Path-integrals for bosons

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Outline

Coherent state

- Coherent states $|\phi\rangle$ are eigenvectors of the annihilation operator a_i
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- Matrix elements of normal-ordered operator

$$\langle\phi|\mathrm{A}\left(a_\alpha^\dagger, a_\alpha\right)|\phi'\rangle = \mathrm{A}\left(\phi_\alpha^*, \phi'_\alpha\right) e^{\sum_\alpha \phi_\alpha^* \phi'_\alpha}$$

Path-integral for bosonic particles

- Break finite time interval into M infinitesimal steps ϵ

$$e^{-\frac{i}{\hbar}H(t_f - t_i)} = \left(e^{-\frac{i}{\hbar}\epsilon H}\right)^M \quad ; \quad \epsilon = \frac{t_f - t_i}{M}$$

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$$\begin{aligned}
K(\phi^*_{\alpha,f} t_f; \phi_{\alpha,i} t_i) &= \langle \phi_f | e^{-\frac{i}{\hbar}H\left(t_f - t_i\right)} | \phi_i \rangle \\
&= \lim_{M\to\infty} \langle \phi_f | e^{-\frac{i}{\hbar}\epsilon H} \mathbb{1}_{M-1} e^{-\frac{i}{\hbar}\epsilon H} \mathbb{1}_{M-2} \ldots \mathbb{1}_1 e^{-\frac{i}{\hbar}\epsilon H} | \phi_i \rangle \\
&= \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k} \phi_{\alpha,k}} \\
&\quad \times \prod_{k=1}^{M} \langle \phi_k | e^{-\frac{i\epsilon}{\hbar} H\left(a^\dagger_\alpha, a_\alpha\right)} | \phi_{k-1} \rangle
\end{aligned}$$

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$$\times \prod_{k=1}^{M} \langle \phi_k | : e^{-\frac{i\epsilon}{\hbar} H\left(a^\dagger_\alpha, a_\alpha\right)} : + \cancelto{0}{\mathcal{O}\left(\epsilon^2\right)} \, |\phi_{k-1}\rangle$$

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$$
\begin{aligned}
K(\phi^*_{\alpha,f} t_f ; \phi_{\alpha,i} t_i) &= \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k} \phi_{\alpha,k}} \\
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&= \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \\
&\quad \times e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k} \phi_{\alpha,k} + \sum_{k=1}^{M} \phi^*_{\alpha,k} \phi_{\alpha,k-1} - \frac{i\epsilon}{\hbar} \sum_{k=1}^{M} H\left(\phi^*_{\alpha,k}, \phi_{\alpha,k-1}\right)}
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$$= \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} e^{S(M,\phi^*,\phi)}$$

Trajectory and continuous notation:

$$\phi^*_{\alpha,k}\frac{(\phi_{\alpha,k}-\phi_{\alpha,k-1})}{\epsilon} \equiv \phi^*_\alpha(t)\frac{\partial}{\partial t}\phi_\alpha(t)\,; \qquad \lim_{M\to\infty}\epsilon\sum_{k=1}^{M-1} \to \int_{t_i}^{t_f} dt$$

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$$= \sum_\alpha \phi^*_\alpha(t_f)\phi_\alpha(t_f) + \frac{i}{\hbar}\int_{t_i}^{t_f} dt\,\underbrace{\left[i\hbar\sum_\alpha \phi^*_\alpha(t)\frac{\partial\phi_\alpha(t)}{\partial t} - \mathrm{H}\left(\phi^*_\alpha(t),\phi_\alpha(t)\right)\right]}_{L(\phi^*(t),\phi(t))}$$

Propagator

$$K(\phi^*_{\alpha,f} t_f; \phi_{\alpha,i} t_i) = \int_{\phi_\alpha(t_i)=\phi_{\alpha,i}}^{\phi_\alpha(t_f)=\phi_{\alpha,f}} D\left[\phi^*_\alpha(t)\phi_\alpha(t)\right] e^{\sum_\alpha \phi^*_\alpha(t_f)\phi_\alpha(t_f)}$$

$$\times e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\left[i\hbar\sum_\alpha \phi^*_\alpha(t)\frac{\partial\phi_\alpha(t)}{\partial t} - \mathrm{H}\left(\phi^*_\alpha(t),\phi_\alpha(t)\right)\right]}$$

Measure

$$D\left[\phi^*_\alpha(t)\phi_\alpha(t)\right] = \lim_{M\to\infty}\int\prod_{k=1}^{M-1}\prod_\alpha \frac{d\phi^*_{\alpha,k}d\phi_{\alpha,k}}{2\pi i}$$

Partition function for bosonic many-particle systems

- Grand canonical ensemble:
  $\Rightarrow$ Hamiltonian = (H - μN) with $N = \sum_\alpha a^\dagger_\alpha a_\alpha$

$$Z = tr\left(e^{-\beta(H-\mu N)}\right)$$

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- Wick-Rotation: t = -iτ ; $\tau_f = -i\beta\hbar$; τ_i=0
- Comparison: $e^{-\frac{i}{\hbar}H\Delta t} \longleftrightarrow e^{-\beta(H-\mu N)}$

$$Z = \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} \prod_{\alpha} \frac{d\phi^*_{\alpha} d\phi_{\alpha}}{2\pi i} e^{-\sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k}\phi_{\alpha,k} - \sum_{\alpha} \phi^*_{\alpha}\phi_{\alpha}}$$

$$\times \prod_{k=1}^{M} \langle \phi_k | e^{-\epsilon\left(H\left(a^\dagger_\alpha, a_\alpha\right) - \mu N\right)} | \phi_{k-1} \rangle$$

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$$e^{-\sum\limits_{k=1}^{M}\sum\limits_{\alpha}\left[\epsilon H\left(\phi^*_{\alpha,k}, \phi_{\alpha,k-1}\right) - (1+\epsilon\mu)\phi^*_{\alpha,k}\phi_{\alpha,k-1}\right]}$$

$$Z = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i}$$
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<u>Rewrite exponent:</u>

$$-\lim_{M\to\infty} \sum_{k=1}^{M} \sum_{\alpha} \left[\phi^*_{\alpha,k}\phi_{\alpha,k} + \epsilon H\left(\phi^*_{\alpha,k}, \phi_{\alpha,k-1}\right) - (1+\epsilon\mu)\phi^*_{\alpha,k}\phi_{\alpha,k-1}\right]$$

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$$= -\lim_{M\to\infty} \epsilon \sum_{k=1}^{M}\sum_{\alpha}\left[H\left(\phi^*_{\alpha,k}, \phi_{\alpha,k-1}\right) + \phi^*_{\alpha,k}\left(\frac{\phi_{\alpha,k} - \phi_{\alpha,k-1}}{\epsilon} - \mu\phi_{\alpha,k-1}\right)\right]$$

$$= -\int_0^\beta d\tau \sum_{\alpha}\left[H\left(\phi^*_\alpha(\tau), \phi_\alpha(\tau)\right) + \phi^*_\alpha(\tau)\left(\frac{\partial}{\partial\tau} - \mu\right)\phi_\alpha(\tau)\right]$$

Partition function

$$Z = \int\limits_{\phi_\alpha(\beta)=\phi_\alpha(0)} D\left[\phi^*_\alpha(\tau), \phi_\alpha(\tau)\right]$$
$$\times\, e^{-\int\limits_0^\beta d\tau \sum\limits_\alpha \left[H\left(\phi^*_\alpha(\tau),\phi_\alpha(\tau)\right)+\phi^*_\alpha(\tau)\left(\frac{\partial}{\partial\tau}-\mu\right)\phi_\alpha(\tau)\right]}$$

Measure

$$D\left[\phi^*_\alpha(\tau)\phi_\alpha(\tau)\right] = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} \frac{d\phi^*_{\alpha,k}\, d\phi_{\alpha,k}}{2\pi i}$$

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$$\times\, e^{-\sum_{k=1}^{M}\sum_\alpha\left[\frac{\beta}{M}H\left(\phi^*_{\alpha,k},\phi_{\alpha,k-1}\right)-\left(1+\frac{\beta}{M}\mu\right)\phi^*_{\alpha,k}\phi_{\alpha,k-1}\right]}$$

- Change exponent representation:

$$e^{-\sum_{k=1}^{M}\sum_\alpha\left[\phi^*_{\alpha,k}\phi_{\alpha,k}+\left(\frac{\beta}{M}(\epsilon_\alpha-\mu)-1\right)\phi^*_{\alpha,k}\phi_{\alpha,k-1}\right]}$$

Example: Partition function of non-interacting bosons

- Hamiltonian: $H = \sum_\alpha \epsilon_\alpha a^\dagger_\alpha a_\alpha$
- Grand canonical ensemble:

$$Z = tr\left(e^{-\beta(H-\mu N)}\right)$$

$$= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_\alpha \frac{d\phi^*_{\alpha,k} d\phi_{\alpha,k}}{2\pi i} e^{-\sum_{k=1}^{M} \sum_\alpha \phi^*_{\alpha,k}\phi_{\alpha,k}}$$

$$\times\, e^{-\sum_{k=1}^{M} \sum_\alpha \left[\frac{\beta}{M} H\left(\phi^*_{\alpha,k}, \phi_{\alpha,k-1}\right) - \left(1+\frac{\beta}{M}\mu\right)\phi^*_{\alpha,k}\phi_{\alpha,k-1}\right]}$$

- Change exponent representation:

$$e^{-\sum_{k=1}^{M} \sum_\alpha \left[\phi^*_{\alpha,k}\phi_{\alpha,k} + \left(\frac{\beta}{M}(\epsilon_\alpha - \mu) - 1\right)\phi^*_{\alpha,k}\phi_{\alpha,k-1}\right]} = e^{-\boldsymbol{\phi}^*_{\boldsymbol{\alpha}} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{\phi}_{\boldsymbol{\alpha}}}$$

- Notation: $a = 1 - \frac{\beta}{M}(\epsilon_\alpha - \mu)$

$$\boldsymbol{\phi_\alpha} = \begin{pmatrix} \phi_{\alpha,1} \\ \phi_{\alpha,2} \\ \phi_{\alpha,3} \\ \vdots \\ \vdots \\ \phi_{\alpha,M} \end{pmatrix}, \quad \boldsymbol{S_\alpha} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -a \\ -a & 1 & 0 & \ddots & \vdots & 0 \\ 0 & -a & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & -a & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -a & 1 \end{pmatrix}$$

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- Transformation to the eigenbasis of $\boldsymbol{S}_{\boldsymbol{\alpha}}$:

$$\boldsymbol{M}\boldsymbol{M}^\dagger = \mathbb{1}\,; \qquad \boldsymbol{M}^\dagger \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{M} = \boldsymbol{S}_{\boldsymbol{\alpha}}^{\boldsymbol{D}}$$

$$\boldsymbol{\phi}_{\boldsymbol{\alpha}} \to \tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}} = \boldsymbol{M}^\dagger \boldsymbol{\phi}_{\boldsymbol{\alpha}}\,; \qquad \boldsymbol{\phi}_{\boldsymbol{\alpha}}^* \to \tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}^* = \boldsymbol{\phi}_{\boldsymbol{\alpha}}^* \boldsymbol{M}$$

$$\int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{\phi}_{\boldsymbol{\alpha}}} = \int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} \boldsymbol{M}\boldsymbol{M}^{\dagger} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{M}\boldsymbol{M}^{\dagger} \boldsymbol{\phi}_{\boldsymbol{\alpha}}}$$

$$\int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{\phi}_{\boldsymbol{\alpha}}} = \int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} \boldsymbol{M}\boldsymbol{M}^{\dagger} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{M}\boldsymbol{M}^{\dagger} \boldsymbol{\phi}_{\boldsymbol{\alpha}}}$$

$$= \int \prod_{k=1}^{M} \frac{d\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}^{*} d\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}^{*} \boldsymbol{S}_{\boldsymbol{\alpha}}^{\boldsymbol{D}} \tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}}$$

$$\int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{\phi}_{\boldsymbol{\alpha}}} = \int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} \boldsymbol{M}\boldsymbol{M}^{\dagger} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{M}\boldsymbol{M}^{\dagger} \boldsymbol{\phi}_{\boldsymbol{\alpha}}}$$

$$= \int \prod_{k=1}^{M} \frac{d\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}^{*} d\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}^{*} \boldsymbol{S}_{\boldsymbol{\alpha}}^{\boldsymbol{D}} \tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}}$$

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$$\int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{\phi}_{\boldsymbol{\alpha}}} = \int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} \boldsymbol{M}\boldsymbol{M}^{\dagger} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{M}\boldsymbol{M}^{\dagger} \boldsymbol{\phi}_{\boldsymbol{\alpha}}}$$

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$$= \int \prod_{k=1}^{M} \frac{d\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}^{*} d\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\sum_{k=1}^{M} s_k \tilde{\phi}_{\alpha,k}^{*} \tilde{\phi}_{\alpha,k}}$$

$\tilde{\phi}_{\alpha} = \mathrm{Re}\left(\tilde{\phi}_{\alpha}\right) + \mathrm{i}\,\mathrm{Im}\left(\tilde{\phi}_{\alpha}\right)$; Jacobian determinant equals 2i

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$$\int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*}\mathbf{S}_{\boldsymbol{\alpha}}\boldsymbol{\phi}_{\boldsymbol{\alpha}}} = \int \prod_{k=1}^{M} \frac{d\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\boldsymbol{\phi}_{\boldsymbol{\alpha}}^{*}\boldsymbol{M}\boldsymbol{M}^{\dagger}\mathbf{S}_{\boldsymbol{\alpha}}\boldsymbol{M}\boldsymbol{M}^{\dagger}\boldsymbol{\phi}_{\boldsymbol{\alpha}}}$$

$$= \int \prod_{k=1}^{M} \frac{d\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}^{*} d\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}}{2\pi i} e^{-\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}^{*}\mathbf{S}_{\boldsymbol{\alpha}}^{\boldsymbol{D}}\tilde{\boldsymbol{\phi}}_{\boldsymbol{\alpha}}}$$

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$$= \prod_{k=1}^{M} \frac{1}{\pi}\sqrt{\frac{\pi}{s_k}}\sqrt{\frac{\pi}{s_k}} = \frac{1}{\det \mathbf{S}_\alpha}$$

$$\det \mathbf{S}_\alpha = \det \begin{vmatrix} 1 & 0 & \cdots & 0 & -a \\ -a & 1 & \ddots & 0 & \\ 0 & -a & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & 0 & \cdots & -a & 1 \end{vmatrix}$$

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$$= 1 + (-1)^{M-1}(-a) \cdot \det \begin{vmatrix} -a & 1 & 0 & 0 \\ 0 & -a & \ddots & \vdots \\ \vdots & 0 & \ddots & 1 \\ 0 & \cdots & 0 & -a \end{vmatrix}$$

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$$= 1 + (-1)^{M-1}(-a)^M = 1 - \left(1 - \frac{\beta(\epsilon_\alpha - \mu)}{M}\right)^M$$

$$\xrightarrow{M\to\infty} \mathbf{1 - e^{-\boldsymbol{\beta}(\boldsymbol{\epsilon}_{\boldsymbol{\alpha}} - \boldsymbol{\mu})}}$$

$$Z = \lim_{M\to\infty} \prod_{\alpha} \frac{1}{\det \mathbf{S}_{\alpha}}$$

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$$= \sum_\alpha \frac{1}{e^{-\beta(\epsilon_\alpha - \mu)} - 1}$$

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$\hookrightarrow$ Boson occupation probability

Summary

Path-Integral for bosonic particles

- Time evolution operator: Break finite time interval into steps
- Coherent states
- Trajectory and continuous notation

$$K(\phi^*_{\alpha,f} t_f; \phi_{\alpha,i} t_i) = \int\limits_{\phi_\alpha(t_i)=\phi_{\alpha,i}}^{\phi_\alpha(t_f)=\phi_{\alpha,f}} D\left[\phi^*_\alpha(t)\phi_\alpha(t)\right] e^{\sum_\alpha \phi^*_\alpha(t_f)\phi_\alpha(t_f)} \times e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\left[i\hbar\sum_\alpha \phi^*_\alpha(t)\frac{\partial\phi_\alpha(t)}{\partial t} - \mathrm{H}\left(\phi^*_\alpha(t),\phi_\alpha(t)\right)\right]}$$

Summary

Partition function for bosonic many-particle systems

- Same ansatz as for path-integrals
- Grand canonical ensemble
- Trace of imaginary time evolution operator
- Wick-rotation

$$Z = \int_{\phi_\alpha(\beta)=\phi_\alpha(0)} D\left[\phi_\alpha^*(\tau), \phi_\alpha(\tau)\right]$$

$$\times\, e^{-\int_0^\beta d\tau \sum_\alpha \phi_\alpha^*(\tau)\left(\frac{\partial}{\partial\tau}-\mu\right)\phi_\alpha(\tau) + H\left(\phi_\alpha^*(\tau),\phi_\alpha(\tau)\right)}$$

References

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