Quantum field-theory of low dimensional systems

Coherent states for fermions

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Overview

Motivation

- Coherent state representation needed for **path integral formalism** of many-particle systems
- Construct **coherent states** $|\phi\rangle$ for fermions analogue to coherent states for bosons
  - Eigenstates of the **fermionic** annihilation operator f

$$f_\alpha|\phi\rangle = \phi_\alpha|\phi\rangle$$

- **Problem:** dealing with **anticommuting** behaviour of fermionic creation $f^\dagger$ and annihilation f operators

$$\{f_i, f_j^\dagger\} = \delta_{ij} \qquad \{f_i, f_j\} = \{f_i^\dagger, f_j^\dagger\} = 0$$

$$f_\alpha f_\beta|\phi\rangle = \phi_\alpha\phi_\beta|\phi\rangle \qquad -f_\beta f_\alpha|\phi\rangle = -\phi_\beta\phi_\alpha|\phi\rangle$$

  - Anticommuting eigenvalues: **Grassmann numbers**

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Creation and annihilation operators

- f and $f^\dagger$ can be used to construct many-particle states and serve as **basis** for many-body operators

Definition

$f_i^\dagger$ and f_i create or annihilate a fermion in state $|\alpha_i\rangle$. The **occupation number** in each state can only be **either 0 or 1**. They operate in **Fock space**.

$$\begin{aligned} f_i^\dagger |n_1 n_2 \ldots n_i \ldots\rangle &= (1-n_i)\,|n_1 n_2 \ldots n_i + 1 \ldots\rangle \\ f_i |n_1 n_2 \ldots n_i \ldots\rangle &= n_i\,|n_1 n_2 \ldots n_i - 1 \ldots\rangle \\ f_i^\dagger f_i = \hat{n}_i \Rightarrow \hat{n}_i |n_1 n_2 \ldots n_i \ldots\rangle &= n_i\,|n_1 n_2 \ldots n_i \ldots\rangle \end{aligned}$$

Grassmann algebra

- An algebra over a field is a vector space equipped with a product
- Grassmann algebras can be defined over the fields $\mathbb{R}$ or $\mathbb{C}$
- Grassmann algebra defined by a set of **anticommuting generators** $\{\xi_\alpha\}$, $\alpha = 1, \ldots, n$

$$\xi_\alpha \xi_\beta + \xi_\beta \xi_\alpha = 0 \qquad \xi_\alpha^2 = 0$$

- **Basis** of the algebra made of all distinct products $\{1, \xi_1, \xi_2, \xi_3 \ldots, \xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3, \ldots, \xi_1\xi_2\xi_3, \ldots\}$
  - Dimension 2^n with n generators

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Grassmann algebra

- The generated Grassmann algebra contains all linear combinations of the basis elements with complex coefficients $f_{\{i\}}$

$$f(\xi) = f_0 + \sum_i f_i \xi_i + \sum_{i<j} f_{ij} \xi_i \xi_j + \sum_{i<j<k} f_{ijk} \xi_i \xi_j \xi_k + \dots$$

- In an algebra with even number of generators $n = 2p$, we define a **conjugation**
  - Select p generators ξ_α, associate $\xi_\alpha^\star$ to each of them

$$(\xi_\alpha)^\star = \xi_\alpha^\star \qquad (\xi_\alpha^\star)^\star = \xi_\alpha$$

- For simplicity: use 2 generators and the basis: $\{1, \xi, \xi^\star, \xi^\star \xi\}$

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Functions and operators

- Any analytic function f is linear on a Grassmann algebra due to $\xi_\alpha^2 = 0$
  - Form of the coherent state representation of a wave function

$$f(\xi) = f_0 + f_1 \xi$$

- Operators are a function of $\xi^\star$ and ξ and have bilinear form

$$A(\xi^\star, \xi) = a_0 + a_1 \xi + a_2 \xi^\star + a_{12} \xi^\star \xi$$

- **Derivative** is defined identical to the complex derivative; variable has to be adjacent to the derivative operator

$$\frac{\partial}{\partial \xi} \xi = 1 \qquad \frac{\partial}{\partial \xi} (\xi^\star \xi) = \frac{\partial}{\partial \xi} (-\xi \xi^\star) = -\xi^\star$$

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Integral

- No analog of the sum motivated Riemann integral possible
- Integration over Grassmann variables: **linear mapping** with fundamental property that integral of exact differential form is zero over functions vanishing at infinity

$$\int d\xi \, 1 = 0 \qquad \int d\xi \, \xi = 1$$

- Similar to Grassmann differentiation

$$\frac{\partial}{\partial \xi} 1 = 0 \qquad \frac{\partial}{\partial \xi} \xi = 1$$

- Analogous definition for conjugated variables

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Applications of the integral

- Grassmann δ-function

$$\delta(\xi,\xi') = \int d\eta\, e^{-\eta(\xi-\xi')} = \int d\eta\,(1-\eta(\xi-\xi')) = -(\xi-\xi')$$

$$\int d\xi'\,\delta(\xi,\xi')f(\xi') = -\int d\xi'\,(\xi-\xi')(f_0+f_1\xi') = f_0+f_1\xi = f(\xi)$$

- Scalar product of Grassmann algebra

$$\langle f|g\rangle = \int d\xi d\xi^\star\, e^{-\xi\xi^\star} f^\star(\xi^\star)g(\xi)$$

$$\langle f|g\rangle = \int d\xi d\xi^\star\,(1-\xi\xi^\star)(f_0^\star+f_1^\star\xi^\star)(g_0+g_1\xi)$$

$$= -\int d\xi d\xi^\star\,\xi\xi^\star f_0^\star g_0 + \int d\xi d\xi^\star\,\xi^\star\xi f_1^\star g_1$$

$$= f_0^\star g_0 + f_1^\star g_1$$

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Coherent states for fermions

- Expansion of states as linear combination of states of the Fock space $\mathcal{F}$ with Grassmann numbers as coefficients

$$|\psi\rangle = \sum_{\alpha} \chi_\alpha |\phi_\alpha\rangle$$

- Generator ξ_α associated to annihilation operator f_α and $\xi_\alpha^\star$ associated with creation operator $f_\alpha^\dagger$

$$\{\tilde{\xi}, \tilde{f}\} = 0 \qquad (\tilde{\xi}\tilde{f})^\dagger = \xi^\star \tilde{f}^\dagger$$

- Defintion of fermionic coherent states $|\xi\rangle$ analogous to bosonic coherent states

$$|\xi\rangle = e^{-\sum_\alpha \xi_\alpha f_\alpha^\dagger}|0\rangle = \prod_\alpha (1 - \xi_\alpha f_\alpha^\dagger)|0\rangle$$

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Coherent states for fermions

Proof: Coherent states are eigenstates of the annihilation operator f_α

For a single state α:

$$f_\alpha(1-\xi_\alpha f_\alpha^\dagger)|0\rangle = +\xi_\alpha|0\rangle = \xi_\alpha(1-\xi_\alpha f_\alpha^\dagger)|0\rangle$$

$$\begin{aligned} f_\alpha|\xi\rangle &= f_\alpha \prod_\beta (1-\xi_\beta f_\beta^\dagger)|0\rangle = \prod_{\beta\neq\alpha}(1-\xi_\beta f_\beta^\dagger) f_\alpha (1-\xi_\alpha f_\alpha^\dagger)|0\rangle \\ &= \prod_{\beta\neq\alpha}(1-\xi_\beta f_\beta^\dagger)\xi_\alpha(1-\xi_\alpha f_\alpha^\dagger)|0\rangle = \xi_\alpha \prod_\beta (1-\xi_\beta f_\beta^\dagger)|0\rangle \\ &= \xi_\alpha|\xi\rangle \end{aligned}$$

where f_α and ξ_α commute with $\xi_\beta f_\beta$ for $\beta \neq \alpha$.

Similarly, adjoint coherent state

$$\langle\xi| = \langle 0|e^{-\sum_\alpha f_\alpha \xi_\alpha^\star} \qquad \langle\xi|f_\alpha^\dagger = \langle\xi|\xi_\alpha^\star$$

Properties of coherent states

- Action of $f_\alpha^\dagger$ on a coherent state is a derivative analogous to the results for Bosons

$$f_\alpha^\dagger|\xi\rangle = f_\alpha^\dagger(1-\xi_\alpha f_\alpha^\dagger)\prod_{\beta\neq\alpha}(1-\xi_\beta f_\beta^\dagger)|0\rangle = f_\alpha^\dagger\prod_{\beta\neq\alpha}(1-\xi_\beta f_\beta^\dagger)|0\rangle$$

$$= -\frac{\partial}{\partial\xi_\alpha}(1-\xi_\alpha f_\alpha^\dagger)\prod_{\beta\neq\alpha}(1-\xi_\beta f_\beta^\dagger)|0\rangle$$

$$= -\frac{\partial}{\partial\xi_\alpha}|\xi\rangle$$

Properties of coherent states

- Action of creation/annihilation operators on coherent states

$$f_\alpha|\xi\rangle = \xi_\alpha|\xi\rangle \qquad f_\alpha^\dagger|\xi\rangle = -\frac{\partial}{\partial\xi_\alpha}|\xi\rangle$$

$$\langle\xi|f_\alpha^\dagger = \langle\xi|\xi_\alpha^\star \qquad \langle\xi|f_\alpha = +\frac{\partial}{\partial\xi_\alpha^\star}\langle\xi|$$

- Overlap of two coherent states

$$\langle\xi|\xi'\rangle = \langle 0|\prod_\alpha(1+\xi_\alpha^\star f_\alpha)(1-\xi'_\alpha f_\alpha^\dagger)|0\rangle$$

$$= \prod_\alpha(1+\xi_\alpha^\star\xi'_\alpha) = e^{\sum_\alpha \xi_\alpha^\star\xi'_\alpha}$$

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Properties of coherent states

- Matrix element between two coherent states

$$\langle \xi | A(f_\alpha^\dagger, f_\alpha) | \xi' \rangle = e^{\sum_\alpha \xi_\alpha^\star \xi'_\alpha} A(\xi_\alpha^\star, \xi_\alpha)$$

- Expectation value of number operator N is not a real value

$$\frac{\langle \xi | N | \xi \rangle}{\langle \xi | \xi \rangle} = \frac{\sum_\alpha \langle \xi | f_\alpha^\dagger f_\alpha | \xi \rangle}{\langle \xi | \xi \rangle} = \sum_\alpha \xi_\alpha^\star \xi_\alpha$$

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Proof of closure relation:

$$A = \int \prod_\alpha \mathrm{d}\xi^\star_\alpha \mathrm{d}\xi_\alpha \, \mathrm{e}^{-\sum_\alpha \xi^\star_\alpha \xi_\alpha} |\xi\rangle\langle\xi| = \mathbb{I}_\mathcal{F}$$

We need to prove

$$\langle \alpha_1 \dots \alpha_n | A | \beta_1 \dots \beta_m \rangle = \langle \alpha_1 \dots \alpha_n | \beta_1 \dots \beta_m \rangle$$

The overlap of the considered states is

$$\langle \alpha_1 \dots \alpha_n | \beta_1 \dots \beta_m \rangle = \delta_{nm} (-1^p) \delta_{\alpha_{i_1} \beta_1} \dots \delta_{\alpha_{i_n} \beta_m}$$

with parity p of the permutation $P \begin{pmatrix} \alpha_1, \dots, \alpha_n \\ \alpha_{i_1}, \dots, \alpha_{i_n} \end{pmatrix}$

Overcompleteness of coherent states

Using the eigenvalue property we obtain

$$\langle \alpha_1 \dots \alpha_n | \xi \rangle = \langle 0 | f_{\alpha_n} \dots f_{\alpha_1} | \xi \rangle = \xi_{\alpha_n} \dots \xi_{\alpha_1}$$

This leads to the matrix element

$$\begin{aligned} \langle \alpha_1 \dots \alpha_n | A | \beta_1 \dots \beta_m \rangle &= \\ &= \int \prod_\alpha \mathrm{d}\xi_\alpha^\star \mathrm{d}\xi_\alpha \, \mathrm{e}^{-\sum_\alpha \xi_\alpha^\star \xi_\alpha} \langle \alpha_1 \dots \alpha_n | \xi \rangle \langle \xi | \beta_1 \dots \beta_n \rangle \\ &= \int \prod_\alpha \mathrm{d}\xi_\alpha^\star \mathrm{d}\xi_\alpha \prod_\alpha (1 - \xi_\alpha^\star \xi_\alpha) \xi_{\alpha_n} \dots \xi_{\alpha_1} \xi_{\beta_1}^\star \dots \xi_{\beta_n}^\star \end{aligned}$$

with the following integrals arising for a particular state γ

$$\int \mathrm{d}\xi_\gamma^\star \mathrm{d}\xi_\gamma \, (1 - \xi_\gamma^\star \xi_\gamma) \begin{Bmatrix} \xi_\gamma^\star \xi_\gamma \\ \xi_\gamma^\star \\ \xi_\gamma \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

Overcompleteness of coherent states

- Integral non-vanishing if each state γ is either occupied in both $\langle \alpha_1 \dots \alpha_n|$ and $|\beta_1 \dots \beta_m\rangle$ or unoccupied in both states
  - $\Rightarrow$ $m = n$ and $\{\alpha_1 \dots \alpha_n\}$ has to be a permutation P of $\{\beta_1 \dots \beta_n\}$

$$\xi_{\alpha_n} \dots \xi_{\alpha_1} \xi^\star_{\beta_1} \dots \xi^\star_{\beta_n} = (-1)^P \xi_{\alpha_n} \dots \xi_{\alpha_1} \xi^\star_{\alpha_1} \dots \xi^\star_{\alpha_n}$$

- Even number of anticommutations needed to bring the ξ_γ and $\xi^\star_\gamma$ adjacent to the integral
  - $\Rightarrow$ Integral evaluates to $(-1)^P$, equal to the value of the overlap
- Resolution of unity for fermionic coherent states

$$\int \prod_\alpha \mathrm{d}\xi^\star_\alpha \mathrm{d}\xi_\alpha \, \mathrm{e}^{-\sum_\alpha \xi^\star_\alpha \xi_\alpha} |\xi\rangle\langle\xi| = \mathbb{I}_\mathcal{F}$$

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Coherent state representation

- Overcompleteness allows to define Grassmann coherent state representation, where $\langle \xi | \psi \rangle = \psi(\xi^\star)$

$$|\psi\rangle = \int \prod_\alpha \mathrm{d}\xi_\alpha^\star \mathrm{d}\xi_\alpha \, \mathrm{e}^{-\sum_\alpha \xi_\alpha^\star \xi_\alpha} \psi(\xi^\star) |\xi\rangle$$

- The creation and annihilation operators in this representation are given by

$$\langle \xi | f_\alpha | \psi \rangle = \frac{\partial}{\partial \xi_\alpha^\star} \psi(\xi^\star)$$
$$\langle \xi | f_\alpha^\dagger | \psi \rangle = \xi_\alpha^\star \psi(\xi^\star)$$

- Anticommutation relation between operators $\frac{\partial}{\partial \xi_\alpha^\star}$ and $\xi_\alpha^\star$

$$\left\{ \frac{\partial}{\partial \xi_\alpha^\star}, \xi_\beta^\star \right\} = \delta_{\alpha\beta}$$

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$$\langle \xi | f^\dagger_\alpha | \psi \rangle = \xi^\star_\alpha \psi(\xi^\star)$$

- Anticommutation relation between operators $\frac{\partial}{\partial \xi^\star_\alpha}$ and $\xi^\star_\alpha$

$$\left\{ \frac{\partial}{\partial \xi^\star_\alpha}, \xi^\star_\beta \right\} = \delta_{\alpha\beta}$$

Coherent state representation

- Overcompleteness allows to define Grassmann coherent state representation, where $\langle \xi | \psi \rangle = \psi(\xi^\star)$

$$|\psi\rangle = \int \prod_\alpha \mathrm{d}\xi_\alpha^\star \mathrm{d}\xi_\alpha \, \mathrm{e}^{-\sum_\alpha \xi_\alpha^\star \xi_\alpha} \psi(\xi^\star) |\xi\rangle$$

- The creation and annihilation operators in this representation are given by

$$\langle \xi | f_\alpha | \psi \rangle = \frac{\partial}{\partial \xi_\alpha^\star} \psi(\xi^\star)$$

$$\langle \xi | f_\alpha^\dagger | \psi \rangle = \xi_\alpha^\star \psi(\xi^\star)$$

- Anticommutation relation between operators $\frac{\partial}{\partial \xi_\alpha^\star}$ and $\xi_\alpha^\star$

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Gaussian integrals

- Matrix elements of the evolution operator give rise to Gaussian integrals in case of quadratic forms
- For a Hermitian operator H, integrals of those form lead to

$$\int \prod_{i=1}^{n} \mathrm{d}\eta_i^\star \,\mathrm{d}\eta_i \, \mathrm{e}^{-\eta_i^\star H_{ij}\eta_j + \xi_i^\star \eta_i + \xi_i \eta_i^\star} = \det(H)\mathrm{e}^{\xi_i^\star H_{ij}^{-1} \xi_j}$$

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Gaussian integrals

- Gaussian integral for a single pair of conjugate Grassmann variables

$$\int \mathrm{d}\xi^\star \mathrm{d}\xi \, \mathrm{e}^{-\xi^\star a \xi} = \int \mathrm{d}\xi^\star \mathrm{d}\xi \, (1 - \xi^\star a \xi) = a$$

- Law for integral transformation under change of Grassmann variables

$$\int \mathrm{d}\xi_1^\star \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n^\star \mathrm{d}\xi_n \, P(\xi^\star, \xi) = \left| \frac{\partial(\eta^\star, \eta)}{\partial(\xi^\star, \xi)} \right| \times \int \mathrm{d}\eta_1^\star \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n^\star \mathrm{d}\eta_n \, P(\xi^\star(\eta^\star, \eta), \xi(\eta^\star, \eta))$$

! Be careful: Inverse Jacobian

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Gaussian integrals

- Proof of Gaussian integral formula
  - Define the transformations $\rho_i = \eta_i - H^{-1}_{ij}\xi_j$ and $\rho^\star_i = \eta^\star_i - H^{-1}_{ij}\xi^\star_j$
  - Diagonalize H with a unitary transformation U
  - Define $\zeta_i = U^{-1}_{ij}\rho_j$ and $\zeta^\star_i = U^{-1\star}_{ij}\rho^\star_j$
  - All Jacobians are unity in this case

$$
\begin{aligned}
\int \prod_{i=1}^{n} \mathrm{d}\eta^\star_i \, \mathrm{d}\eta_i \, \mathrm{e}^{-\eta^\star_i H_{ij}\eta_j + \xi^\star_i \eta_i + \xi_i \eta^\star_i - \xi^\star_i H^{-1}_{ij}\xi_j} \\
= \int \prod_{i=1}^{n} \mathrm{d}\rho^\star_i \, \mathrm{d}\rho_i \, \mathrm{e}^{-\rho^\star_i H_{ij}\rho_j} \\
= \int \prod_{i=1}^{n} \mathrm{d}\zeta^\star_i \, \mathrm{d}\zeta_i \, \mathrm{e}^{h_i \zeta^\star_i \zeta_i} \\
= \prod_{m=1}^{n} h_m = \det(H)
\end{aligned}
$$

Comparison with coherent states for bosons

Bosons	**Fermions**
Eigenstates of the bosonic/fermionic **annihilation operator** b_α / f_α	
$\lvert\phi\rangle = e^{\sum_\alpha \phi_\alpha b_\alpha^\dagger}\lvert 0\rangle$	$\lvert\xi\rangle = e^{-\sum_\alpha \xi_\alpha f_\alpha^\dagger}\lvert 0\rangle$
Provide **overcomplete** basis of bosonic/fermionic Fock space closure relation	
$\int \prod_\alpha \frac{d\phi_\alpha^\star d\phi_\alpha}{2i\pi} e^{-\sum_\alpha \phi_\alpha^\star \phi_\alpha}\lvert\phi\rangle\langle\phi\rvert = \mathbb{I}_\mathcal{B}$	$\int \prod_\alpha d\xi_\alpha^\star d\xi_\alpha \, e^{-\sum_\alpha \xi_\alpha^\star \xi_\alpha}\lvert\xi\rangle\langle\xi\rvert = \mathbb{I}_\mathcal{F}$
Coherent state **representation**	
$\langle\phi\rvert b_\alpha \lvert\psi\rangle = \phi_\alpha^\star \psi(\phi^\star)$ $\langle\phi\rvert b_\alpha^\dagger \lvert\psi\rangle = \frac{\partial}{\partial\phi_\alpha^\star}\psi(\phi^\star)$	$\langle\xi\rvert f_\alpha \lvert\psi\rangle = \xi_\alpha^\star \psi(\xi^\star)$ $\langle\xi\rvert f_\alpha^\dagger \lvert\psi\rangle = \frac{\partial}{\partial\xi_\alpha^\star}\psi(\xi^\star)$
Expectation value of **number operator**	
$\frac{\langle\phi\rvert\hat{N}\lvert\phi\rangle}{\langle\phi\vert\phi\rangle} = \sum_\alpha \phi_\alpha^\star \phi_\alpha \in \mathbb{R}$	$\frac{\langle\xi\rvert\hat{N}\lvert\xi\rangle}{\langle\xi\vert\xi\rangle} = \sum_\alpha \xi_\alpha^\star \xi_\alpha \notin \mathbb{R}$

Comparison with coherent states for bosons

Bosons	**Fermions**
Physical states in sense of the classical limit Most classical state	Not physically observable but useful uniformication No analogon
Classical field $\phi(x)$ e.g. electromagnetic field as coherent state of photons	No classical field for fermions
Provide representation for **path integral formalism**	
Important to relate **statistical physics** and **quantum mechanics** Enables calculation of thermodynamic values for quantum many-particle systems	

References

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