Quantum field-theory of low dimensional systems

– Coherent states for bosons –

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Contents

Motivation

- **Single particle:** express operators in terms of $\hat{x}$ and $\hat{p}$

  $\longrightarrow$ natural representation = **coordinate representation** (eigenfunctions of **position** operator $\hat{x}$)

- **System of identical particles:** 2nd quantization, express operators in terms of $b^\dagger$ and b

  $\longrightarrow$ natural representation = **coherent state representation** (eigenstates of **annihilation** operator b)

- Coherent states provide important representation for **path integral formalism** of many-particle systems

- Coherent states represent the **most classical-like states** (e.g.: $\Delta\hat{x}\,\Delta\hat{p} = \frac{\hbar}{2}$, $\langle\hat{\vec{E}}\rangle \neq 0$)

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Systems of identical particles

- **Hilbert space** for N particles: $\mathcal{H}_N = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}}_{N \text{ times}}$
- in nature: only totally symmetric **or** antisymmetric states are observed $\Rightarrow \mathcal{H}_N = \mathcal{B}_N \oplus \mathcal{F}_N$
- construct $\mathcal{B}_N$ by **symmetrization operator**: $\mathcal{B}_N = \mathcal{P}_B \mathcal{H}_N$
- physical bosonic system: **no restriction** on total number of particles $N = \sum_{i=1}^{\infty} n_{\alpha_i}$ nor on particles n_{α_i} in state $|\alpha_i\rangle$

$\longrightarrow$ **Fock space:** $\mathcal{B} \equiv \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \ldots = \oplus_{n=0}^{\infty} \mathcal{B}_n$

$\longrightarrow$ ONB for $\mathcal{B}$: $\{|0\rangle, |\alpha_1\rangle, |\alpha_1\alpha_2\rangle, \ldots |\alpha_1\alpha_2\ldots\alpha_i\ldots\rangle, \ldots\}$
$= \{|n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_i} \ldots\rangle\} \equiv \{|n_1 n_2 \ldots n_i \ldots\rangle\}$

$\longrightarrow$ construction: $|\alpha_1\alpha_2\ldots\alpha_N\rangle = \frac{1}{\sqrt{\prod_\alpha n_\alpha!}} b_1^\dagger b_2^\dagger \cdots b_N^\dagger |0\rangle$

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Creation and annihilation operators

- $b^\dagger$ and b **provide representation** for many-particle states **and basis** for many-body operators
- **Definitions:**

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\begin{aligned}
& b_i^\dagger \,|n_1 n_2 \ldots n_i \ldots\rangle \equiv \sqrt{n_i+1}\,|n_1 n_2 \ldots n_i+1 \ldots\rangle \\
b_i \equiv (b_i^\dagger)^\dagger \quad \Rightarrow \quad & b_i \,|n_1 n_2 \ldots n_i \ldots\rangle = \sqrt{n_i}\,|n_1 n_2 \ldots n_i-1 \ldots\rangle \\
\hat{n}_i \equiv b_i^\dagger b_i \quad \Rightarrow \quad & \hat{n}_i \,|n_1 n_2 \ldots n_i \ldots\rangle = n_i\,|n_1 n_2 \ldots n_i \ldots\rangle
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- $b^\dagger$ and b **operate in Fock space** instead of Hilbert space
- symmetry property $\Rightarrow$ **commutation relations:**

$$[b_i^\dagger, b_j^\dagger] = 0 = [b_i, b_j] \qquad \text{and} \qquad [b_i, b_j^\dagger] = \delta_{ij}$$

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Motivation

- normally: basis of **position** eigenstates $|\vec{r}\rangle$

  $\longrightarrow$ **permanents** as natural basis of $\mathcal{B}$

$$\psi_{\alpha_1 \dots \alpha_N}(\vec{r}_1, \dots \vec{r}_N) = (\vec{r}_1 \dots \vec{r}_N | \alpha_1 \dots \alpha_N\rangle = \frac{Per(\psi_{\alpha_i}(\vec{r}_j))}{\sqrt{N! \prod_\alpha n_\alpha!}}$$

- now second quantization: $(\hat{x}, \hat{p}) \longrightarrow (b^\dagger_\alpha, b_\alpha)$

  $\longrightarrow$ basis of eigenstates $|\Phi\rangle$ of **annihilation operator** b

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Why eigenstates of b instead of $b^\dagger$?

consider general state $|\Phi\rangle$ of Fock space $\mathcal{B}$:

$$|\Phi\rangle = \sum_{n=0}^{\infty} \sum_{\alpha_1 \ldots \alpha_n} \Phi_{\alpha_1 \ldots \alpha_n} \; |\alpha_1 \; \ldots \; \alpha_n\rangle$$

$$= \sum_{n_{\alpha_1}, \ldots n_{\alpha_i}, \ldots} \Phi_{n_{\alpha_1} \ldots n_{\alpha_i} \ldots} \; |n_{\alpha_1} \; \ldots \; n_{\alpha_i} \; \ldots\rangle$$

$\longrightarrow$ $|\Phi'\rangle \equiv b^\dagger_{\alpha_j} |\Phi\rangle \;\Rightarrow\; n'_{\alpha_j} = n_{\alpha_j} + 1$ for every component

$\Rightarrow$ (component of $|\Phi\rangle$ with lowest N) $\notin$ $|\Phi'\rangle$ anymore

$\Rightarrow$ $|\Phi'\rangle = b^\dagger_{\alpha_j} |\Phi\rangle \;\propto\; |\Phi\rangle$ **is not possible**

$\Rightarrow$ $b^\dagger_\alpha$ **cannot** have an eigenstate

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using $|n_{\alpha_1}\ldots n_{\alpha_i}\ldots\rangle = \frac{b_{\alpha_1}^{\dagger n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}}\cdots\frac{b_{\alpha_i}^{\dagger n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}}\cdots|0\rangle$:

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$$\Rightarrow \Phi_{n_{\alpha_1}\dots n_{\alpha_i}\dots} = \frac{\Phi_{\alpha_i}^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}}\,\Phi_{n_{\alpha_1}\dots 0\dots} = \frac{\Phi_{\alpha_1}^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}}\cdots\frac{\Phi_{\alpha_i}^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}}\cdots$$

using $|n_{\alpha_1}\dots n_{\alpha_i}\dots\rangle = \frac{b^{\dagger\, n_{\alpha_1}}_{\alpha_1}}{\sqrt{n_{\alpha_1}!}}\cdots\frac{b^{\dagger\, n_{\alpha_i}}_{\alpha_i}}{\sqrt{n_{\alpha_i}!}}\cdots|0\rangle$:

$$\Rightarrow |\Phi\rangle = \sum_{n_{\alpha_1},\dots n_{\alpha_i},\dots}\prod_{i=1}^{\infty}\frac{(\Phi_{\alpha_i}b^\dagger_{\alpha_i})^{n_{\alpha_i}}}{n_{\alpha_i}!}|0\rangle = \prod_{i=1}^{\infty}\sum_{n_{\alpha_i}}\frac{(\Phi_{\alpha_i}b^\dagger_{\alpha_i})^{n_{\alpha_i}}}{n_{\alpha_i}!}|0\rangle$$

Construction

assuming $|\Phi\rangle$ is eigenstate of any b_α $\Rightarrow$ $b_{\alpha_i}|\Phi\rangle = \Phi_{\alpha_i}|\Phi\rangle$

convenient expansion: $|\Phi\rangle = \sum_{n_{\alpha_1},\dots n_{\alpha_i},\dots} \Phi_{n_{\alpha_1}\dots n_{\alpha_i}\dots} |n_{\alpha_1}\dots n_{\alpha_i}\dots\rangle$

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$$\Rightarrow |\Phi\rangle = \sum_{n_{\alpha_1},\dots n_{\alpha_i},\dots}\prod_{i=1}^{\infty}\frac{(\Phi_{\alpha_i}b_{\alpha_i}^\dagger)^{n_{\alpha_i}}}{n_{\alpha_i}!}|0\rangle = \prod_{i=1}^{\infty}\sum_{n_{\alpha_i}}\frac{(\Phi_{\alpha_i}b_{\alpha_i}^\dagger)^{n_{\alpha_i}}}{n_{\alpha_i}!}|0\rangle$$

$$= \prod_\alpha \exp\left(\Phi_\alpha b_\alpha^\dagger\right)|0\rangle = \boxed{\exp\left(\sum_\alpha \Phi_\alpha b_\alpha^\dagger\right)|0\rangle}$$

Properties and consequences

Useful relations

$$b_\alpha \left|\Phi\right\rangle = \Phi_\alpha \left|\Phi\right\rangle \quad \Rightarrow \quad \left|\Phi\right\rangle = \exp\left(\sum_\alpha \Phi_\alpha b_\alpha^\dagger\right)\left|0\right\rangle$$

$\Rightarrow$ adjoint expression:

$$\left\langle\Phi\right| b_\alpha^\dagger = \left\langle\Phi\right| \Phi_\alpha^* \quad \Rightarrow \quad \left\langle\Phi\right| = \left\langle 0\right| \exp\left(\sum_\alpha \Phi_\alpha^* b_\alpha\right)$$

action of creation operators on coherent states:

$$b_\alpha^\dagger \left|\Phi\right\rangle = b_\alpha^\dagger \exp\left(\sum_{\alpha'} \Phi_{\alpha'} b_{\alpha'}^\dagger\right)\left|0\right\rangle = \frac{\partial}{\partial \Phi_\alpha}\left|\Phi\right\rangle$$

$\Rightarrow$ adjoint expression:

$$\left\langle\Phi\right| b_\alpha = \left\langle\Phi\right| \exp\left(\sum_{\alpha'} \Phi_{\alpha'}^* b_{\alpha'}\right) b_\alpha = \frac{\partial}{\partial \Phi_\alpha^*}\left|\Phi\right\rangle$$

Properties and consequences

Useful relations

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Properties and consequences

Overlap

$$\langle \Phi | \Phi' \rangle = \sum_{\substack{n_{\alpha_1},\ldots n_{\alpha_i},\ldots \\ n'_{\alpha_1},\ldots n'_{\alpha_i},\ldots}} \frac{{\Phi^*}_{\alpha_1}^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \cdots \frac{{\Phi^*}_{\alpha_i}^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}} \cdots \frac{{\Phi'}_{\alpha_1}^{n'_{\alpha_1}}}{\sqrt{n'_{\alpha_1}!}} \cdots \frac{{\Phi'}_{\alpha_i}^{n'_{\alpha_i}}}{\sqrt{n'_{\alpha_i}!}} \cdots$$

$$\times \underbrace{\langle n_{\alpha_1} \ldots n_{\alpha_i} \ldots | n'_{\alpha_1} \ldots n'_{\alpha_i} \ldots \rangle}_{\prod_{i=1}^{\infty} \delta_{n_{\alpha_i} n'_{\alpha_i}}}$$

Properties and consequences

Overlap

$$\langle \Phi | \Phi' \rangle = \sum_{\substack{n_{\alpha_1},\ldots n_{\alpha_i},\ldots \\ n'_{\alpha_1},\ldots n'_{\alpha_i},\ldots}} \frac{{\Phi^*}_{\alpha_1}^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \cdots \frac{{\Phi^*}_{\alpha_i}^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}} \cdots \frac{{\Phi'}_{\alpha_1}^{n'_{\alpha_1}}}{\sqrt{n'_{\alpha_1}!}} \cdots \frac{{\Phi'}_{\alpha_i}^{n'_{\alpha_i}}}{\sqrt{n'_{\alpha_i}!}} \cdots$$

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$$= \sum_{n_{\alpha_1},\ldots n_{\alpha_i},\ldots} \frac{({\Phi^*}_{\alpha_1} {\Phi'}_{\alpha_1})^{n_{\alpha_1}}}{n_{\alpha_1}!} \cdots \frac{({\Phi^*}_{\alpha_i} {\Phi'}_{\alpha_i})^{n_{\alpha_i}}}{n_{\alpha_i}!} \cdots$$

Properties and consequences

Overlap

$$\langle \Phi | \Phi' \rangle = \sum_{\substack{n_{\alpha_1},\ldots n_{\alpha_i},\ldots \\ n'_{\alpha_1},\ldots n'_{\alpha_i},\ldots}} \frac{{\Phi^*}_{\alpha_1}^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \cdots \frac{{\Phi^*}_{\alpha_i}^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}} \cdots \frac{{\Phi'}_{\alpha_1}^{n'_{\alpha_1}}}{\sqrt{n'_{\alpha_1}!}} \cdots \frac{{\Phi'}_{\alpha_i}^{n'_{\alpha_i}}}{\sqrt{n'_{\alpha_i}!}} \cdots$$

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$$= \sum_{n_{\alpha_1},\ldots n_{\alpha_i},\ldots} \frac{({\Phi^*}_{\alpha_1} {\Phi'}_{\alpha_1})^{n_{\alpha_1}}}{n_{\alpha_1}!} \cdots \frac{({\Phi^*}_{\alpha_i} {\Phi'}_{\alpha_i})^{n_{\alpha_i}}}{n_{\alpha_i}!} \cdots$$

$$= \prod_{\alpha} \sum_{n_\alpha} \frac{({\Phi^*}_{\alpha} {\Phi'}_{\alpha})^{n_\alpha}}{n_\alpha!} = \exp\left(\sum_\alpha {\Phi^*}_\alpha {\Phi'}_\alpha \right)$$

Properties and consequences

Overlap

$$\langle \Phi | \Phi' \rangle = \sum_{\substack{n_{\alpha_1},\ldots n_{\alpha_i},\ldots \\ n'_{\alpha_1},\ldots n'_{\alpha_i},\ldots}} \frac{{\Phi^*}_{\alpha_1}^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \cdots \frac{{\Phi^*}_{\alpha_i}^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}!}} \cdots \frac{{\Phi'}_{\alpha_1}^{n'_{\alpha_1}}}{\sqrt{n'_{\alpha_1}!}} \cdots \frac{{\Phi'}_{\alpha_i}^{n'_{\alpha_i}}}{\sqrt{n'_{\alpha_i}!}} \cdots$$

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$\Rightarrow$ coherent states are **neither orthogonal nor normalized**

Properties and consequences

Overcompleteness

closure relation:

$$\prod_\alpha \int_{\mathbb{C}} \frac{\mathrm{d}\Phi_\alpha^* \mathrm{d}\Phi_\alpha}{2\imath\pi} \exp\left(-\sum_\alpha \Phi_\alpha^* \Phi_\alpha\right) |\Phi\rangle \langle\Phi| = \mathbb{1}_{\mathcal{B}}$$

Properties and consequences

Overcompleteness

closure relation:

$$\prod_\alpha \int_{\mathbb{C}} \frac{d\Phi^*_\alpha d\Phi_\alpha}{2\imath\pi} \exp\left(-\sum_\alpha \Phi^*_\alpha \Phi_\alpha\right) |\Phi\rangle \langle\Phi| \; = \; \mathbb{1}_{\mathcal{B}}$$

Proof:

1) Start with state $|n\rangle \in \mathcal{B}$ representing n particles in state $|\alpha\rangle$

$$\int_{\mathbb{C}} \frac{d\Phi^* d\Phi}{2\imath\pi} e^{-\Phi^*\Phi} |\Phi\rangle \langle\Phi| \; = \; \int_{\mathbb{C}} \frac{d\Phi^* d\Phi}{2\imath\pi} e^{-\Phi^*\Phi} \sum_{n,m} \frac{\Phi^n}{\sqrt{n!}} |n\rangle \langle m| \frac{{\Phi^*}^m}{\sqrt{m!}}$$

Properties and consequences

Overcompleteness

closure relation:

$$\prod_\alpha \int_{\mathbb{C}} \frac{d\Phi_\alpha^* d\Phi_\alpha}{2\imath\pi} \exp\left(-\sum_\alpha \Phi_\alpha^* \Phi_\alpha\right) |\Phi\rangle \langle\Phi| = \mathbb{1}_{\mathcal{B}}$$

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Properties and consequences

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$$\overset{\Phi=\rho e^{\imath\theta}}{=} \int\int \frac{\rho\, d\rho\, d\theta}{\pi} e^{-\rho^2} \sum_{n,m} \frac{(\rho e^{\imath\theta})^n}{\sqrt{n!}} \frac{(\rho e^{-\imath\theta})^m}{\sqrt{m!}} |n\rangle \langle m|$$

Properties and consequences

Overcompleteness

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$$= \int \frac{\rho}{\pi} d\rho\; e^{-\rho^2} \sum_{n,m} \frac{\rho^{n+m}}{\sqrt{n!m!}} \int d\theta\; e^{\imath(n-m)\theta} |n\rangle\langle m|$$

Properties and consequences

Overcompleteness

closure relation:

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Properties and consequences

Overcompleteness

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$$= \sum_n \frac{2}{n!} \int d\rho\, e^{-\rho^2} \rho^{2n+1} |n\rangle\langle n|$$

Properties and consequences

Overcompleteness

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$$= \sum_n \frac{2}{n!} \underbrace{\int d\rho\, e^{-\rho^2} \rho^{2n+1}}_{=\frac{1}{2}\Gamma(n+1) = \frac{1}{2}n! \text{ for } n\in\mathbb{N}} |n\rangle\langle n|$$

Properties and consequences

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Properties and consequences

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2) now: set of states $\{|\alpha_k\rangle\} \Rightarrow |n\rangle \to |n_{\alpha_1} \dots n_{\alpha_i} \dots\rangle$

$$\prod_\alpha \int_{\mathbb{C}} \frac{d\Phi_\alpha^* d\Phi_\alpha}{2\imath\pi} e^{-\sum_\alpha \Phi_\alpha^* \Phi_\alpha} |\Phi\rangle \langle\Phi|$$

Properties and consequences

Overcompleteness

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$$= \sum_{\{n_\alpha\}} |n_{\alpha_1} \dots n_{\alpha_i} \dots\rangle \langle n_{\alpha_1} \dots n_{\alpha_i} \dots|$$

Properties and consequences

Overcompleteness

closure relation:

$$\prod_\alpha \int_{\mathbb{C}} \frac{d\Phi^*_\alpha d\Phi_\alpha}{2\imath\pi} \exp\left(-\sum_\alpha \Phi^*_\alpha \Phi_\alpha\right) |\Phi\rangle\langle\Phi| = \mathbb{1}_{\mathcal{B}}$$

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$$= \sum_{\{n_\alpha\}} |n_{\alpha_1} \dots n_{\alpha_i} \dots\rangle\langle n_{\alpha_1} \dots n_{\alpha_i} \dots| = \mathbb{1}_{\mathcal{B}}$$

Properties and consequences

coherent state representation

vs. coordinate representation

- representation of arbitrary state $|\psi\rangle$:

$$|\psi\rangle = \mathbb{1} \cdot |\psi\rangle = \begin{cases} \prod\limits_{\alpha} \int\limits_{\mathbb{C}} \frac{\mathrm{d}\Phi_\alpha^* \mathrm{d}\Phi_\alpha}{2\imath\pi} \exp\left(- \sum\limits_{\alpha} \Phi_\alpha^* \Phi_\alpha \right) \langle \Phi | \psi \rangle \, |\Phi\rangle \\ \int \mathrm{d}r^3 \, \langle \vec{r} | \psi \rangle \, |\vec{r}\rangle \end{cases}$$

with $\langle \Phi | \psi \rangle \equiv \psi(\Phi^*)$: coherent state representation of $|\psi\rangle$

cf.: $\langle \vec{r} | \psi \rangle \equiv \psi(\vec{r})$: coordinate representation of $|\psi\rangle$

Properties and consequences

coherent state representation
vs. coordinate representation

- representation of arbitrary state $|\psi\rangle$:

$$|\psi\rangle = \mathbb{1} \cdot |\psi\rangle = \begin{cases} \prod_\alpha \int_{\mathbb{C}} \frac{d\Phi^*_\alpha d\Phi_\alpha}{2\imath\pi} \exp\left(-\sum_\alpha \Phi^*_\alpha \Phi_\alpha\right) \langle\Phi|\psi\rangle\,|\Phi\rangle \\ \int dr^3 \,\langle\vec{r}|\psi\rangle\,|\vec{r}\rangle \end{cases}$$

with $\langle\Phi|\psi\rangle \equiv \psi(\Phi^*)$: coherent state representation of $|\psi\rangle$
cf.: $\langle\vec{r}|\psi\rangle \equiv \psi(\vec{r})$: coordinate representation of $|\psi\rangle$

- representation of $b^\dagger_\alpha, b_\alpha$ and $\hat{x}, \hat{p}$ respectively:

$$\langle\Phi|b^\dagger_\alpha|\psi\rangle = \Phi^*_\alpha\,\langle\Phi|\psi\rangle = \Phi^*_\alpha\,\psi(\Phi^*) \Rightarrow b^\dagger_\alpha \triangleq \Phi^*_\alpha$$

$$\langle\Phi|b_\alpha|\psi\rangle = \frac{\partial}{\partial\Phi^*_\alpha}\langle\Phi|\psi\rangle = \frac{\partial}{\partial\Phi^*_\alpha}\psi(\Phi^*) \Rightarrow b_\alpha \triangleq \frac{\partial}{\partial\Phi^*_\alpha}$$

cf.: $\hat{x} \triangleq \vec{r}$ and $\hat{p} \triangleq -\imath\hbar\frac{\partial}{\partial\vec{r}}$ (in coordinate representation)

Properties and consequences

Matrix elements

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simple form of **matrix elements** of **normal-ordered operators** $A(b^\dagger_\alpha, b_\alpha)$ *(normal order: all* $b^\dagger_\alpha$ *to the left of all* b_α*)*:

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- relative width: $\frac{\sigma}{\overline{N}} = \frac{1}{\sqrt{\overline{N}}} \xrightarrow{\overline{N} \to \infty} 0$

  $\hookrightarrow$ in TD-limit: coherent states **sharply peaked** around $\overline{N}$

Summary

- 2nd quantization: $(\hat{x}, \hat{p}) \longrightarrow (b^\dagger_\alpha, b_\alpha)$
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