Path-Integrals for Fermions

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Hauptseminar:
Quantum field-theory of low dimensional systems

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Overview

Recapitulation: path-integrals

- Feynman path-integral:

$$K(x_f,t_f;x_i,t_i) = \langle x_f|\, U(t_f,t_i)\, |x_i\rangle = \sum_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \exp\left(\frac{i}{\hbar}\, S\left[x(t)\right]\right)$$

with $S\left[x(t)\right] = \int_{t_i}^{t_f} dt\ L(x,\dot{x}) = \int_{t_i}^{t_f} dt\ \dot{x}p - H(x,p)$

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- Feynman path-integral:

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$$\text{with} \quad S\left[x(t)\right] = \int_{t_i}^{t_f} dt \; L(x,\dot{x}) = \int_{t_i}^{t_f} dt \; \dot{x}p - H(x,p)$$

- Connection to statistical physics: $t \longrightarrow -i\tau$

$$\mathcal{Z} = tr\left(e^{-\beta(H-\mu N)} \right) = \int_{-\infty}^{\infty} dx \; K(x, t = -i\beta\hbar; x, 0)$$

Recapitulation: coherent states for fermions

- Fermions: antisymmetric wavefunction

$$\Longrightarrow [a_i, a_j^\dagger]_+ = a_i\, a_j^\dagger + a_j^\dagger\, a_i = \delta_{ij}$$

$$\Longrightarrow [a_i, a_j]_+ = [a_i^\dagger, a_j^\dagger]_+ = 0$$

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- Grassmann-algebra: generators ϕ_α with $\alpha = 1,...,n$

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- Grassmann-integrals:

$$\int d\phi\, 1 = 0\,, \qquad \int d\phi^*\, 1 \;= 0$$
$$\int d\phi\, \phi = 1\,, \qquad \int d\phi^*\, \phi^* = 1$$

Recapitulation: coherent states for fermions

- Fermion coherent states:

$$|\phi\rangle = \exp\left(-\sum_\alpha \phi_\alpha\, a_\alpha^\dagger\right)|0\rangle = \prod_\alpha (1 - \phi_\alpha\, a_\alpha^\dagger)\,|0\rangle$$

with $\langle \phi \mid \phi' \rangle = e^{\sum_\alpha \phi_\alpha^* \, \phi'_\alpha}$

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- Closure relation:

$$\mathbb{1} = \int \prod_{\alpha} d\phi_{\alpha}^{*} d\phi_{\alpha} \exp\left(-\sum_{\alpha} \phi_{\alpha}^{*}\, \phi_{\alpha}\right)|\phi\rangle\langle\phi|$$

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- Trace of an operator:

$$\mathrm{tr}(A) = \int \prod_\alpha d\phi_\alpha^* d\phi_\alpha \exp\left(-\sum_\alpha \phi_\alpha^*\, \phi_\alpha\right)\langle -\phi|\, A\, |\phi\rangle$$

Recapitulation: coherent states for fermions

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&= \int \prod_\alpha d\phi_\alpha^* d\phi_\alpha \exp\left(-\sum_\alpha \phi_\alpha^* \phi_\alpha\right) \sum_n \langle -\phi| A |n\rangle \langle n \mid \phi\rangle \\
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\end{aligned}$$

with $\langle -\phi| = \langle 0| \exp\left(-\sum_\alpha \phi_\alpha^* a_\alpha\right)$

Path-Integrals for fermions I

- Discrete time steps of the length $\epsilon = \frac{t_f - t_i}{M}$

$$\text{intital state}\, |\phi_i\rangle : \phi_{\alpha,0} \qquad \text{final state}\, \langle\phi_f| : \phi^*_{\alpha,M}$$

$$\Longrightarrow \phi_{\alpha,k} \text{ with } k = 0, ..., M$$

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- Separation of the exponential function:

$$e^{-\frac{i}{\hbar}H(t_f - t_i)} = e^{-\frac{i\epsilon}{\hbar}H} \cdot e^{-\frac{i\epsilon}{\hbar}H} \cdots e^{-\frac{i\epsilon}{\hbar}H}$$

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- Normalorder:

$$e^{-\frac{i\epsilon}{\hbar}H(a^\dagger, a)} = :e^{-\frac{i\epsilon}{\hbar}H(a^\dagger, a)}: + \mathcal{O}(\epsilon^2)$$

Path-Integrals for fermions II

$$K(\phi_f, t_f; \phi_i, t_i) = \langle\phi_f|\, e^{-\frac{i}{\hbar}H(t_f - t_i)}\,|\phi_i\rangle$$

Path-Integrals for fermions II

$$K(\phi_f, t_f; \phi_i, t_i) = \langle\phi_f|\, e^{-\frac{i}{\hbar}H(t_f - t_i)}\, |\phi_i\rangle$$
$$= \langle\phi_f|\, e^{-\frac{i\epsilon}{\hbar}H}\, \mathbb{1}_{M-1}\, e^{-\frac{i\epsilon}{\hbar}H}\, \mathbb{1}_{M-2} \cdots \mathbb{1}_1\, e^{-\frac{i\epsilon}{\hbar}H}\, |\phi_i\rangle$$

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$$= \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_{\alpha} d\phi^*_{\alpha,k}\, d\phi_{\alpha,k}\ \exp\left(- \sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k}\, \phi_{\alpha,k} \right)$$

$$\times \prod_{k=1}^{M} \langle\phi_k|\, :e^{-\frac{i\epsilon}{\hbar}H(a^\dagger, a)}:\, |\phi_{k-1}\rangle$$

Path-Integrals for fermions II

$$
\begin{aligned}
K(\phi_f, t_f; \phi_i, t_i) &= \langle \phi_f | \, e^{-\frac{i}{\hbar} H (t_f - t_i)} \, |\phi_i\rangle \\
&= \langle \phi_f | \, e^{-\frac{i\epsilon}{\hbar} H} \, \mathbb{1}_{M-1} \, e^{-\frac{i\epsilon}{\hbar} H} \, \mathbb{1}_{M-2} \cdots \mathbb{1}_1 \, e^{-\frac{i\epsilon}{\hbar} H} \, |\phi_i\rangle \\
&= \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_{\alpha} d\phi^*_{\alpha,k} d\phi_{\alpha,k} \, \exp\left(- \sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k} \, \phi_{\alpha,k} \right) \\
&\qquad \times \prod_{k=1}^{M} \langle \phi_k | \, : e^{-\frac{i\epsilon}{\hbar} H(a^\dagger, a)} : \, |\phi_{k-1}\rangle \\
&= \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_{\alpha} d\phi^*_{\alpha,k} d\phi_{\alpha,k} \, \exp\left(- \sum_{k=1}^{M-1} \sum_{\alpha} \phi^*_{\alpha,k} \, \phi_{\alpha,k} \right) \\
&\qquad \times \exp\left(\sum_{k=1}^{M} \sum_{\alpha} \phi^*_{\alpha,k} \, \phi_{\alpha,k-1} - \frac{i\epsilon}{\hbar} H(\phi^*_{\alpha,k}, \phi_{\alpha,k-1}) \right)
\end{aligned}
$$

Path-Integrals for fermions III

- Rewrite the exponent:

$$\sum_\alpha \phi^*_{\alpha,M}\,\phi_{\alpha,M-1} - \frac{i\epsilon}{\hbar}\,H(\phi^*_{\alpha,M},\phi_{\alpha,M-1}) + i\epsilon\sum_{k=1}^{M-1}\left(\sum_\alpha i\,\phi^*_{\alpha,k}\,\frac{\phi_{\alpha,k} - \phi_{\alpha,k-1}}{\epsilon} - \frac{1}{\hbar}\,H(\phi^*_{\alpha,k},\phi_{\alpha,k-1})\right)$$

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- Introduce a trajectory $\phi_\alpha(t)$ for a given set $\{\phi_{\alpha,1}, \phi_{\alpha,2}, ..., \phi_{\alpha,M}\}$

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- Introduce a trajectory $\phi_\alpha(t)$ for a given set $\{\phi_{\alpha,1},\phi_{\alpha,2},...,\phi_{\alpha,M}\}$
- Notation:

$$\phi^*_{\alpha,k}\;\frac{\phi_{\alpha,k}-\phi_{\alpha,k-1}}{\epsilon}\;\equiv\;\phi^*_\alpha(t)\,\frac{\partial}{\partial t}\phi_\alpha(t)$$

$$H(\phi^*_{\alpha,k},\phi_{\alpha,k-1})\;\equiv\;H(\phi^*_\alpha(t),\phi_\alpha(t))$$

- Continuous notation ($M \to \infty$) of the exponent:

$$\sum_\alpha \phi^*_{\alpha,M}\,\phi_{\alpha,M-1} - \frac{i\epsilon}{\hbar}\,H(\phi^*_{\alpha,M},\phi_{\alpha,M-1}) + i\epsilon\sum_{k=1}^{M-1}\left(\sum_\alpha i\,\phi^*_{\alpha,k}\,\frac{\phi_{\alpha,k} - \phi_{\alpha,k-1}}{\epsilon} - \frac{1}{\hbar}\,H(\phi^*_{\alpha,k},\phi_{\alpha,k-1})\right)$$

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$$= \sum_\alpha \phi^*_\alpha(t_f)\,\phi_\alpha(t_f) + \frac{i}{\hbar}\int_{t_i}^{t_f} dt\;\sum_\alpha i\hbar\,\phi^*_\alpha(t)\,\frac{\partial\,\phi_\alpha(t)}{\partial t} - H(\phi^*_\alpha(t),\phi_\alpha(t))$$

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Path-Integrals for fermions V

Summary

$$K(\phi_f, t_f; \phi_i, t_i) = \int\limits_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} \mathcal{D}\left[\phi_\alpha^*(t), \phi_\alpha(t)\right] \; \exp\left(\sum_\alpha \phi_\alpha^*(t_f)\, \phi_\alpha(t_f)\right)$$
$$\times \exp\left(\frac{i}{\hbar} \int\limits_{t_i}^{t_f} dt \; \sum_\alpha i\hbar \, \phi_\alpha^*(t) \, \frac{\partial \, \phi_\alpha(t)}{\partial t} - H(\phi_\alpha^*(t), \phi_\alpha(t))\right)$$

$$\mathcal{D}\left[\phi_\alpha^*(t), \phi_\alpha(t)\right] \equiv \lim_{M\to\infty} \int \prod_{k=1}^{M-1} \prod_\alpha d\phi_{\alpha,k}^* d\phi_{\alpha,k}$$

Partition function I

- Partition function in the grand canonical ensemble:

$$\mathcal{Z} = tr\left(e^{-\beta(H-\mu N)}\right)$$

from now on $\hbar \equiv 1$

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- Grand canonical ensemble: $H - \mu N$

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Partition function II

- Notation: $\epsilon = \frac{\beta}{M}$
- antiperiodic boundary conditions: $\phi_{\alpha,0} = \phi_\alpha\,, \quad \phi^*_{\alpha,M} = -\phi^*_\alpha$

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$$\times\; e^{\sum\limits_{k=1}^{M}\sum\limits_{\alpha} \phi^*_{\alpha,k}\,\phi_{\alpha,k-1} - \epsilon\left(H(\phi^*_{\alpha,k},\phi_{\alpha,k-1}) - \mu\,\phi^*_{\alpha,k}\,\phi_{\alpha,k-1}\right)}$$

Partition function III

$$\mathcal{Z} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\phi^*_{\alpha,k} d\phi_{\alpha,k} \; e^{-\sum_{k=1}^{M} \sum_{\alpha} \phi^*_{\alpha,k} \phi_{\alpha,k}}$$

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where $\phi_{\alpha,0} = -\phi_{\alpha,M} \implies$ antiperiodic boundary conditions

Partition function IV

- Discrete notation:

$$\mathcal{Z} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\phi^*_{\alpha,k} d\phi_{\alpha,k}$$
$$\times\ e^{-\epsilon \sum\limits_{k=1}^{M} \left(\sum\limits_{\alpha} \phi^*_{\alpha,k} \left(\frac{\phi_{\alpha,k} - \phi_{\alpha,k-1}}{\epsilon} - \mu\, \phi_{\alpha,k-1} \right) + H(\phi^*_{\alpha,k}, \phi_{\alpha,k-1}) \right)}$$

- Trajectory notation:

$$\mathcal{Z} = \int\limits_{\phi_\alpha(\beta) = -\phi_\alpha(0)} \mathcal{D}\left[\phi^*_\alpha(\tau), \phi_\alpha(\tau)\right]\ e^{-\int\limits_0^\beta d\tau \sum\limits_\alpha \phi^*_\alpha(\tau) \left(\frac{\partial}{\partial\tau} - \mu\right) \phi_\alpha(\tau) + H(\phi^*_\alpha(\tau), \phi_\alpha(\tau))}$$

Non-interacting fermions I

- Non-interacting Hamiltonian: $H = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha$

$$\Longrightarrow H - \mu N = \sum_\alpha (\epsilon_\alpha - \mu) a_\alpha^\dagger a_\alpha$$

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$$\implies H - \mu N = \sum_\alpha (\epsilon_\alpha - \mu) a_\alpha^\dagger a_\alpha$$

- Partition function:

$$\mathcal{Z} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_\alpha d\phi_{\alpha,k}^* d\phi_{\alpha,k}$$
$$\times\ e^{-\frac{\beta}{M} \sum_{k=1}^{M} \sum_\alpha \left(\phi_{\alpha,k}^* \left(\frac{\phi_{\alpha,k} - \phi_{\alpha,k-1}}{\beta/M} - \mu\, \phi_{\alpha,k-1} \right) + \epsilon_\alpha\, \phi_{\alpha,k}^* \phi_{\alpha,k-1} \right)}$$

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$$\Longrightarrow H - \mu N = \sum_\alpha (\epsilon_\alpha - \mu) a_\alpha^\dagger a_\alpha$$

- Partition function:

$$\mathcal{Z} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_\alpha d\phi_{\alpha,k}^* d\phi_{\alpha,k} \times e^{-\frac{\beta}{M} \sum_{k=1}^{M} \sum_\alpha \left(\phi_{\alpha,k}^* \left(\frac{\phi_{\alpha,k} - \phi_{\alpha,k-1}}{\beta/M} - \mu\, \phi_{\alpha,k-1} \right) + \epsilon_\alpha\, \phi_{\alpha,k}^* \phi_{\alpha,k-1} \right)}$$

$$= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_\alpha d\phi_{\alpha,k}^* d\phi_{\alpha,k} \times e^{-\sum_{k=1}^{M} \sum_\alpha \left(\left(\frac{\beta}{M}(\epsilon_\alpha - \mu) - 1 \right) \phi_{\alpha,k}^* \phi_{\alpha,k-1} + \phi_{\alpha,k}^* \phi_{\alpha,k} \right)}$$

Non interacting fermions II

- Notation: $a = 1 - \frac{\beta}{M}(\epsilon_\alpha - \mu)$

$$\boldsymbol{\phi_\alpha} = \begin{pmatrix} \phi_{\alpha,1} \\ \phi_{\alpha,2} \\ \phi_{\alpha,3} \\ \vdots \\ \vdots \\ \phi_{\alpha,M} \end{pmatrix} \quad , \quad \boldsymbol{S^{(\alpha)}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & a \\ -a & 1 & 0 & \ddots & \vdots & 0 \\ 0 & -a & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & -a & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -a & 1 \end{pmatrix}$$

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$$\Longrightarrow \mathcal{Z} = \lim_{M \to \infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\phi}^*_{\boldsymbol{\alpha}} d\boldsymbol{\phi}_{\boldsymbol{\alpha}} \; e^{-\boldsymbol{\phi}^*_{\boldsymbol{\alpha}} \, \boldsymbol{S^{(\alpha)}} \, \boldsymbol{\phi}_{\boldsymbol{\alpha}}}$$

Non-interacting fermions III

- Transformation to the eigenbasis of $\boldsymbol{S}^{(\alpha)}$: $\boldsymbol{U}^\dagger \boldsymbol{U} = \mathbb{1}$
  $\boldsymbol{\phi}_{\boldsymbol{\alpha}} \to \boldsymbol{\eta}_{\boldsymbol{\alpha}} \equiv \boldsymbol{U} \boldsymbol{\phi}_{\boldsymbol{\alpha}}$, $\boldsymbol{\phi}^*_{\boldsymbol{\alpha}} \to \boldsymbol{\eta}^*_{\boldsymbol{\alpha}} \equiv \boldsymbol{\phi}^*_{\boldsymbol{\alpha}} \boldsymbol{U}^\dagger$, $\boldsymbol{U} \boldsymbol{S}^{(\alpha)} \boldsymbol{U}^\dagger \equiv \boldsymbol{D}^{(\alpha)}$

Non-interacting fermions III

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$$\mathcal{Z} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\phi}^*_{\boldsymbol{\alpha}} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}\ e^{-\boldsymbol{\phi}^*_{\boldsymbol{\alpha}}\,\boldsymbol{S}^{(\boldsymbol{\alpha})}\,\boldsymbol{\phi}_{\boldsymbol{\alpha}}}$$

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$$\begin{aligned}
\mathcal{Z} &= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\phi^*_\alpha} d\boldsymbol{\phi_\alpha}\ e^{-\boldsymbol{\phi^*_\alpha}\,\boldsymbol{S^{(\alpha)}}\,\boldsymbol{\phi_\alpha}} \\
&= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\phi^*_\alpha} d\boldsymbol{\phi_\alpha}\ e^{-\boldsymbol{\phi^*_\alpha}\,\boldsymbol{U}^\dagger\,\boldsymbol{U}\,\boldsymbol{S^{(\alpha)}}\,\boldsymbol{U}^\dagger\,\boldsymbol{U}\,\boldsymbol{\phi_\alpha}}
\end{aligned}$$

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\mathcal{Z} &= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\phi}^*_{\boldsymbol{\alpha}} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}\; e^{-\boldsymbol{\phi}^*_{\boldsymbol{\alpha}}\,\boldsymbol{S}^{(\alpha)}\,\boldsymbol{\phi}_{\boldsymbol{\alpha}}} \\
&= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\phi}^*_{\boldsymbol{\alpha}} d\boldsymbol{\phi}_{\boldsymbol{\alpha}}\; e^{-\boldsymbol{\phi}^*_{\boldsymbol{\alpha}}\,\boldsymbol{U}^\dagger\,\boldsymbol{U}\,\boldsymbol{S}^{(\alpha)}\,\boldsymbol{U}^\dagger\,\boldsymbol{U}\,\boldsymbol{\phi}_{\boldsymbol{\alpha}}} \\
&= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\eta}^*_{\boldsymbol{\alpha}} d\boldsymbol{\eta}_{\boldsymbol{\alpha}}\; e^{-\boldsymbol{\eta}^*_{\boldsymbol{\alpha}}\,\boldsymbol{D}^{(\alpha)}\,\boldsymbol{\eta}_{\boldsymbol{\alpha}}}
\end{aligned}$$

Non-interacting fermions III

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$$
\begin{aligned}
\mathcal{Z} &= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\phi^*_\alpha} d\boldsymbol{\phi_\alpha}\; e^{-\boldsymbol{\phi^*_\alpha}\,\boldsymbol{S^{(\alpha)}}\,\boldsymbol{\phi_\alpha}} \\
&= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\phi^*_\alpha} d\boldsymbol{\phi_\alpha}\; e^{-\boldsymbol{\phi^*_\alpha}\,\boldsymbol{U}^\dagger\,\boldsymbol{U}\,\boldsymbol{S^{(\alpha)}}\,\boldsymbol{U}^\dagger\,\boldsymbol{U}\,\boldsymbol{\phi_\alpha}} \\
&= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\eta^*_\alpha} d\boldsymbol{\eta_\alpha}\; e^{-\boldsymbol{\eta^*_\alpha}\,\boldsymbol{D^{(\alpha)}}\,\boldsymbol{\eta_\alpha}} \\
&= \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\eta^*_\alpha} d\boldsymbol{\eta_\alpha}\; e^{-\sum\limits_{k=1}^{M} d_k\,\boldsymbol{\eta^*_\alpha}\,\boldsymbol{\eta_\alpha}}
\end{aligned}
$$

Non-interacting fermions IV

- Grassmann-integrals:

$$\int d\eta\, 1 = 0\,, \qquad \int d\eta^*\, 1 \;\; = 0$$
$$\int d\eta\, \eta = 1\,, \qquad \int d\eta^*\, \eta^* = 1$$

Non-interacting fermions IV

- Grassmann-integrals:

$$\int d\eta \, 1 = 0 \, , \qquad \int d\eta^* \, 1 \;\; = 0$$
$$\int d\eta \, \eta = 1 \, , \qquad \int d\eta^* \, \eta^* = 1$$

- Gaussian Grassmann-integral:

$$\int d\eta^* d\eta \; e^{-\eta^* a \, \eta}$$

Non-interacting fermions IV

- Grassmann-integrals:

$$\int d\eta\, 1 = 0\,, \qquad \int d\eta^*\, 1 \;= 0$$
$$\int d\eta\, \eta = 1\,, \qquad \int d\eta^*\, \eta^* = 1$$

- Gaussian Grassmann-integral:

$$\int d\eta^* d\eta\; e^{-\eta^* a\, \eta} = \int d\eta^* d\eta\; (1 - \eta^* a\, \eta)$$

Non-interacting fermions IV

- Grassmann-integrals:

$$\int d\eta\, 1 = 0\,, \qquad \int d\eta^*\, 1 \ = 0$$
$$\int d\eta\, \eta = 1\,, \qquad \int d\eta^*\, \eta^* = 1$$

- Gaussian Grassmann-integral:

$$\int d\eta^* d\eta\; e^{-\eta^* a\, \eta} = \int d\eta^* d\eta\; (1 - \eta^* a\, \eta)$$
$$= \int d\eta^* d\eta\; \eta a\, \eta^*$$

Non-interacting fermions IV

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$$\int d\eta \, 1 = 0 \,, \qquad \int d\eta^* \, 1 \; = 0$$
$$\int d\eta \, \eta = 1 \,, \qquad \int d\eta^* \, \eta^* = 1$$

- Gaussian Grassmann-integral:

$$\begin{aligned}\int d\eta^* d\eta \; e^{-\eta^* a \, \eta} &= \int d\eta^* d\eta \; (1 - \eta^* a \, \eta) \\ &= \int d\eta^* d\eta \; \eta a \, \eta^* \\ &= a\end{aligned}$$

Non-interacting fermions V

- Partition function:

$$\mathcal{Z} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\eta}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\eta}_{\boldsymbol{\alpha}} \; e^{-d_k \, \boldsymbol{\eta}_{\boldsymbol{\alpha}}^{*} \boldsymbol{\eta}_{\boldsymbol{\alpha}}}$$

Non-interacting fermions V

- Partition function:

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$$= \lim_{M\to\infty} \prod_{\alpha} \prod_{k=1}^{M} d_k$$

Non-interacting fermions V

- Partition function:

$$\mathcal{Z} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\eta}^*_{\boldsymbol{\alpha}} d\boldsymbol{\eta}_{\boldsymbol{\alpha}} \; e^{-d_k \, \boldsymbol{\eta}^*_{\boldsymbol{\alpha}} \boldsymbol{\eta}_{\boldsymbol{\alpha}}}$$

$$= \lim_{M\to\infty} \prod_{\alpha} \prod_{k=1}^{M} d_k = \lim_{M\to\infty} \prod_{\alpha} \det \boldsymbol{S}^{(\boldsymbol{\alpha})}$$

Non-interacting fermions V

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- Determinant:

$$\lim_{M\to\infty} \det \boldsymbol{S}^{(\boldsymbol{\alpha})} = \lim_{M\to\infty} \left(1 - (-1)^{M-1}(-a)^{M}\right)$$

Non-interacting fermions V

- Partition function:

$$\mathcal{Z} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\eta}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\eta}_{\boldsymbol{\alpha}} \; e^{-d_k \, \boldsymbol{\eta}_{\boldsymbol{\alpha}}^{*} \, \boldsymbol{\eta}_{\boldsymbol{\alpha}}}$$

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$$= \lim_{M\to\infty} \left(1 + \left(1 - \frac{\beta}{M}(\epsilon_\alpha - \mu)\right)^{M}\right)$$

Non-interacting fermions V

- Partition function:

$$\mathcal{Z} = \lim_{M\to\infty} \int \prod_{k=1}^{M} \prod_{\alpha} d\boldsymbol{\eta}_{\boldsymbol{\alpha}}^{*} d\boldsymbol{\eta}_{\boldsymbol{\alpha}} \; e^{-d_k \; \boldsymbol{\eta}_{\boldsymbol{\alpha}}^{*} \boldsymbol{\eta}_{\boldsymbol{\alpha}}}$$

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$$= 1 + e^{-\beta(\epsilon_\alpha - \mu)}$$

Non-interacting fermions VI

$$\Longrightarrow \mathcal{Z} = \prod_{\alpha} \left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right)$$

Non-interacting fermions VI

$$\implies \mathcal{Z} = \prod_\alpha \left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right)$$

- Expectation value of $N = \sum_\alpha a_\alpha^\dagger a_\alpha$:

$$\langle N \rangle = k_B T \, \frac{\partial}{\partial \mu} \, \ln \mathcal{Z}$$

Non-interacting fermions VI

$$\Longrightarrow \mathcal{Z} = \prod_{\alpha} \left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right)$$

- Expectation value of $N = \sum_{\alpha} a_\alpha^\dagger a_\alpha$:

$$\begin{aligned} \langle N \rangle &= k_B T \, \frac{\partial}{\partial \mu} \, \ln \mathcal{Z} \\ &= \frac{1}{\beta} \frac{\partial}{\partial \mu} \, \sum_{\alpha} \ln \left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right) \end{aligned}$$

Non-interacting fermions VI

$$\Longrightarrow \mathcal{Z} = \prod_\alpha \left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right)$$

- Expectation value of $N = \sum_\alpha a_\alpha^\dagger a_\alpha$:

$$\begin{aligned}
\langle N \rangle &= k_B T \, \frac{\partial}{\partial \mu} \, \ln \mathcal{Z} \\
&= \frac{1}{\beta} \frac{\partial}{\partial \mu} \, \sum_\alpha \ln \left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right) \\
&= \sum_\alpha \, \frac{1}{1 + e^{\beta(\epsilon_\alpha - \mu)}} \\
&\quad \Longrightarrow \text{Fermi–Dirac distribution}
\end{aligned}$$

Conclusion

Summary

- Path-Integral for fermionic particles expressed by coherent states
- Grassmann variables needed for the coherent states due to the anticommuting of the annihilation operator a
- Derivation similar to the case of bosonic particles

Conclusion

Summary

- Path-Integral for fermionic particles expressed by coherent states
- Grassmann variables needed for the coherent states due to the anticommuting of the annihilation operator a
- Derivation similar to the case of bosonic particles

- Partition function can be expressed through the path-integral formalism
- Antiperiodic boundary conditions for fermionic particles
- Non-interacting Hamiltonian $\Longrightarrow$ Fermi-Dirac distribution

Thank you for your attention!

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