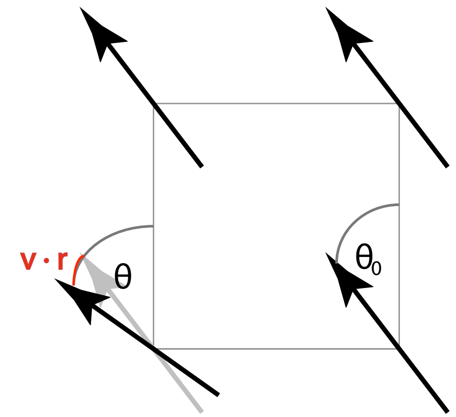
Superfluid density ρ_s

- Correlation function implies phase transition
- Activated vortex&antivortex-pairs responsible
- Superfluid density measures the energy change caused by a twist
- Order parameter

→ Take a look

Superfluid density ρ_s

- Start with applying a twist: $\theta(\mathbf{r}) = \theta_0(\mathbf{r}) + \mathbf{v}_{ex} \cdot \mathbf{r}$
- Increase: $F(\mathbf{v}_{ex}) - F(0) = \frac{1}{2} V \rho_s^R v_{ex}^2$



- With some algebra
  and the vortex density $n(\mathbf{r}) = \sum_\alpha n_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha)$:

$$\rho_s^R = \rho_s + \frac{\rho_s^2}{T} \int d^2 r \langle \mathbf{n}(\mathbf{r}) \cdot \mathbf{n}(0) \rangle$$

- Note: integral is negative for vortex & anti-vortex pairs
  → reduce superfluid density

Superfluid density ρ_s

- One Finds:

$$\rho_s^R = \begin{cases} \rho_s^R(T_{KT}^-)[1 + const.(T_{KT} - T)^{1/2}] & \text{for } T < T_{KT} \\ 0 & \text{for } T > T_{KT} \end{cases}$$

- Universal for 2D systems:

$$\rho_s^R(T_{KT}^-)/T_{KT} = 2/\pi$$

- Note: Critical temperature differ for one sample to another

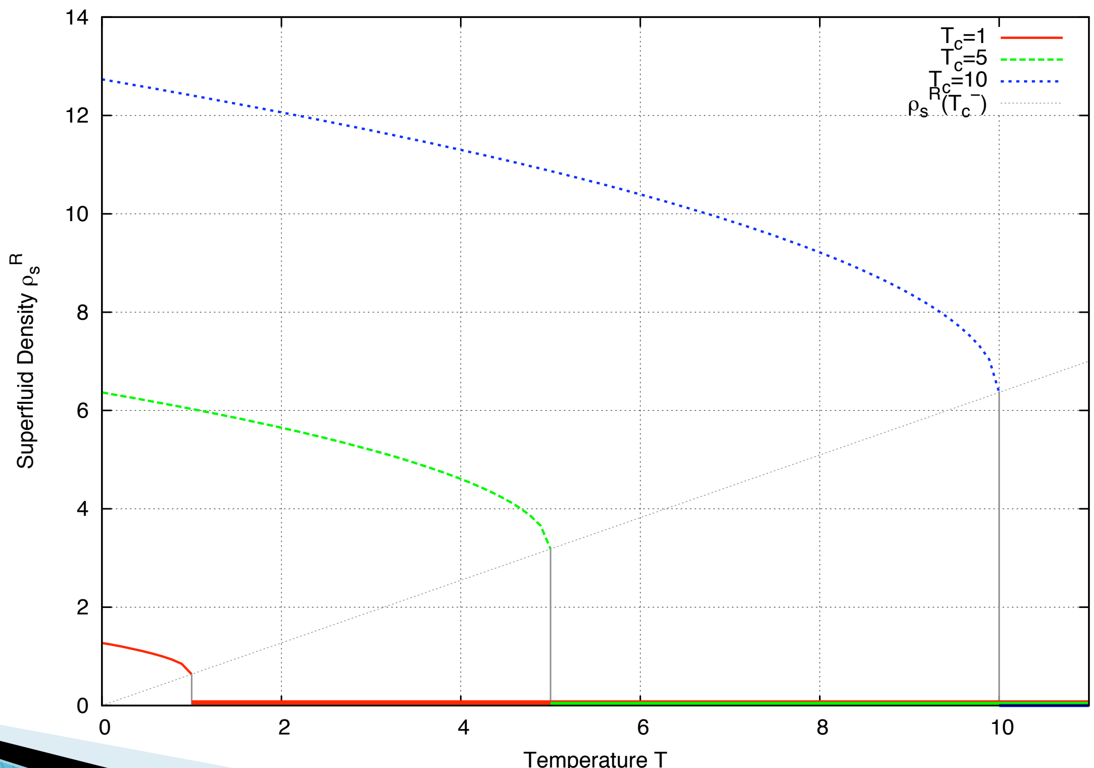
Kosterlitz & Thouless Phase Transition

- Phenomenological summary:
  - Increasing the temperature ($0 < T < T_{KT}$) activates vortex pairs
  - Superfluid density decreases
  - Phase field becomes distorted as temperature increases
  - At high temperature vortex pairs unbind

Kosterlitz & Thouless Phase Transition

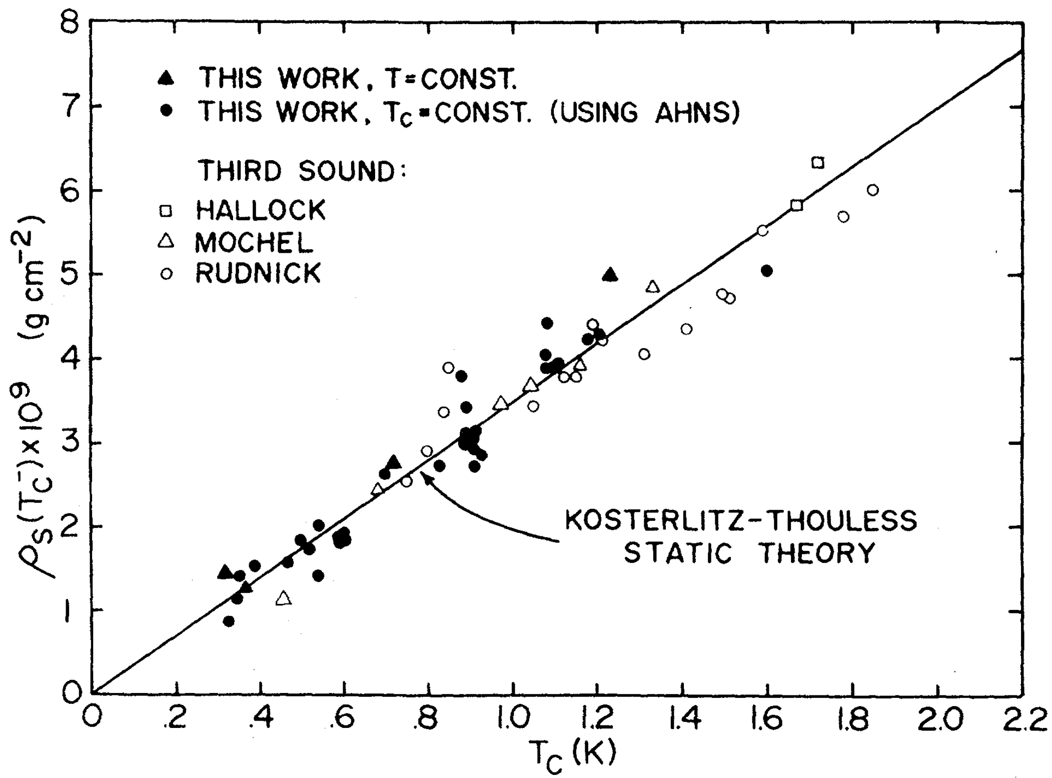
- Superfluid density:

$$\rho_s^R = \begin{cases} \rho_s^R(T_{KT}^-)[1 + const.(T_{KT} - T)^{1/2}] & \text{for } T < T_{KT} \\ 0 & \text{for } T > T_{KT} \end{cases}$$



Experiment

- Superfluid ^{4}He films:



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Conclusion & Outlook

- Phase transition in 2D XY-model
- Note: No typical transition in terms of Ginzburg Landau argument (Ferromagnetism→Paramagnetism)
- Transition linked to unbinding and binding vortices
- Two behaviours for correlation function
- Superfluid density order parameter

Questions?

Gibt es Fragen?

References

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Kosterlitz & Thouless Phase Transition

- Correlation length:

$$\xi(T) \sim \exp\left(\frac{const.}{(T - T_{KT})^{1/2}}\right) \text{for } T > T_{KT}.$$

- Approache critical temperature from above
  → Correlation length diverges exponential:

  - Field distorted by unbound vortices
  - Not screened by nearby antivortices
  - At critical temperature antivortex and vortex pairs screen each other → Correlation length diverges to infinity