BEC in one dimension

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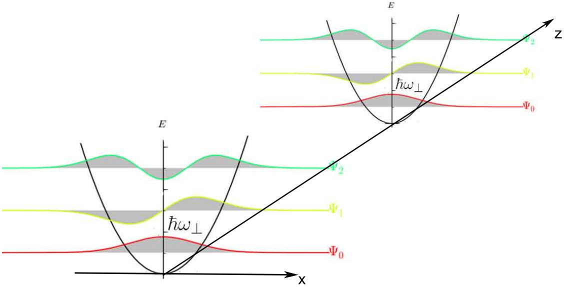
11. Juni 2013

Outline

1. one-dimensional BEC
2. theoretical description
   - Tonks-Girardeau gas
   - Interaction
   - exact solution (Lieb and Liniger)
3. experimental realization
4. conclusion

how to realize 1D: cylindrically symmetric traps

- strong confinement in transverse direction, weak confinement along longitutinal direction



condition for 1D

$k_B T \ll \hbar\omega_\perp$

excitations in transverse directions are frozen out

BEC in 1D

- reduced dimensionality $\Rightarrow$ absence of long range order and true BEC
- finite size L of the system
  - $L > \xi$: thermal gas (high T)
  - $L < \xi$: system is smaller than the correlation function decays $\Rightarrow$ thermal- and quantum fluctuations in the size of the system quasi-condensate

description as Luttinger Liquid

- effective theory for low energy excitations for bosons
- density-phase representation of $\Psi_B^\dagger$: $\Psi_B^\dagger = \sqrt{\rho(x)}e^{-i\varphi(x)}$
- local fluctuation field $\Pi(x)$: $\rho(x) \propto \rho_0 + \Pi(x)$
- $\Pi(x), \varphi(x)$ are conjugate conanical fields, satisfying $[\varphi(x), \Pi(x')] = i\delta(x - x')$

$$H \approx \frac{\hbar^2}{2m} \int dx \left[v_J \left(\nabla\varphi(x)\right)^2 + v_N \Pi(x)^2 \right]$$

$v_J = \frac{\pi\hbar\rho_0}{m}$,$v_N = \frac{k}{\pi\hbar\rho_0^2}$
$\Rightarrow$ 1D: always fluctuations

correlation functions

$$\left\langle \Psi^\dagger(x)\Psi(0) \right\rangle \propto x^{-\frac{1}{\eta}}$$

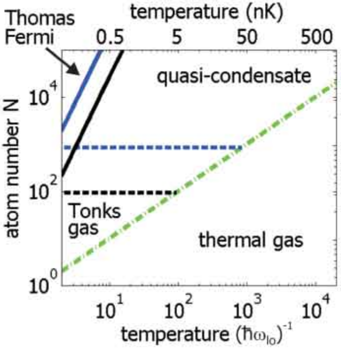
correlation exponent $\eta = 2\sqrt{\frac{v_J}{v_N}}$

- in a LL the correlation function decays algebraically
- same proportionality for bosons and fermions

different regimes

$$\left\langle \Psi^\dagger(x)\Psi(0) \right\rangle \propto x^{-\frac{1}{\eta}}$$

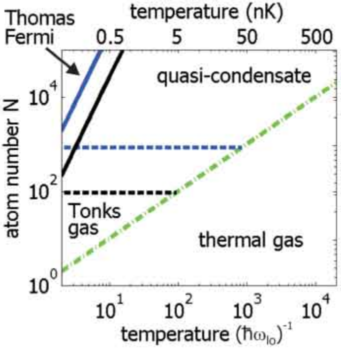
- high T: exponential decay of the correlation function: $e^{-\frac{x}{\xi T}}$
  $\Rightarrow$ thermal fluctuations
- low T: algebraically decay: $x^{-\frac{1}{\eta}}$
  $\Rightarrow$ quantum fluctuations



different regimes

three regimes below the degeneracy temperature $T_d \approx N\hbar\omega$ (green)

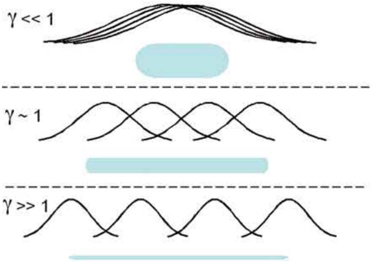
- BEC
- quasi-condensate
- Tonks-Girardeau gas of impenetrable bosons



different regimes

weakly and strong interacting regime

- weakly interacting regime: $\xi \gg \frac{1}{n}$
  $\Rightarrow$ small parameter $\gamma = \sqrt{\frac{1}{\xi n}} \ll 1$
- strong interacting regime
  $\Rightarrow \gamma \gg 1$



characteristic coherence length ξ
mean particle separation $\frac{1}{n}$

one important parameter γ

- interaction energy: $E_{int} = n_{1D} g_{1D}$
- kinetic energy: $E_{kin} = \frac{\hbar^2 n_{1D}^2}{m}$

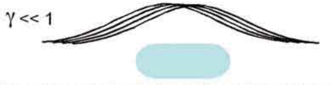
$$\gamma = \frac{E_{int}}{E_{kin}} = \frac{m g_{1D}}{\hbar^2 n_{1D}}$$

γ characterizes the behavior of trapped 1D-gases

one important parameter γ

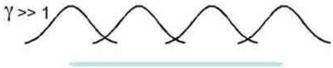
Thomas Fermi regime ($\gamma \ll 1$)

- high density
- weakly interaction
- mean field regime well described by the GPE
- BEC is possible
- the system retains its 3D feature



Tonks-Girardeau regime ($\gamma \gg 1$)

- low density
- strong interaction
- fermionic properties



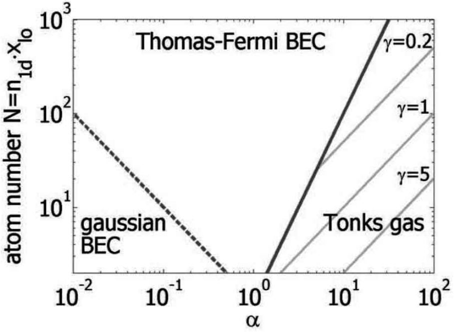
regimes

weakly interacting regime

- $\gamma \ll 1$
- reducing N
  $\Rightarrow$ macroscopic occupation of the ground state

strongly interacting regime

- $\gamma \gg 1$
- reducing N
  $\Rightarrow$ strongly interacting Tonks-Girardeau gas



$\alpha = \frac{mgl}{\hbar^2}$

- Tonks gas and Bose-Fermi mapping:
  $\Psi_B(x_1, ..., x_n) = |\Psi_F(x_1, ..., x_n)|$
- interaction: repulsive zero-range force
- Lieb and Liniger (1963): exact solution as a mathematical problem

Tonks-Girardeau gas, $\gamma \to \infty$

- interaction: impenetrable core: $\Psi(x_1, ..., x_N) = 0$ if $x_i = x_j$
- $\Psi_B = A\Psi_F$
- the relationship permits comparison of approximation methods designed for Fermi systems

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- groundstate: $\Psi_0^B = \left|\Psi_0^F\right| \propto \left|det[\varphi_i(x_j)]\right| \propto \Pi_{j>l} \left|\sin\left[\frac{\pi}{L}(x_j - x_l)\right]\right|$

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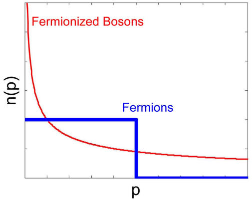
Tonks-Girardeau gas, $\gamma \to \infty$

- ground state energy: $E = \sum_n \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 (\pi \rho_0)^2}{6\pi m}$
- density distribution $\left|\Psi_0^B(x)\right|^2 = \left|\Psi_0^F(x)\right|^2$
- pair correlation function
  $\left\langle \Psi^\dagger(x)\Psi^\dagger(0)\Psi(0)\Psi(x)\right\rangle \overset{x \ll L}{\approx} 1 - \left(\frac{\sin(\pi\rho x)}{\pi\rho x}\right)^2$
- correlation function
  $g(x) = \left\langle \Psi_B^\dagger(0)\Psi_B(x)\right\rangle \neq \left\langle \Psi_F^\dagger(0)\Psi_F(x)\right\rangle$
  (absolute value of det matters)
- momentum distribution $n(p) \approx \int e^{-ipx} g(x) dx$
  different to the one of free fermions



Interaction

we want to solve now the one-dimensional problem in general
how is the interaction?

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Interaction

model to describe the binary collisions between cold atoms:

a) axially 2D harmonic potential of a frequency $\omega_\perp$
b) Atomic motion along the Z axis is free
c) pseudopotential $U(r) = g\delta(r)\left(\frac{\partial}{\partial r} r\right)$
d) atomic motion is cooled down below the transverse vibrational energy $\hbar\omega_\perp$

Schrödinger equation:

$$\left[\frac{p_z}{2\mu} + g\delta(r)\left(\frac{\partial}{\partial r} r\right) + H_\perp(p_x, p_y, x, y)\right]\Psi = E\Psi$$

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$g = \frac{2\pi\hbar^2 a}{\mu}$, $H_\perp = \frac{p_x^2+p_y^2}{2\mu} + \frac{\mu\omega_\perp^2(x^2+y^2)}{2}$

- incident wave: particle in the groundstate of $H_\perp$: $e^{ik_z z}\Phi_{n=0,m_z=0}(\rho)$
- longitutinal kinetic energy: $\frac{\hbar^2 k_z^2}{2\mu} < E_{n=2,m_z=0} - E_{n=0,m_z=0} = 2\hbar\omega_\perp$
  $E_{n,m_z} = \hbar\omega_\perp(n+1)$: energy of the 2D harmonic oscillator

$$\Psi(z,\rho) \to [e^{ik_z z} + f_{even} e^{ik_z|z|} + f_{odd} e^{ik_z|z|}]\Phi_{0,0}(\rho)$$

Interaction

- one-dimensional scattering amplitudes can be calculated analytically for the potential $U(r) = g\delta(r)\left(\frac{\partial}{\partial r} r\right)$
- $f(k_z) = -\frac{1}{1+ik_z a_{1D} - \mathcal{O}((k_z a_\perp)^3)} \approx -\frac{1}{1+ik_z a_{1D}}$
- scattering length: $a_{1D} = -\frac{a_\perp^2}{2a}\left(1 - C\frac{a}{a_\perp}\right)$
- calculate a scattering amplitude for a 1D δ-potential $U_{1D}(z) = g_{1D}\delta(z)$
  $\Rightarrow$ spherical scattering process reduces to 1D description with the same phase shift

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exact solution

Bethe ansatz:

$$\Psi(x_1, ..., x_N) = \sum_P a(P) e^{i \sum_n k_{P(n)} x_n}$$

for $x_1 < x_2 < ... < x_N$
the P's are the $N!$ possible permutations of the set $1, ..., N$.
physical interpretation:

- when the particle coordinates are all distinct $\Rightarrow$ potential energy term vanishes
  $\Rightarrow$ eigenstates: linear combination of single particle plane waves
- if 2 particles n and m have the same coordinate: collision
- considering all possible sequences of two-body collisions leads to the wavefunction.

exact solution

when permuatations P and P' only differ by the transposition of 1 and 2

$$a(P) = \frac{k_1 - k_2 + ic}{k_1 - k_2 - ic} a(P')$$

$\Rightarrow$ the coefficients are fully determined by two-body collisions.

exact solution

the momenta k_n are determined by requiring that the wf obeys periodic boundary conditions:

$$e^{ik_nL} = \prod_{m=1,m\neq n}^{N} \frac{k_n - k_m + ic}{k_n - k_m - ic}$$

for each $1 \leq n \leq N$.
taking the logarithm $\Rightarrow$ the eigenstates are labeled by a set of integers I_n

$$k_n = \frac{2\pi I_n}{L} + \frac{1}{L}\sum_m log\left(\frac{k_n - k_m + ic}{k_n - k_m - ic}\right)$$

ground state: filling the pseudo Fermi-sea of the I_n variables.

exact solution

$$k_n = \frac{2\pi I_n}{L} + \frac{1}{L}\sum_m log\left(\frac{k_n - k_m + ic}{k_n - k_m - ic}\right)$$

in the contimuum limit the sum becomes an integral for the density

$$\rho(k_n) = \frac{1}{L(k_{n+1} - k_n)}$$

$$2\pi\rho(k) = 1 + 2\int_{-q_0}^{q_0} \frac{c\rho(k')}{c^2 + (k - k')^2} dx$$

with $\rho(k) = 0$ for $|k| > q_0$
and the normalization

$$\rho_o = \int_{-q_0}^{q_0} dk\rho(k)$$

exact solution

by changing to dimensionless variables ($g(u) = \rho(q_0 x)$), this leads to the three equations:

$$1 + 2\lambda \int_{-1}^{1} \frac{g(u')}{\lambda^2 + (u - u')^2} du' = 2\pi g(u) \tag{1}$$

$$e(\gamma) = \frac{\gamma^3}{\lambda^3} \int_{-1}^{1} g(u) u^2 du \tag{2}$$

$$\gamma \int_{-1}^{1} g(u) du = \lambda \tag{3}$$

with $\gamma = \frac{c}{\rho_0}$, $\lambda = \frac{c}{q_0}$, $g(u) = \rho(q_0 x)$
$E_0 = N\rho^2 e(\gamma)$

exact solution

in the limit $c \to \infty$

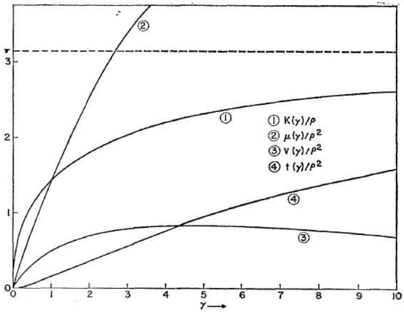
$$1 + 2\lambda \int_{-1}^{1} \frac{g(u')}{\lambda^2 + (u-u')^2} du' = 2\pi g(u) \Rightarrow g(u) \to \frac{1}{2\pi} \tag{4}$$

$$e(\gamma) = \frac{\gamma^3}{\lambda^3} \int g(u) u^2 du \Rightarrow \int u^2 du \tag{5}$$

In the limit of strong interaction the ground state energy becomes that of the TG gas

discussion

1. cutoff momentum $K \overset{\gamma=\infty}{\rightarrow} \pi\rho$
2. chemical potential: $\mu = \frac{\partial E_0}{\partial N} = \rho^2 \left(3e - \gamma \frac{de}{d\gamma}\right) \rightarrow \pi^2\rho^2$
3. potential energy: $v = \frac{c}{N}\frac{\partial}{\partial c}E_0 = \rho^2\gamma\frac{de}{d\gamma} \rightarrow 0$
4. kinetic energy: $t = \frac{1}{N}E_0 - v = \rho^2 \left(e - \gamma \frac{de}{d\gamma}\right) \rightarrow \frac{\pi^2\rho^2}{3}$

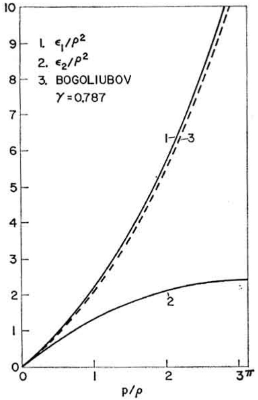


discussion

- physical properties of the Lieb-Liniger gas depend only on the dimensionless ratio $\gamma = \frac{c}{\rho_0}$
- $\gamma \to \infty$: fermionic properties (TG gas)
- low density corresponds to strong interaction, which is the reverse in 3D

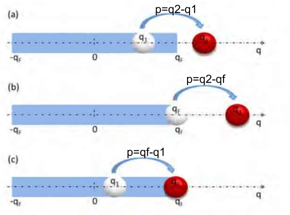
excitation spectrum

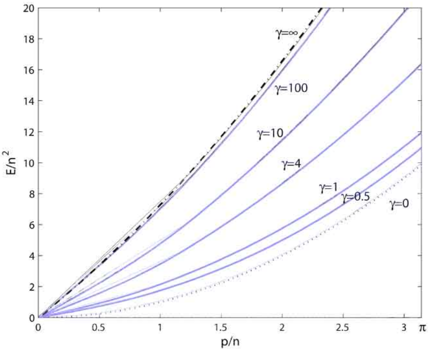
- double spectrum
- Bogoliubov's perturbation theory:
  - quiet accurately for a weak potential
  - second spectrum entirely unaccounted
- for small excitations: linear spectrum
  $\epsilon(p) = v_s p$
  velocity of sound:
  $v_s = 2\left(\mu(\gamma) - \frac{1}{2}\gamma\frac{\partial\mu(\gamma)}{\partial\gamma}\right)^{\frac{1}{2}}$



excitation spectrum

fermi sea



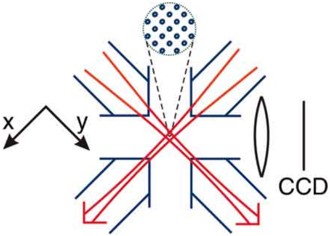


- $\gamma = 0$: $E(p) = \frac{p^2}{2m} \Rightarrow$ free particles
- $\gamma = \infty$: $E(p) = \frac{p^2}{2m} + 2\pi np$
  $\Rightarrow$ fermi gas

experimental setup

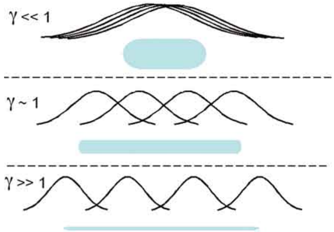
Weiss, Kinoshita:
enter the TG regime with cold ^{87}Rb atoms by trapping them with two light traps

- blue-detuned beams form 2D optical lattice
- atoms are confined in 1D tubes
- red-detuned waves trap the atoms axially.



the two light traps are independent

- transverse confinement can be made tighter $\Rightarrow$ increases γ
- strengthening the axial confinement decreases γ
- $\Rightarrow$ scan γ an make the atoms either BEC-like or TG like

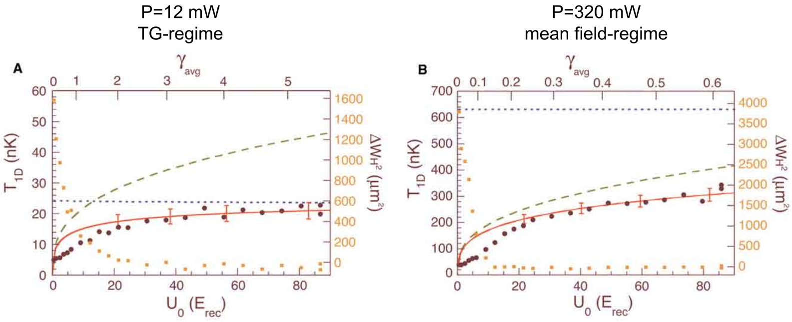


measurements

measurement of ϵ

- suddenly turn off crossed dipole trap
- atoms expand ballistically
- take images after 7 ms and 17 ms
- $\epsilon = \frac{k_B T_{1D}}{2}$ as a function of U_0

measurements



- $U_0 > 20E_{rec}$: only vertical expansion
- ΔW: transverse width (squares)
- green: exact mean-filed theory, $\gamma \ll 1$
- blue: exact TG-theory, $\gamma \gg 1$

conclusion

- reduced dimensions strongly enhances quantum fluctuations
- completely new features in 1D
- two regimes: weakly interacing ($\gamma \ll 1$), strong interacting ($\gamma \gg 1$)
- the interaction can be described by a δ-function potential with a one-dimensional coupling strength g_{1D}, which is linked to 3D parameters
- in the strong interacting regime ($\gamma \gg 1$) the bosons get Fermi-like features
- experimental realization with optical lattices