Coherent states for spins

Felix Engel

Motivation

CS for harmo

CS for spins

Analogue spin states Alternative parametrization Using group SU(2)

Summary

Coherent states for spins

Seminar: Quantum field-theory of low dimensional systems

Felix Engel

Universität Stuttgart Institut für Theoretische Physik III Prof. Dr. Alejandro Muramatsu

May 20, 2014

Coherent states for spins

Felix Engel

Motivatio

CS for harmon

CS for spin

Analogue spin states Alternative parametrization Using group SU(2)

Summary

- Motivation
- 2 Coherent states for harmonic oscillator (reminder)
 - Coherent states for spins
 - Analogue for spin states
 - Alternative parametrization
 - Using properties of the group SU(2)
- 4 Summary

Coherent states for spins

Felix Engel

Analogue spin

- Motivation
- 2 Coherent states for harmonic oscillator (reminder)
 - - Analogue for spin states
 - Alternative parametrization
 - Using properties of the group SU(2)

Coherent states for spins

Felix Engel

Analogue spin

- Motivation
- 2 Coherent states for harmonic oscillator (reminder)
 - Coherent states for spins
 - Analogue for spin states
 - Alternative parametrization
 - Using properties of the group SU(2)

Coherent states for spins

Felix Engel

Motivation

CS for harmon

CS for spins
Analogue spin
states
Alternative
parametrization
Using group

Summary

- Motivation
- 2 Coherent states for harmonic oscillator (reminder)
 - Coherent states for spins
 - Analogue for spin states
 - Alternative parametrization
 - Using properties of the group SU(2)
 - Summary

Coherent states for spins

Felix Engel

Motivation

CS for harmo

CS for spins
Analogue spin states
Alternative parametrization
Using group

Using group SU(2)

Motivation

- 2 Coherent states for harmonic oscillator (reminder)
 - 3 Coherent states for spins
 - Analogue for spin states
 - Alternative parametrization
 - Using properties of the group SU(2)
 - 4 Summary

Coherent states for spins

Felix Engel

Motivation

Analogue spin

Why do we need coherent states for spins?

• We can describe magnetic systems in which we are interested in

$$H = -\sum_{n,m} J(n,m) \boldsymbol{S}_n \cdot \boldsymbol{S}_m$$

- By using the coherent states of spins we are able to construct
- We obtain a more "classical-like" picture of spins by using the

Coherent states for spins

Felix Engel

Motivation

motivation

CS for har

CS for spin

Analogue spin states Alternative parametrization Using group SU(2)

Summary

Why do we need coherent states for spins?

 We can describe magnetic systems in which we are interested in the degree of freedom of the spins

$$H = -\sum_{n,m} J(n,m) \boldsymbol{S}_n \cdot \boldsymbol{S}_m$$

- By using the coherent states of spins we are able to construct the path-integral for them
- We obtain a more "classical-like" picture of spins by using the coherent spin states

Coherent states for spins

Felix Engel

Motivation

Analogue spin

Why do we need coherent states for spins?

• We can describe magnetic systems in which we are interested in the degree of freedom of the spins

$$H = -\sum_{n,m} J(n,m) \boldsymbol{S}_n \cdot \boldsymbol{S}_m$$

- By using the coherent states of spins we are able to construct the path-integral for them
- We obtain a more "classical-like" picture of spins by using the

Coherent states for spins

Felix Engel

Motivation

Analogue spin

Why do we need coherent states for spins?

• We can describe magnetic systems in which we are interested in the degree of freedom of the spins

$$H = -\sum_{n,m} J(n,m) \boldsymbol{S}_n \cdot \boldsymbol{S}_m$$

- By using the coherent states of spins we are able to construct the path-integral for them
- We obtain a more "classical-like" picture of spins by using the coherent spin states

Coherent states for spins

Felix Engel

Motivation

CS for harmonic oscillator

CS for spins
Analogue spin states
Alternative parametrization
Using group
SU(2)

Summary

- Motivation
- 2 Coherent states for harmonic oscillator (reminder)
 - Coherent states for spins
 - Analogue for spin states
 - Alternative parametrization
 - Using properties of the group SU(2)
 - 4 Summary

Coherent states for a harmonic oscillator I

Coherent states for spins

Felix Engel

Motivotion

CS for harmonic

CS for spins
Analogue spin states
Alternative parametrization
Using group

Summary

A coherent state (CS) $|\alpha\rangle$ is completely defined by $\alpha\in\mathbb{C}$

$$\begin{aligned} |\alpha\rangle &= \pi^{-1/2} \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\ &= \pi^{-1/2} \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\left(\alpha a^{\dagger}\right)^n}{n!} |0\rangle \\ &= \pi^{-1/2} \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \mathrm{e}^{\alpha a^{\dagger}} |0\rangle \end{aligned}$$

- We create the state $|n\rangle$ from the vacuum state $|0\rangle$ by using the creation operator
- Therefore, we end up with the creation operator in the exponential

Coherent states for a harmonic oscillator II

Coherent states for spins

Felix Engel

CS for harmonic oscillator

Analogue spin

Coherent state for a harmonic oscillator

$$|\alpha\rangle = \pi^{-1/2} e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^{\dagger}} |0\rangle, \ \alpha \in \mathbb{C}$$

These states build an overcomplete set

$$\int_{\mathbb{C}} \mathrm{d}\alpha \left| \alpha \right\rangle \left\langle \alpha \right| = \sum_{n=0}^{\infty} \left| n \right\rangle \left\langle n \right| = \mathbb{1}$$

 As you have already seen we can calculate quantities like the partition function

$$Z = \int \prod \mathsf{d}\Phi_{lpha}^* \mathsf{d}\Phi_{lpha} \mathsf{e}^{-\sum_{lpha}\Phi_{lpha}^*\Phi_{lpha}} \left\langle \Phi \right| \mathsf{e}^{-eta(\hat{H}-\mu\hat{N})} \left| \Phi
ight
angle$$

Coherent states for spins

Felix Engel

Motivatio

CS for harmon oscillator

CS for spins

Analogue spin states Alternative parametrization Using group SU(2)

Summary

- Motivation
- 2 Coherent states for harmonic oscillator (reminder)
 - 3 Coherent states for spins
 - Analogue for spin states
 - Alternative parametrization
 - Using properties of the group SU(2)
 - 4 Summary

Coherent states for spins

Felix Engel

Motivation

CS for harmo

CS for spin

Analogue spin

Alternative parametrization Using group

SU(2)

Eigenstates for operators: S_z , S^2 (m = -S, -S + 1, ..., S)

$$S_z |S, m\rangle = m |S, m\rangle$$
 $S^2 |S, m\rangle = S(S+1) |S, m\rangle$

Creation & annihilation operator: $S_{\pm} \equiv S_x \pm iS_y$

$$|S_{+}|S,m\rangle = \sqrt{(S+m+1)(S-m)}|S,m+1\rangle$$

$$S_{-}|S,m\rangle = \sqrt{(S-m+1)(S+m)}|S,m-1\rangle$$

Commutation relations:

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$
 $[S_z, S_{\pm}] = \pm S_{\pm}$ $[S_+, S_-] = 2S_z$

Coherent states for spins

Felix Engel

Analogue spin

states

Eigenstates for operators: S_z , S^2 (m = -S, -S + 1, ..., S)

$$S_z |S, m\rangle = m |S, m\rangle$$
 $S^2 |S, m\rangle = S(S+1) |S, m\rangle$

Creation & annihilation operator: $S_{\pm} \equiv S_x \pm iS_y$

$$|S_+|S,m\rangle = \sqrt{(S+m+1)(S-m)} |S,m+1\rangle$$

$$S_{-}|S,m\rangle = \sqrt{(S-m+1)(S+m)}|S,m-1\rangle$$

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$
 $[S_z, S_{\pm}] = \pm S_{\pm}$ $[S_+, S_-] = 2S_z$

Coherent states for spins

Felix Engel

Eigenstates for operators: S_z , S^2 (m = -S, -S + 1, ..., S)

 $\langle S_z | S, m \rangle = m \langle S, m \rangle$ $\langle S^2 | S, m \rangle = S(S+1) \langle S, m \rangle$

Analogue spin

states

Creation & annihilation operator: $S_{\pm} \equiv S_x \pm iS_y$

$$S_{+}|S,m\rangle = \sqrt{(S+m+1)(S-m)}|S,m+1\rangle$$

$$S_{-}\left|S,m\right\rangle = \sqrt{(S-m+1)(S+m)}\left|S,m-1\right\rangle$$

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$
 $[S_z, S_{\pm}] = \pm S_{\pm}$ $[S_+, S_-] = 2S_z$

Coherent states for spins

Felix Engel

Analogue spin states

Eigenstates for operators: S_z , S^2 (m = -S, -S + 1, ..., S)

$$S_z |S, m\rangle = m |S, m\rangle$$
 $S^2 |S, m\rangle = S(S+1) |S, m\rangle$

Creation & annihilation operator: $S_{\pm} \equiv S_x \pm iS_y$

$$S_{+}\left|S,m\right\rangle = \sqrt{(S+m+1)(S-m)}\left|S,m+1\right\rangle$$

$$S_{-}|S,m\rangle = \sqrt{(S-m+1)(S+m)}|S,m-1\rangle$$

Commutation relations:

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$
 $[S_z, S_{\pm}] = \pm S_{\pm}$ $[S_+, S_-] = 2S_z$

Choice of the "vacuum state"

Coherent states for spins

Felix Engel

Analogue spin

states

Minimize the uncertainty relation

$$\left\langle \Delta S_{x}\right
angle _{\psi_{0}}\left\langle \Delta S_{y}
ight
angle _{\psi_{0}}\geq\frac{1}{2}\left|\left\langle S_{z}
ight
angle _{\psi_{0}}\right|$$

$$\begin{split} \left\langle \Delta S_{x} \right\rangle_{\left|S,m\right\rangle} \left\langle \Delta S_{y} \right\rangle_{\left|S,m\right\rangle} &= \left[\left(\left\langle S_{x}^{2} \right\rangle - \left\langle S_{x} \right\rangle^{2} \right) \left(\left\langle S_{y}^{2} \right\rangle - \left\langle S_{y} \right\rangle^{2} \right) \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left[\left\langle \left(S_{+} + S_{-} \right)^{2} \right\rangle - \left(\left\langle S_{+} + S_{-} \right\rangle \right)^{2} \right]^{\frac{1}{2}} \\ &\times \left[\left(\left\langle S_{+} - S_{-} \right\rangle \right)^{2} - \left\langle \left(S_{+} - S_{-} \right)^{2} \right\rangle \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left\langle S, m \middle| S_{+} S_{-} + S_{-} S_{+} \middle| S, m \right\rangle \\ &= \frac{1}{4} \left[\left(S_{-} m + 1 \right) \left(S_{$$

Coherent states for spins

Felix Engel

Motivat

CS for harm

CS for spin

CS for spins
Analogue spin

Alternative parametrization Using group

Summanı

Minimize the uncertainty relation

$$\left\langle \Delta S_{x}\right\rangle _{\psi_{0}}\left\langle \Delta S_{y}\right\rangle _{\psi_{0}}\geq\frac{1}{2}\left|\left\langle S_{z}\right\rangle _{\psi_{0}}\right|$$

Consider the left side:

$$\begin{split} \langle \Delta S_{x} \rangle_{|S,m\rangle} \, \langle \Delta S_{y} \rangle_{|S,m\rangle} &= \left[\left(\langle S_{x}^{2} \rangle - \langle S_{x} \rangle^{2} \right) \left(\langle S_{y}^{2} \rangle - \langle S_{y} \rangle^{2} \right) \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left[\left\langle (S_{+} + S_{-})^{2} \rangle - (\langle S_{+} + S_{-} \rangle)^{2} \right]^{\frac{1}{2}} \\ &\times \left[(\langle S_{+} - S_{-} \rangle)^{2} - \langle (S_{+} - S_{-})^{2} \rangle \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left\langle S, m | S_{+} S_{-} + S_{-} S_{+} | S, m \rangle \\ &= \frac{1}{4} \left[(S - m + 1)(S + m) + (S + m + 1)(S - m) \right] \end{split}$$

 \Rightarrow minimum at: m = +

Coherent states for spins

Felix Engel

Motivat

CS for harm

CS for coins

CS for spins
Analogue spin

Alternative parametrizatio Using group

Summanı

Minimize the uncertainty relation

$$\left\langle \Delta S_{x}\right\rangle _{\psi_{0}}\left\langle \Delta S_{y}\right\rangle _{\psi_{0}}\geq\frac{1}{2}\left|\left\langle S_{z}\right\rangle _{\psi_{0}}\right|$$

Consider the left side:

$$\begin{split} \left\langle \Delta S_{x} \right\rangle_{|S,m\rangle} \left\langle \Delta S_{y} \right\rangle_{|S,m\rangle} &= \left[\left(\left\langle S_{x}^{2} \right\rangle - \left\langle S_{x} \right\rangle^{2} \right) \left(\left\langle S_{y}^{2} \right\rangle - \left\langle S_{y} \right\rangle^{2} \right) \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left[\left\langle (S_{+} + S_{-})^{2} \right\rangle - \left(\left\langle S_{+} + S_{-} \right\rangle \right)^{2} \right]^{\frac{1}{2}} \\ &\times \left[\left(\left\langle S_{+} - S_{-} \right\rangle \right)^{2} - \left\langle (S_{+} - S_{-})^{2} \right\rangle \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left\langle S, m | S_{+} S_{-} + S_{-} S_{+} | S, m \right\rangle \\ &= \frac{1}{4} \left[\left(S - m + 1 \right) \left(S + m \right) + \left(S + m + 1 \right) \left(S - m \right) \end{split}$$

 \Rightarrow minimum at: m = +

Coherent states for spins

Felix Engel

Motivat

CS for harm

CS for spin

Analogue spin states

Alternative parametrizatio Using group

Summanı

Minimize the uncertainty relation

$$\left\langle \Delta S_{x}\right\rangle _{\psi_{0}}\left\langle \Delta S_{y}\right\rangle _{\psi_{0}}\geq\frac{1}{2}\left|\left\langle S_{z}\right\rangle _{\psi_{0}}\right|$$

Consider the left side:

$$\begin{split} \left\langle \Delta S_{x} \right\rangle_{\left|S,m\right\rangle} \left\langle \Delta S_{y} \right\rangle_{\left|S,m\right\rangle} &= \left[\left(\left\langle S_{x}^{2} \right\rangle - \left\langle S_{x} \right\rangle^{2} \right) \left(\left\langle S_{y}^{2} \right\rangle - \left\langle S_{y} \right\rangle^{2} \right) \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left[\left\langle \left(S_{+} + S_{-} \right)^{2} \right\rangle - \left(\left\langle S_{+} + S_{-} \right\rangle \right)^{2} \right]^{\frac{1}{2}} \\ &\times \left[\left(\left\langle S_{+} - S_{-} \right\rangle \right)^{2} - \left\langle \left(S_{+} - S_{-} \right)^{2} \right\rangle \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left\langle S, m | S_{+} S_{-} + S_{-} S_{+} | S, m \right\rangle \\ &= \frac{1}{4} \left[\left(S - m + 1 \right) \left(S + m \right) + \left(S + m + 1 \right) \left(S - m \right) \right] \end{split}$$

 \Rightarrow minimum at: $m = \pm 3$

Coherent states for spins

Felix Engel

Motivat

CS for harm

CS for spin

Analogue spin states

Alternative parametrizatio Using group

Summanı

Minimize the uncertainty relation

$$\langle \Delta S_{\mathsf{x}} \rangle_{\psi_0} \langle \Delta S_{\mathsf{y}} \rangle_{\psi_0} \geq \frac{1}{2} \left| \langle S_{\mathsf{z}} \rangle_{\psi_0} \right|$$

Consider the left side:

$$\begin{split} \left\langle \Delta S_{x} \right\rangle_{\left|S,m\right\rangle} \left\langle \Delta S_{y} \right\rangle_{\left|S,m\right\rangle} &= \left[\left(\left\langle S_{x}^{2} \right\rangle - \left\langle S_{x} \right\rangle^{2} \right) \left(\left\langle S_{y}^{2} \right\rangle - \left\langle S_{y} \right\rangle^{2} \right) \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left[\left\langle \left(S_{+} + S_{-} \right)^{2} \right\rangle - \left(\left\langle S_{+} + S_{-} \right\rangle \right)^{2} \right]^{\frac{1}{2}} \\ &\qquad \times \left[\left(\left\langle S_{+} - S_{-} \right\rangle \right)^{2} - \left\langle \left(S_{+} - S_{-} \right)^{2} \right\rangle \right]^{\frac{1}{2}} \\ &= \frac{1}{4} \left\langle S, m | S_{+} S_{-} + S_{-} S_{+} | S, m \right\rangle \\ &= \frac{1}{4} \left[\left(S - m + 1 \right) \left(S + m \right) + \left(S + m + 1 \right) \left(S - m \right) \right] \end{split}$$

 \Rightarrow minimum at: $m = \pm S$

Coherent states for spins

Felix Engel

Analogue spin

states

For reasons of simplicity we use the following definition

Definition

$$S_z |p\rangle \equiv (S-p)|p\rangle, \quad 0 \le p \le 2S$$

by using the annihilation operator p times

$$(S_{-})^{p}|0\rangle = [(S - S + 1)(S - (S - 1) + 1) \dots (S - (S - p + 1) + 1)]^{1/2} \times [(S + S)(S + S - 1) \dots (S + S - p + 1)]^{1/2} |S, S - p\rangle$$

$$= \left[\frac{p!(2S)!}{(2S - p)!}\right]^{1/2} |p\rangle$$

Coherent states for spins

Felix Engel

Analogue spin states

For reasons of simplicity we use the following definition

Definition

$$S_z |p\rangle \equiv (S-p)|p\rangle, \quad 0 \le p \le 2S$$

Now we **construct** the state $|p\rangle$ from "vacuum state" $|0\rangle \equiv |S, S\rangle$ by using the **annihilation operator** p times

$$(S_{-})^{p}|0\rangle = [(S - S + 1)(S - (S - 1) + 1) \dots (S - (S - p + 1) + 1)]^{1/2} \times [(S + S)(S + S - 1) \dots (S + S - p + 1)]^{1/2} |S, S - p\rangle$$

$$= \left[\frac{p!(2S)!}{(2S - p)!}\right]^{1/2} |p\rangle$$

Choice of the "vacuum state" |0 | II

Coherent states for spins

Felix Engel

Analogue spin

states

For reasons of simplicity we use the following definition

Definition

$$S_{z}\left|p\right\rangle \equiv\left(S-p\right)\left|p\right\rangle ,\quad0\leq p\leq2S$$

Now we **construct** the state $|p\rangle$ from "vacuum state" $|0\rangle \equiv |S, S\rangle$ by using the **annihilation operator** p times

$$(S_{-})^{p}|0\rangle = [(S - S + 1)(S - (S - 1) + 1) \dots (S - (S - p + 1) + 1)]^{1/2}$$

$$\times [(S + S)(S + S - 1) \dots (S + S - p + 1)]^{1/2} |S, S - p\rangle$$

$$= \left[\frac{p!(2S)!}{(2S - p)!}\right]^{1/2} |p\rangle$$

Consider single particle of spin S

Coherent states for spins

Felix Engel

Motivatio

CS for harmon

CS for spins Analogue spin

Alternative parametrization Using group SU(2)

Summary

Analogue to the CS of a harmonic oscillator

Coherent state HO (reminder)

$$\left|\alpha\right\rangle = \pi^{-1/2} \mathrm{e}^{-\frac{1}{2}\left|\alpha\right|^2} \mathrm{e}^{\alpha \mathbf{a}^\dagger} \left|\mathbf{0}\right\rangle, \, \alpha \in \mathbb{C}$$

where we create the state $|\alpha\rangle$ using the creation operator ${\it a^{\dag}}$ on the vacuum state $|0\rangle$

We **introduce** a CS $|\mu\rangle$ for a single particle of spin S by

Definition of CS for spin S

$$\left|\mu\right\rangle = \frac{1}{\sqrt{N}} \cdot \mathrm{e}^{\mu S_{-}} \left|0\right\rangle, \, \mu \in \mathbb{C}$$

where we create the state $|\mu\rangle$ using the annihilation operator S_- on the "vacuum state" $|0\rangle = |S,S\rangle$

Consider single particle of spin S

Coherent states for spins

Felix Engel

Motivatio

S for harmon scillator

CS for spins
Analogue spin
states

Alternative parametrization Using group SU(2)

Summar

Analogue to the CS of a harmonic oscillator

Coherent state HO (reminder)

$$\left|\alpha\right\rangle = \pi^{-1/2} \mathrm{e}^{-\frac{1}{2}\left|\alpha\right|^2} \mathrm{e}^{\alpha \mathsf{a}^\dagger} \left|0\right\rangle, \, \alpha \in \mathbb{C}$$

where we create the state $|\alpha\rangle$ using the creation operator a^{\dagger} on the vacuum state $|0\rangle$

We **introduce** a CS $|\mu\rangle$ for a single particle of spin *S* by:

Definition of CS for spin S

$$\ket{\mu} = rac{1}{\sqrt{N}} \cdot \mathrm{e}^{\mu S_{-}} \ket{0}, \, \mu \in \mathbb{C}$$

where we create the state $|\mu\rangle$ using the annihilation operator S_- on the "vacuum state" $|0\rangle = |S,S\rangle$

Calculation of normalization factor N

Coherent states for spins

Felix Engel

Motivatio

CS for harmo

CS for spins

Analogue spin

Alternative parametrizatio Using group SU(2)

Summary

Series expansion of the exponential function

$$|\mu\rangle = \frac{\mathrm{e}^{\mu S_{-}}}{\sqrt{N}} \, |0\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{2S} \frac{\mu^{p}}{p!} (S_{-})^{p} \, |0\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{2S} \mu^{p} \sqrt{\frac{(2S)!}{(2S-p)!p!}} \, |p\rangle$$

States $|p\rangle$ build an orthogonal basis this leads us to:

$$\langle \mu | \mu \rangle = \frac{1}{N} \sum_{p,p'=0}^{2S} \left[\left(\frac{(2S)!}{(2S-p)!p!} \right) \left(\frac{(2S)!}{(2S-p')!p'!} \right) \right]^{1/2} \mu^{p} (\mu^{*})^{p'} \delta_{p,p}$$

$$= \frac{1}{N} \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} |\mu|^{2p} = \frac{1}{N} \sum_{p=0}^{2S} {2S \choose p} (|\mu|^{2})^{p}$$

$$= \frac{1}{N} (1 + |\mu|^{2})^{2S}$$

Calculation of normalization factor N

Coherent states for spins

Felix Engel

Motivatio

CS for harmon

CS for spins

Analogue spin states Alternative

parametrizatio Using group SU(2)

Summary

Series expansion of the exponential function

$$|\mu\rangle = \frac{\mathrm{e}^{\mu S_{-}}}{\sqrt{N}} |0\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{2S} \frac{\mu^{p}}{p!} (S_{-})^{p} |0\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{2S} \mu^{p} \sqrt{\frac{(2S)!}{(2S-p)!p!}} |p\rangle$$

States $|p\rangle$ build an orthogonal basis this leads us to:

$$\langle \mu | \mu \rangle = \frac{1}{N} \sum_{p,p'=0}^{2S} \left[\left(\frac{(2S)!}{(2S-p)!p!} \right) \left(\frac{(2S)!}{(2S-p')!p'!} \right) \right]^{1/2} \mu^{p} (\mu^{*})^{p'} \delta_{p,p'}$$

$$= \frac{1}{N} \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} |\mu|^{2p} = \frac{1}{N} \sum_{p=0}^{2S} \binom{2S}{p} \left(|\mu|^{2} \right)^{p}$$

$$= \frac{1}{N} (1 + |\mu|^{2})^{2S}$$

Completeness relation of coherent spin states I

Coherent states for spins

Felix Engel

Motivation

MOLIVALION

CS for harmoni

CS for spin

Analogue spin

Alternative parametrization Using group SU(2)

Summary

Coherent spin states $|\mu\rangle$ build an overcomplete set of states

$$\mathbb{1} = \sum_{p=0}^{2S} \ket{p}\bra{p} = \int_{\mathbb{C}} \mathrm{d}\mu \ket{\mu}\bra{\mu} \cdot w(\ket{\mu}^2)$$

By using the substitution $\mu = \rho \cdot e^{i\phi}$ we obtain:

$$\int_{\mathbb{C}} \mathrm{d}\mu \left| \mu \right\rangle \left\langle \mu \right| \cdot w(|\mu|^2) = \sum_{p=0}^{2S} \left| p \right\rangle \left\langle p \right| \frac{(2S)!}{(2S-p)!p!} \underbrace{2\pi \int_{0}^{\infty} \frac{w(\rho^2)\rho^{2p+1}}{(1+\rho^2)^{2S}} \mathrm{d}\rho}_{\equiv \mathrm{I}(S,p)}$$

Therefore, we need a weight function $w(\rho^2)$ which fulfills the condition:

$$I(S, p) = \frac{(2S - p)! p!}{(2S)!}$$

Completeness relation of coherent spin states I

Coherent states for spins

Felix Engel

Mativation

IVIOLIVACIOII

CS for spins

Analogue spin

Alternative parametrization Using group SU(2)

Summary

Coherent spin states $|\mu\rangle$ build an overcomplete set of states

$$\mathbb{1} = \sum_{p=0}^{2S} \ket{p}\bra{p} = \int_{\mathbb{C}} \mathrm{d}\mu \ket{\mu}\bra{\mu} \cdot w(\ket{\mu}^2)$$

By using the substitution $\mu = \rho \cdot e^{i\phi}$ we obtain:

$$\int_{\mathbb{C}} \mathrm{d}\mu \left| \mu \right\rangle \left\langle \mu \right| \cdot w(|\mu|^2) = \sum_{p=0}^{2S} \left| p \right\rangle \left\langle p \right| \frac{(2S)!}{(2S-p)! p!} \underbrace{2\pi \int_{0}^{\infty} \frac{w(\rho^2) \rho^{2p+1}}{(1+\rho^2)^{2S}} \mathrm{d}\rho}_{\equiv \mathrm{I}(S,p)}$$

Therefore, we need a weight function $w(\rho^2)$ which fulfills the condition:

$$I(S, p) = \frac{(2S - p)!p}{(2S)!}$$

Completeness relation of coherent spin states I

Coherent states for spins

Felix Engel

Analogue spin

states

Coherent spin states $|\mu\rangle$ build an overcomplete set of states

$$\mathbb{1} = \sum_{p=0}^{2S} \ket{p}\bra{p} = \int_{\mathbb{C}} \mathrm{d}\mu \ket{\mu}\bra{\mu} \cdot w(\ket{\mu}^2)$$

By using the substitution $\mu = \rho \cdot e^{i\phi}$ we obtain:

$$\int_{\mathbb{C}} \mathrm{d}\mu \left| \mu \right\rangle \left\langle \mu \right| \cdot w(\left| \mu \right|^2) = \sum_{p=0}^{2S} \left| p \right\rangle \left\langle p \right| \frac{(2S)!}{(2S-p)! p!} \underbrace{2\pi \int_{0}^{\infty} \frac{w(\rho^2) \rho^{2p+1}}{(1+\rho^2)^{2S}} \mathrm{d}\rho}_{\equiv \mathrm{I}(S,p)}$$

Therefore, we need a weight function $w(\rho^2)$ which fulfills the condition:

$$I(S,p) = \frac{(2S-p)!p!}{(2S)!}$$

Completeness relation of coherent spin states II

Coherent states for spins

Felix Engel

Motivatio

S for harmon scillator

CS for spins

Analogue spin states

Alternative parametrizatio Using group

Summary

A good choice seems to be $w(\rho^2) = \frac{2S+1}{\pi} \frac{1}{(1+\rho^2)^2}$

$$I(S,p) = 2\pi \int_0^\infty \frac{w(\rho^2)\rho^{2p+1}}{(1+\rho^2)^{2S}} d\rho \stackrel{[\rho^2=\sigma]}{=} \pi \int_0^\infty \frac{w(\sigma)\sigma^p}{(1+\sigma)^{2S}} d\sigma =$$

$$= (2S+1) \int_0^\infty \frac{\sigma^p d\sigma}{(1+\sigma)^{2S+2S}} d\sigma = 0$$

$$= (25+1) \int_0^{\infty} \frac{(1+\sigma)^{2S+2}}{(1+\sigma)^{2S+2}}$$

$$= (2S+1) \left[\frac{\sigma^{p}}{-(2S+1)(1+\sigma)^{2S+1}} \right]_{0}^{\infty} - \frac{(2S+1)p}{-(2S+1)} \int_{0}^{\infty} \frac{\sigma^{p-1} d\sigma}{(1+\sigma)^{2S+1}}$$

$$= \dots = \frac{(2S+1)p(p-1)\dots 1}{(2S+1)(2S)\dots (2S-p+1)} = \frac{p!}{(2S)!/(2S-p)!}$$

where we determine I(S, p) using the integration by parts p times

$$\Rightarrow \quad \mathbb{I} = \sum_{c=0}^{2S} \left| p \right\rangle \left\langle p \right| = \frac{2S+1}{\pi} \int_{\mathbb{C}} \mathrm{d}\mu \left| \mu \right\rangle \left\langle \mu \right| \cdot \frac{1}{(1+|\mu|^2)^2}$$

Completeness relation of coherent spin states II

Coherent states for spins

Felix Engel

Motivatio

CS for harmon

CS for spins

Analogue spin states

Alternative parametrization Using group

Summary

A good choice seems to be
$$w(\rho^2) = \frac{2S+1}{\pi} \frac{1}{(1+\rho^2)^2}$$

$$\begin{split} & \mathrm{I}(S, \rho) = 2\pi \int_0^\infty \frac{w(\rho^2)\rho^{2\rho+1}}{(1+\rho^2)^{2S}} \mathrm{d}\rho \stackrel{[\rho^2=\sigma]}{=} \pi \int_0^\infty \frac{w(\sigma)\sigma^\rho}{(1+\sigma)^{2S}} \mathrm{d}\sigma = \\ & = & (2S+1) \int_0^\infty \frac{\sigma^\rho \, \mathrm{d}\sigma}{(1+\sigma)^{2S+2}} \end{split}$$

$$= (2S+1) \underbrace{\left[\frac{\sigma^p}{-(2S+1)(1+\sigma)^{2S+1}} \right]_0^{\infty} - \frac{(2S+1)p}{-(2S+1)} \int_0^{\infty} \frac{\sigma^{p-1} d\sigma}{(1+\sigma)^{2S+1}}}_{=0}$$

$$= \dots = \frac{(2S+1)p(p-1)\dots 1}{(2S+1)(2S)\dots (2S-p+1)} = \frac{p!}{(2S)!/(2S-p)!}$$

where we determine I(S, p) using the integration by parts p times

$$\Rightarrow \quad \mathbb{1} = \sum_{p=0}^{2S} |p\rangle \langle p| = \frac{2S+1}{\pi} \int_{\mathbb{C}} \mathrm{d}\mu |\mu\rangle \langle \mu| \cdot \frac{1}{(1+|\mu|^2)^2}$$

Completeness relation of coherent spin states II

Coherent states for spins

Felix Engel

Motivatio

CS for harmo

CS for spins

Analogue spin states

Alternative parametrization Using group

Summary

A good choice seems to be
$$w(\rho^2) = \frac{2S+1}{\pi} \frac{1}{(1+\rho^2)^2}$$

$$I(S, p) = 2\pi \int_{0}^{\infty} \frac{w(\rho^{2})\rho^{2p+1}}{(1+\rho^{2})^{2S}} d\rho \stackrel{[\rho^{2}=\sigma]}{=} \pi \int_{0}^{\infty} \frac{w(\sigma)\sigma^{p}}{(1+\sigma)^{2S}} d\sigma =$$

$$= (2S+1) \int_{0}^{\infty} \frac{\sigma^{p} d\sigma}{(1+\sigma)^{2S+2}}$$

$$= (2S+1) \underbrace{\left[\frac{\sigma^{p}}{-(2S+1)(1+\sigma)^{2S+1}} \right]_{0}^{\infty} - \frac{(2S+1)p}{-(2S+1)} \int_{0}^{\infty} \frac{\sigma^{p-1} d\sigma}{(1+\sigma)^{2S+1}}}_{=0}$$

$$= \dots = \frac{(2S+1)p(p-1)\dots 1}{(2S+1)(2S)\dots(2S-p+1)} = \frac{p!}{(2S)!/(2S-p)!}$$

where we determine I(S, p) using the integration by parts p times

$$\Rightarrow \quad \mathbb{1} = \sum_{p=0}^{2S} |p\rangle \langle p| = \frac{2S+1}{\pi} \int_{\mathbb{C}} \mathrm{d}\mu |\mu\rangle \langle \mu| \cdot \frac{1}{(1+|\mu|^2)^2}$$

Completeness relation of coherent spin states II

Coherent states for spins

Felix Engel

Motivatio

CS for harmo oscillator

CS for spins
Analogue spin

Alternative parametrization Using group

Summary

A good choice seems to be $w(\rho^2) = \frac{2S+1}{\pi} \frac{1}{(1+\rho^2)^2}$

$$I(S, p) = 2\pi \int_{0}^{\infty} \frac{w(\rho^{2})\rho^{2p+1}}{(1+\rho^{2})^{2S}} d\rho \stackrel{[\rho^{2}=\sigma]}{=} \pi \int_{0}^{\infty} \frac{w(\sigma)\sigma^{p}}{(1+\sigma)^{2S}} d\sigma =$$

$$= (2S+1) \int_{0}^{\infty} \frac{\sigma^{p} d\sigma}{(1+\sigma)^{2S+2}}$$

$$= (2S+1) \underbrace{\left[\frac{\sigma^{p}}{-(2S+1)(1+\sigma)^{2S+1}}\right]_{0}^{\infty} - \frac{(2S+1)p}{-(2S+1)} \int_{0}^{\infty} \frac{\sigma^{p-1} d\sigma}{(1+\sigma)^{2S+1}}}_{=0}$$

$$= \dots = \underbrace{\frac{(2S+1)p(p-1)\dots 1}{(2S+1)(2S)\dots (2S-p+1)}}_{=0} = \frac{p!}{(2S)!/(2S-p)!}$$

where we determine I(S, p) using the integration by parts p times

$$\Rightarrow \quad \mathbb{1} = \sum_{p=0}^{2S} \ket{p} \bra{p} = \frac{2S+1}{\pi} \int_{\mathbb{C}} \mathsf{d}\mu \ket{\mu} \bra{\mu} \cdot \frac{1}{(1+|\mu|^2)^2}$$

Overlap integral of two different states $|\mu\rangle\,, |\nu\rangle$

Coherent states for spins

Felix Engel

Motivation

oscillator

CS for spins

Analogue spin

Alternative parametrizatio
Using group

Summar

Now we are able to calculate the overlap integral $\langle \nu | \mu \rangle$

$$\begin{split} \langle \nu | \mu \rangle &= \frac{1}{\sqrt{N_{\mu} N_{\nu}}} \sum_{p,p'=0}^{2S} \left[\left(\frac{(2S)!}{(2S-p'!)p'!} \frac{(2S)!}{(2S-p!)p!} \right) \right]^{\frac{1}{2}} (\nu^{*})^{p'} \mu^{p} \delta_{p,p'} \\ &= \frac{1}{\sqrt{N_{\mu} N_{\nu}}} \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p!)p!} (\nu^{*}\mu)^{p} = \frac{1}{\sqrt{N_{\mu} N_{\nu}}} \sum_{p=0}^{2S} \binom{2S}{p} (\nu^{*}\mu)^{p} \\ &= \frac{(1+\nu^{*}\mu)^{2S}}{(1+|\nu|^{2})^{S} (1+|\mu|^{2})^{S}} \end{split}$$

For the absolute value we obtain

$$\begin{aligned} \left| \langle \nu | \mu \rangle \right|^2 &= \left| \frac{(1 + \nu^* \mu)^{2S}}{(1 + |\nu|^2)^S (1 + |\mu|^2)^S} \right|^2 = \left(\frac{|1 + \nu^* \mu|^2}{(1 + |\nu|^2)(1 + |\mu|^2)} \right)^{2S} \\ &= \left(\frac{(1 + \nu^* \mu)(1 + \nu \mu^*)}{(1 + |\nu|^2)(1 + |\mu|^2)} \right)^{2S} = \left(1 - \frac{|\nu - \mu|}{(1 + |\nu|^2)(1 + |\mu|^2)} \right)^{2S} \end{aligned}$$

Overlap integral of two different states $|\mu\rangle\,, |\nu\rangle$

Coherent states for spins

Felix Engel

Motivatio

CS for harmo

CS for spins

Analogue spin states
Alternative

parametrizatio Using group SU(2)

Summary

Now we are able to calculate the overlap integral $\langle \nu | \mu \rangle$

$$\begin{split} \langle \nu | \mu \rangle &= \frac{1}{\sqrt{N_{\mu}N_{\nu}}} \sum_{p,p'=0}^{2S} \left[\left(\frac{(2S)!}{(2S-p'!)p'!} \frac{(2S)!}{(2S-p!)p!} \right) \right]^{\frac{1}{2}} (\nu^{*})^{p'} \mu^{p} \delta_{p,p'} \\ &= \frac{1}{\sqrt{N_{\mu}N_{\nu}}} \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p!)p!} (\nu^{*}\mu)^{p} = \frac{1}{\sqrt{N_{\mu}N_{\nu}}} \sum_{p=0}^{2S} \binom{2S}{p} (\nu^{*}\mu)^{p} \\ &= \frac{(1+\nu^{*}\mu)^{2S}}{(1+|\nu|^{2})^{S} (1+|\mu|^{2})^{S}} \end{split}$$

For the absolute value we obtain

$$\begin{aligned} \left| \langle \nu | \mu \rangle \right|^2 &= \left| \frac{(1 + \nu^* \mu)^{2S}}{(1 + |\nu|^2)^S (1 + |\mu|^2)^S} \right|^2 = \left(\frac{|1 + \nu^* \mu|^2}{(1 + |\nu|^2)(1 + |\mu|^2)} \right)^{2S} \\ &= \left(\frac{(1 + \nu^* \mu)(1 + \nu \mu^*)}{(1 + |\nu|^2)(1 + |\mu|^2)} \right)^{2S} = \left(1 - \frac{|\nu - \mu|}{(1 + |\nu|^2)(1 + |\mu|^2)} \right)^{2S} \end{aligned}$$

Overlap integral of two different states $|\mu\rangle\,, |\nu\rangle$

Coherent states for spins

Felix Engel

Motivation

S for harmo

CS for spins
Analogue spin

Alternative parametrization Using group

Summary

Now we are able to calculate the overlap integral $\langle \nu | \mu \rangle$

$$\begin{split} \langle \nu | \mu \rangle &= \frac{1}{\sqrt{N_{\mu}N_{\nu}}} \sum_{p,p'=0}^{2S} \left[\left(\frac{(2S)!}{(2S-p'!)p'!} \frac{(2S)!}{(2S-p!)p!} \right) \right]^{\frac{1}{2}} (\nu^{*})^{p'} \mu^{p} \delta_{p,p'} \\ &= \frac{1}{\sqrt{N_{\mu}N_{\nu}}} \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p!)p!} (\nu^{*}\mu)^{p} = \frac{1}{\sqrt{N_{\mu}N_{\nu}}} \sum_{p=0}^{2S} \binom{2S}{p} (\nu^{*}\mu)^{p} \\ &= \frac{(1+\nu^{*}\mu)^{2S}}{(1+|\nu|^{2})^{S}(1+|\mu|^{2})^{S}} \end{split}$$

For the absolute value we obtain:

$$\begin{split} \left| \left\langle \nu | \mu \right\rangle \right|^2 &= \left| \frac{(1 + \nu^* \mu)^{2S}}{(1 + |\nu|^2)^S (1 + |\mu|^2)^S} \right|^2 = \left(\frac{|1 + \nu^* \mu|^2}{(1 + |\nu|^2)(1 + |\mu|^2)} \right)^{2S} \\ &= \left(\frac{(1 + \nu^* \mu)(1 + \nu \mu^*)}{(1 + |\nu|^2)(1 + |\mu|^2)} \right)^{2S} = \left(1 - \frac{|\nu - \mu|}{(1 + |\nu|^2)(1 + |\mu|^2)} \right)^{2S} \end{split}$$

High spin limit I

Coherent states for spins

Felix Engel

Motivatio

CS for harmoni

CS for spins

Analogue spin

Alternative parametrizatio Using group SU(2)

Summary

We use the *Holstein-Primakoff transformation* to obtain the high spin limit of the coherent spin states

$$S_{+} \equiv \left(\sqrt{2S - a^{\dagger}a}\right)a$$
 $S_{-} \equiv a^{\dagger}\left(\sqrt{2S - a^{\dagger}a}\right)$ $S_{z} \equiv S - a^{\dagger}a$

It is not difficult to see that this definition fulfills the commutation relation:

$$[S_i, S_j] = i\epsilon_{ijk}S_j$$

For $S \gg 1$ we can expand S_{-} in a Taylor series

$$S_- \to (2S)^{1/2} a^{\dagger}$$

 $\Rightarrow \mu \to \alpha/(2S)^{1/2}$

High spin limit I

Coherent states for spins

Felix Engel

Motivation

CS for harmon

CS for spins

Analogue spin states

Alternative parametrization Using group SU(2)

Summary

We use the *Holstein-Primakoff transformation* to obtain the high spin limit of the coherent spin states

$$S_{+} \equiv \left(\sqrt{2S - a^{\dagger}a}\right)a$$
 $S_{-} \equiv a^{\dagger}\left(\sqrt{2S - a^{\dagger}a}\right)$ $S_{z} \equiv S - a^{\dagger}a$

It is not difficult to see that this definition fulfills the commutation relation:

$$[S_i, S_j] = \mathrm{i} \epsilon_{ijk} S_k$$

For $S \gg 1$ we can expand S_{-} in a Taylor series

$$S_- \to (2S)^{1/2} a^{\dagger}$$

 $\Rightarrow \mu \to \alpha/(2S)^{1/2}$

High spin limit I

Coherent states for spins

Felix Engel

Motivation

....

Analogue spin states

Alternative parametrization Using group SU(2)

Summary

We use the *Holstein-Primakoff transformation* to obtain the high spin limit of the coherent spin states

$$S_{+} \equiv \left(\sqrt{2S - a^{\dagger}a}\right)a$$
 $S_{-} \equiv a^{\dagger}\left(\sqrt{2S - a^{\dagger}a}\right)$ $S_{z} \equiv S - a^{\dagger}a$

It is not difficult to see that this definition fulfills the commutation relation:

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk}S_k$$

For $S \gg 1$ we can expand S_{-} in a Taylor series

$$S_- \rightarrow (2S)^{1/2} a^{\dagger}$$

 $\Rightarrow \mu \rightarrow \alpha/(2S)^{1/2}$

High spin limit II

Coherent states for spins

Felix Engel

Motivatio

CS for harmon

CS for s

Analogue spin states

Alternative parametrization Using group SU(2)

Summary

Therefore we obtain:

$$\exp\left(\mu \mathcal{S}_{-}
ight)
ightarrow \exp\left(lpha \mathcal{a}^{\dagger}
ight)$$

Remember a coherent spin state

$$|\mu\rangle = (1+|\mu|^2)^{-S} \mathrm{e}^{\mu S_-} |0\rangle$$

In the limit $S \to \infty$ a normalized spin state $|\alpha\rangle_S$ is:

$$\lim_{S \to \infty} |\alpha\rangle_S = \underbrace{\lim_{S \to \infty} \left(1 + \frac{|\alpha|^2}{2S}\right)^{-S}}_{\to \exp(-|\alpha|^2/2)} e^{\alpha s^{\dagger}} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha s^{\dagger}} |0\rangle$$

Except a factor $\pi^{-1/2}$, this result is equal to the **coherent states of** a harmonic oscillator

High spin limit II

Coherent states for spins

Felix Engel

Motivatio

CS for harmon

CS for s

Analogue spin states

Alternative parametrization Using group SU(2)

Summary

Therefore we obtain:

$$\exp\left(\mu \mathcal{S}_{-}
ight)
ightarrow \exp\left(lpha \mathcal{a}^{\dagger}
ight)$$

Remember a coherent spin state

$$|\mu\rangle = (1 + |\mu|^2)^{-S} e^{\mu S_-} |0\rangle$$

In the limit $S \to \infty$ a normalized spin state $|\alpha\rangle_S$ is:

$$\lim_{S \to \infty} |\alpha\rangle_S = \underbrace{\lim_{S \to \infty} \left(1 + \frac{|\alpha|^2}{2S}\right)^{-S}}_{\rightarrow \exp(-|\alpha|^2/2)} e^{\alpha \mathbf{a}^\dagger} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \mathbf{a}^\dagger} |0\rangle$$

Except a factor $\pi^{-1/2}$, this result is equal to the **coherent states of** a harmonic oscillator

Important expectation values I

Coherent states for spins

Felix Engel

Motivatio

CS for harmo

CS for spins

Analogue spin states

Alternative parametrizatio Using group SU(2)

Summan

The coherent spin states are:

$$|\mu\rangle = (1+|\mu|^2)^{-S} \sum_{p=0}^{2S} \left(\frac{(2S)!}{(2S-p)!p!}\right)^{1/2} \mu^p |p\rangle$$

Remember state $|p\rangle$ is defined as:

$$\hat{S}_{z}|p\rangle \equiv (S-p)|p\rangle, \quad 0 \leq p \leq 2S$$

Therefore we define an operator \hat{p} :

$$\hat{p} \equiv S - \hat{S}_z \quad \Rightarrow \quad \hat{p} |p\rangle = p |p\rangle$$

Expectation value of the operator \hat{p} in the coherent spin states is:

$$\langle \mu | \hat{\rho} | \mu \rangle = (1 + |\mu|^2)^{-2S} \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} |\mu|^{2p} \cdot \mu^{2p}$$

Important expectation values I

Coherent states for spins

Felix Engel

Motivatio

CS for harmo

CS for spins

Analogue spin states

Alternative parametrizatio Using group SU(2)

Summan

The coherent spin states are:

$$|\mu\rangle = (1+|\mu|^2)^{-S} \sum_{p=0}^{2S} \left(\frac{(2S)!}{(2S-p)!p!}\right)^{1/2} \mu^p |p\rangle$$

Remember state $|p\rangle$ is defined as:

$$\hat{S}_z |p\rangle \equiv (S-p)|p\rangle, \quad 0 \le p \le 2S$$

Therefore we define an operator \hat{p} :

$$\hat{p} \equiv S - \hat{S}_z \quad \Rightarrow \quad \hat{p} \ket{p} = p \ket{p}$$

Expectation value of the operator \hat{p} in the coherent spin states is:

$$\langle \mu | \hat{p} | \mu \rangle = (1 + |\mu|^2)^{-2S} \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} |\mu|^{2p} \cdot p$$

Important expectation values II

Coherent states for spins

Felix Engel

Analogue spin

states

Since, we can write the sum as

$$\begin{split} &\sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} \left(|\mu|^2 \right)^p \cdot p = |\mu|^2 \frac{\partial}{\partial (|\mu|^2)} (1+|\mu|^2)^{2S} \\ &= \sum_{p=0}^{2S} \binom{2S}{p} |\mu|^2 \frac{\partial}{\partial (|\mu|^2)} \left(|\mu|^2 \right)^p = \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} \left(|\mu|^2 \right)^p \cdot p \end{split}$$

$$\langle \mu | \hat{\rho} | \mu \rangle = (1 + |\mu|^2)^{-25} |\mu|^2 \frac{\partial}{\partial (|\mu|^2)} (1 + |\mu|^2)^{25}$$

$$= \frac{2S|\mu|^2}{1 + |\mu|^2}$$

Important expectation values II

Coherent states for spins

Felix Engel

Motivation

INIOCIVACIOII

CS for spins

Analogue spin

Alternative parametrization Using group

Summary

Since, we can write the sum as

$$\sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} (|\mu|^2)^p \cdot p = |\mu|^2 \frac{\partial}{\partial (|\mu|^2)} (1+|\mu|^2)^{2S}$$

$$= \sum_{p=0}^{2S} {2S \choose p} |\mu|^2 \frac{\partial}{\partial (|\mu|^2)} (|\mu|^2)^p = \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} (|\mu|^2)^p \cdot p$$

we can easily determine the expectation value

$$\langle \mu | \hat{p} | \mu \rangle = (1 + |\mu|^2)^{-2S} |\mu|^2 \frac{\partial}{\partial (|\mu|^2)} (1 + |\mu|^2)^{2S}$$

$$= \frac{2S|\mu|^2}{1 + |\mu|^2}$$

Important expectation values III

Coherent states for spins

Felix Engel

CS for harmo

CS for spins

Analogue spin

Alternative parametrizatio Using group SU(2)

Summary

Same way to calculate the following expectation values:

$$\langle \mu | S_+ | \mu \rangle = (1 + |\mu|^2)^{-2S} \frac{\partial}{\partial \mu^*} (1 + |\mu|^2)^{2S} = \frac{2S\mu}{1 + |\mu|^2}$$

$$\langle \mu | S_{-} | \mu \rangle = (1 + |\mu|^2)^{-2S} \frac{\partial}{\partial \mu} (1 + |\mu|^2)^{2S} = \frac{2S\mu^*}{1 + |\mu|^2}$$

$$\langle \nu | \hat{\rho} | \mu \rangle = (1 + |\nu|^2)^{-S} (1 + |\mu|^2)^{-S} \mu \frac{\partial}{\partial \mu} (1 + \nu^* \mu)^{2S} = \frac{2S\nu^* \mu}{1 + \nu^* \mu} \langle \nu | \mu \rangle$$

$$\langle \nu | S_+ | \mu \rangle = (1 + |\nu|^2)^{-S} (1 + |\mu|^2)^{-S} \frac{\partial}{\partial \nu^*} (1 + \nu^* \mu)^{2S} = \frac{2S\mu}{1 + \nu^* \mu} \langle \nu | \mu \rangle$$

$$\langle \nu | S_- | \mu \rangle = (1 + |\nu|^2)^{-S} (1 + |\mu|^2)^{-S} \frac{\partial}{\partial \mu} (1 + \nu^* \mu)^{2S} = \frac{2S\nu^*}{1 + \nu^* \mu} \langle \nu | \mu$$

Important expectation values III

Coherent states for spins

Felix Engel

Motivati

CS for harmo

CS for spin

Analogue spin

Alternative parametrizatio Using group SU(2)

Summary

Same way to calculate the following expectation values:

$$\langle \mu | S_+ | \mu \rangle = (1 + |\mu|^2)^{-2S} \frac{\partial}{\partial \mu^*} (1 + |\mu|^2)^{2S} = \frac{2S\mu}{1 + |\mu|^2}$$

$$\langle \mu | S_{-} | \mu \rangle = (1 + |\mu|^2)^{-2S} \frac{\partial}{\partial \mu} (1 + |\mu|^2)^{2S} = \frac{2S\mu^*}{1 + |\mu|^2}$$

$$\langle \nu | \hat{\rho} | \mu \rangle = (1 + |\nu|^2)^{-S} (1 + |\mu|^2)^{-S} \mu \frac{\partial}{\partial \mu} (1 + \nu^* \mu)^{2S} = \frac{2S \nu^* \mu}{1 + \nu^* \mu} \langle \nu | \mu \rangle$$

$$\langle \nu | S_+ | \mu \rangle = (1 + |\nu|^2)^{-S} (1 + |\mu|^2)^{-S} \frac{\partial}{\partial \nu^*} (1 + \nu^* \mu)^{2S} = \frac{2S\mu}{1 + \nu^* \mu} \langle \nu | \mu \rangle$$

$$\langle \nu | S_- | \mu \rangle = (1 + |\nu|^2)^{-S} (1 + |\mu|^2)^{-S} \frac{\partial}{\partial \mu} (1 + \nu^* \mu)^{2S} = \frac{2S\nu^*}{1 + \nu^* \mu} \langle \nu | \mu \rangle$$

Important expectation values III

Coherent states for spins

Felix Engel

. . . .

CC 5---:--

Analogue spin states

Alternative parametrizatio Using group SU(2)

Summary

Same way to calculate the following expectation values:

$$\langle \mu | S_+ | \mu \rangle = (1 + |\mu|^2)^{-2S} \frac{\partial}{\partial \mu^*} (1 + |\mu|^2)^{2S} = \frac{2S\mu}{1 + |\mu|^2}$$

$$\langle \mu | S_{-} | \mu \rangle = (1 + |\mu|^2)^{-2S} \frac{\partial}{\partial \mu} (1 + |\mu|^2)^{2S} = \frac{2S\mu^*}{1 + |\mu|^2}$$

$$\langle \nu | \hat{p} | \mu \rangle = (1 + |\nu|^2)^{-S} (1 + |\mu|^2)^{-S} \mu \frac{\partial}{\partial \mu} (1 + \nu^* \mu)^{2S} = \frac{2S \nu^* \mu}{1 + \nu^* \mu} \langle \nu | \mu \rangle$$

$$\langle
u | \mathcal{S}_{+} | \mu
angle = (1 + |
u|^{2})^{-S} (1 + |
u|^{2})^{-S} \frac{\partial}{\partial
u^{*}} (1 +
u^{*} \mu)^{2S} = \frac{2S\mu}{1 +
u^{*} \mu} \langle
u | \mu
angle$$

$$\langle
u | S_{-} | \mu
angle = (1 + |
u|^2)^{-S} (1 + |
u|^2)^{-S} \frac{\partial}{\partial \mu} (1 +
u^* \mu)^{2S} = \frac{2S
u^*}{1 +
u^* \mu} \langle
u | \mu
angle$$

Coherent states for spins

Felix Engel

Motivation

CS for harmon

CS for spin

Analogue spin

Alternative parametrization

SU(2)

Summary

 In analogy to the coherent states of the harmonic oscillator we construct coherent spin states

$$|\mu\rangle = (1 + |\mu|^2)^{-S} \cdot e^{\mu S_-} |0\rangle$$

• We find a state which minimizes the uncertainty relation

$$\left. \left< \Delta S_{\mathsf{x}} \right>_{\left| 0 \right>} \left< \Delta S_{\mathsf{y}} \right>_{\left| 0 \right>} \geq rac{1}{2} \left| \left< S_{\mathsf{z}} \right>_{\left| 0 \right>}
ight| \quad \Rightarrow \quad \left| 0 \right> = \left| S, \pm S \right>$$

$$\mathbb{1} = \sum_{p=0}^{2S} |p\rangle \langle p| = \frac{2S+1}{\pi} \int_{\mathbb{C}} \mathrm{d}\mu |\mu\rangle \langle \mu| \cdot \frac{1}{(1+|\mu|^2)^2}$$

- As expected the high spin limit of the coherent spin states leads to the coherent states of a harmonic oscillator
- So far we are able to calculate important expectation values in the basis of the constructed coherent spin states

Coherent states for spins

Felix Engel

Motivation

CS for harmon

CS for spin

Analogue spin

Alternative parametrization

SU(2)

Summary

 In analogy to the coherent states of the harmonic oscillator we construct coherent spin states

$$|\mu\rangle = (1+|\mu|^2)^{-S} \cdot \mathrm{e}^{\mu S_-} |0\rangle$$

We find a state which minimizes the uncertainty relation

$$\left. \left< \Delta S_x \right>_{\ket{0}} \left< \Delta S_y \right>_{\ket{0}} \ge rac{1}{2} \left| \left< S_z \right>_{\ket{0}} \right| \quad \Rightarrow \quad \ket{0} = \ket{S, \pm S}$$

$$\mathbb{1} = \sum_{p=0}^{2S} \left| p \right\rangle \left\langle p \right| = \frac{2S+1}{\pi} \int_{\mathbb{C}} \mathrm{d}\mu \left| \mu \right\rangle \left\langle \mu \right| \cdot \frac{1}{(1+|\mu|^2)^2}$$

- As expected the high spin limit of the coherent spin states leads to the coherent states of a harmonic oscillator
- So far we are able to calculate important expectation values in the basis of the constructed coherent spin states

Coherent states for spins

Felix Engel

Analogue spin states

• In analogy to the coherent states of the harmonic oscillator we construct coherent spin states

$$|\mu\rangle = (1+|\mu|^2)^{-S} \cdot \mathrm{e}^{\mu S_-} |0\rangle$$

• We find a state which **minimizes** the uncertainty relation

$$\left\langle \Delta S_{x}\right
angle _{\left|0\right\rangle }\left\langle \Delta S_{y}
ight
angle _{\left|0\right\rangle }\geq rac{1}{2}\left|\left\langle S_{z}
ight
angle _{\left|0\right\rangle }\right|\quad \Rightarrow\quad \left|0\right\rangle =\left|S,\pm S\right\rangle$$

• By further calculations we obtain the completeness relation of

$$\mathbb{1} = \sum_{p=0}^{2S} \left| p \right\rangle \left\langle p \right| = \frac{2S+1}{\pi} \int_{\mathbb{C}} \mathrm{d}\mu \left| \mu \right\rangle \left\langle \mu \right| \cdot \frac{1}{(1+|\mu|^2)^2}$$

- As expected the high spin limit of the coherent spin states
- So far we are able to calculate important expectation values

Coherent states for spins

Felix Engel

Motivati

CS for harmon

CS for spins

Analogue spin states

Alternative parametrization Using group

50(2)

 In analogy to the coherent states of the harmonic oscillator we construct coherent spin states

$$|\mu\rangle = (1+|\mu|^2)^{-S} \cdot e^{\mu S_-} |0\rangle$$

• We find a state which **minimizes** the uncertainty relation

$$\left\langle \Delta S_{x}\right
angle _{\left|0\right\rangle }\left\langle \Delta S_{y}
ight
angle _{\left|0\right\rangle }\geq rac{1}{2}\left|\left\langle S_{z}
ight
angle _{\left|0\right\rangle }\right|\quad \Rightarrow\quad \left|0\right\rangle =\left|S,\pm S\right\rangle$$

$$\mathbb{1} = \sum_{p=0}^{2S} \ket{p} \bra{p} = \frac{2S+1}{\pi} \int_{\mathbb{C}} d\mu \ket{\mu} \bra{\mu} \cdot \frac{1}{(1+|\mu|^2)^2}$$

- As expected the high spin limit of the coherent spin states leads to the coherent states of a harmonic oscillator
- So far we are able to calculate important expectation values in the basis of the constructed coherent spin states

Coherent states for spins

Felix Engel

Motivation

CS for harmo

CS for spins

Analogue spin states

Alternative parametrization Using group

Summary

 In analogy to the coherent states of the harmonic oscillator we construct coherent spin states

$$|\mu\rangle = (1+|\mu|^2)^{-S} \cdot e^{\mu S_-} |0\rangle$$

• We find a state which **minimizes** the uncertainty relation

$$\left\langle \Delta S_{x}\right
angle _{\left|0\right\rangle }\left\langle \Delta S_{y}
ight
angle _{\left|0\right\rangle }\geq rac{1}{2}\left|\left\langle S_{z}
ight
angle _{\left|0\right\rangle }\right|\quad \Rightarrow\quad \left|0\right\rangle =\left|S,\pm S\right\rangle$$

$$\mathbb{1} = \sum_{p=0}^{2S} \ket{p}\bra{p} = \frac{2S+1}{\pi} \int_{\mathbb{C}} \mathrm{d}\mu \ket{\mu}\bra{\mu} \cdot \frac{1}{(1+|\mu|^2)^2}$$

- As expected the high spin limit of the coherent spin states leads to the coherent states of a harmonic oscillator
- So far we are able to calculate important expectation values in the basis of the constructed coherent spin states

Coherent states for spins

Felix Engel

Motivatio

CS for harmon

CS for spins
Analogue spin

Alternative parametrization Using group

SU(2)

 In analogy to the coherent states of the harmonic oscillator we construct coherent spin states

$$|\mu\rangle = (1+|\mu|^2)^{-S} \cdot e^{\mu S_-} |0\rangle$$

• We find a state which **minimizes** the uncertainty relation

$$\left\langle \Delta S_{x}\right\rangle _{\left|0\right\rangle }\left\langle \Delta S_{y}\right\rangle _{\left|0\right\rangle }\geq \frac{1}{2}\left|\left\langle S_{z}\right\rangle _{\left|0\right\rangle }\right|\quad \Rightarrow\quad \left|0\right\rangle =\left|S,\pm S\right\rangle$$

$$\mathbb{1} = \sum_{p=0}^{2S} \left| p \right\rangle \left\langle p \right| = \frac{2S+1}{\pi} \int_{\mathbb{C}} \mathrm{d}\mu \left| \mu \right\rangle \left\langle \mu \right| \cdot \frac{1}{(1+|\mu|^2)^2}$$

- As expected the high spin limit of the coherent spin states leads to the coherent states of a harmonic oscillator
- So far we are able to calculate important expectation values in the basis of the constructed coherent spin states

Coherent states for spins

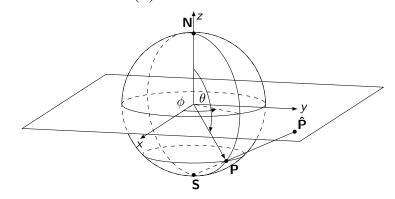
Felix Engel

Analogue spin

Alternative parametrization

An alternative parametrization of μ is the **stereographic projection**

$$\mu \equiv an\left(rac{ heta}{2}
ight) \mathrm{e}^{\mathrm{i}\phi} \quad 0 \leq heta < \pi \quad 0 \leq \phi < 2\pi$$



A coherent state $|\mu\rangle$ can be **associated** with a point on the **unit** sphere

Coherent states for spins

Felix Engel

Motivati

CS for harmo

oscillator

Analogue spin states
Alternative parametrization

Using group SU(2)

Summary

With respect to previous calculations we get the **normalized** coherent spin states

$$| heta,\phi
angle\equiv|\Omega
angle=\left(\cosrac{ heta}{2}
ight)^{2S}{
m e}^{\left(anrac{ heta}{2}{
m e}^{{
m i}\phi}S_{-}
ight)}|0
angle$$

with the completeness relation

$$1 = rac{2S+1}{4\pi}\int \mathrm{d}\phi\,\mathrm{d}\theta\sin heta\,| heta,\phi
angle\,\langle heta,\phi| = (2S+1)\intrac{\mathrm{d}\Omega}{4\pi}\,|\Omega
angle\,\langle\Omega|$$

the **overlap integral** of the states $|\Omega\rangle$, $|\Omega'\rangle$

$$\langle \Omega' | \Omega \rangle = \left[\cos \frac{1}{2} \theta \cos \frac{1}{2} \theta' + \sin \frac{1}{2} \theta \sin \frac{1}{2} \theta' e^{i(\phi - \phi')} \right]^{2S}$$

Further calculations give us

$$|\left\langle \Omega' | \Omega \right\rangle| = \left\lceil \frac{1 + \mathbf{n} \cdot \mathbf{n'}}{2} \right\rceil^S \qquad \mathbf{n} \equiv \left(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta \right)$$

where $\mathbf{n}, \mathbf{n'}$ are unit vectors defined by angles $(\theta, \phi), (\theta', \phi')$

Coherent states for spins

Felix Engel

Analogue spin

Alternative parametrization

With respect to previous calculations we get the **normalized** coherent spin states

$$| heta,\phi
angle\equiv|\Omega
angle=\left(\cosrac{ heta}{2}
ight)^{2S}{
m e}^{\left(anrac{ heta}{2}{
m e}^{{
m i}\phi}S_{-}
ight)}|0
angle$$

with the completeness relation

$$\mathbb{1} = \frac{2S+1}{4\pi} \int \mathsf{d}\phi \, \mathsf{d}\theta \sin\theta \, |\theta,\phi\rangle \, \langle \theta,\phi| = (2S+1) \int \frac{\mathsf{d}\Omega}{4\pi} \, |\Omega\rangle \, \langle \Omega|$$

$$\langle \Omega' | \Omega \rangle = \left[\cos \frac{1}{2} \theta \cos \frac{1}{2} \theta' + \sin \frac{1}{2} \theta \sin \frac{1}{2} \theta' e^{i(\phi - \phi')} \right]^{25}$$

$$|\left\langle \Omega' | \Omega \right\rangle| = \left\lceil \frac{1 + \mathbf{n} \cdot \mathbf{n'}}{2} \right\rceil^S \qquad \mathbf{n} \equiv \left(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta\right)$$

where n, n' are unit vectors defined by angles (θ, ϕ) , (θ', ϕ')

Coherent states for spins

Felix Engel

Motivation

CS for harmo

oscillator

CS for spins Analogue spin

Alternative parametrization Using group

Summary

With respect to previous calculations we get the **normalized coherent spin states**

$$| heta,\phi
angle\equiv|\Omega
angle=\left(\cosrac{ heta}{2}
ight)^{2S}{
m e}^{\left(anrac{ heta}{2}{
m e}^{{
m i}\phi}S_{-}
ight)}|0
angle$$

with the completeness relation

$$\mathbb{1} = \frac{2S+1}{4\pi} \int \mathsf{d}\phi \, \mathsf{d}\theta \sin\theta \, |\theta,\phi\rangle \, \langle \theta,\phi| = (2S+1) \int \frac{\mathsf{d}\Omega}{4\pi} \, |\Omega\rangle \, \langle \Omega|$$

the **overlap integral** of the states $|\Omega\rangle$, $|\Omega'\rangle$

$$\langle \Omega' | \Omega \rangle = \left[\cos \tfrac{1}{2} \theta \cos \tfrac{1}{2} \theta' + \sin \tfrac{1}{2} \theta \sin \tfrac{1}{2} \theta' \mathrm{e}^{\mathrm{i} (\phi - \phi')} \right]^{2S}$$

Further calculations give us

$$|\left\langle \Omega' | \Omega \right\rangle| = \left\lceil \frac{1 + \mathbf{n} \cdot \mathbf{n'}}{2} \right\rceil^S \qquad \mathbf{n} \equiv \left(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta \right)$$

where n, n' are unit vectors defined by angles $(\theta, \phi), (\theta', \phi')$

Coherent states for spins

Felix Engel

Motivat

CS for harm

oscillator

CS for

Analogue spin states
Alternative parametrization

Using group SU(2)

Summary

With respect to previous calculations we get the **normalized coherent spin states**

$$| heta,\phi
angle\equiv|\Omega
angle=\left(\cosrac{ heta}{2}
ight)^{2S}{
m e}^{\left(anrac{ heta}{2}{
m e}^{{
m i}\phi}S_{-}
ight)}|0
angle$$

with the completeness relation

$$\mathbb{1} = \frac{2S+1}{4\pi} \int \mathsf{d}\phi \, \mathsf{d}\theta \sin\theta \, |\theta,\phi\rangle \, \langle \theta,\phi| = (2S+1) \int \frac{\mathsf{d}\Omega}{4\pi} \, |\Omega\rangle \, \langle \Omega|$$

the **overlap integral** of the states $|\Omega\rangle$, $|\Omega'\rangle$

$$\langle \Omega' | \Omega \rangle = \left[\cos \tfrac{1}{2} \theta \cos \tfrac{1}{2} \theta' + \sin \tfrac{1}{2} \theta \sin \tfrac{1}{2} \theta' \mathrm{e}^{\mathrm{i}(\phi - \phi')} \right]^{2S}$$

Further calculations give us

$$|\left\langle \Omega' \middle| \Omega \right\rangle| = \left\lceil \frac{1 + \mathbf{n} \cdot \mathbf{n'}}{2} \right\rceil^S \qquad \mathbf{n} \equiv \left(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta \right)$$

where $\mathbf{n}, \mathbf{n'}$ are unit vectors defined by angles $(\theta, \phi), (\theta', \phi')$

Coherent states for spins

Felix Engel

Motivation

MOLIVALION

CC C 1

CS for spins
Analogue spin

Alternative parametrization

Using group SU(2)

Summary

The expectation value of the spin components

By using previous results we get:

$$\langle \Omega | \hat{\pmb{p}} | \Omega
angle = rac{2S \left(an rac{ heta}{2}
ight)^2}{1 + \left(an rac{ heta}{2}
ight)^2} = S \left(1 - \cos heta
ight)$$

$$\langle \Omega | S_+ | \Omega \rangle = S \sin \theta \mathrm{e}^{\mathrm{i}\phi}$$
 $\langle \Omega | S_- | \Omega \rangle = S \sin \theta \mathrm{e}^{-\mathrm{i}\phi}$

Thus, the expectation value is given by:

$$\langle \Omega | \mathbf{S} | \Omega \rangle = S \mathbf{n}$$
 $\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{pmatrix}$

The expectation value of the spin operator in the coherent spin states corresponds to a vector moving on a sphere of radius S

Coherent states for spins

Felix Engel

Motivation

IVIOTIVATIO

CS for spins
Analogue spin

Analogue spin states
Alternative parametrization
Using group

SU(2)

Summary

The expectation value of the spin components

By using previous results we get:

$$\langle \Omega | \hat{p} | \Omega
angle = rac{2S \left(an rac{ heta}{2}
ight)^2}{1 + \left(an rac{ heta}{2}
ight)^2} = S \left(1 - \cos heta
ight)$$

$$\langle \Omega | S_+ | \Omega \rangle = S \sin \theta \mathrm{e}^{\mathrm{i} \phi} \qquad \qquad \langle \Omega | S_- | \Omega \rangle = S \sin \theta \mathrm{e}^{-\mathrm{i} \phi}$$

Thus, the expectation value is given by:

$$\langle \Omega | \mathbf{S} | \Omega \rangle = S \mathbf{n}$$
 $\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{pmatrix}$

The expectation value of the **spin operator** in the **coherent spin states** corresponds to a **vector moving on a sphere of radius** *S*

Coherent states for spins

Felix Engel

Analogue spin

Using group SU(2)

The elements g of the group SU(2) are 2×2 unitary matrices with determinant 1:

$$g \in \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \right\}, \quad \text{with } |\alpha|^2 + |\beta|^2 = 1$$

$$h \in \left\{ \begin{pmatrix} lpha & 0 \\ 0 & lpha^* \end{pmatrix}
ight\}, \quad ext{with } |lpha|^2 = 1 \ \Rightarrow \ lpha = \mathrm{e}^{\mathrm{i} arphi}$$

generators σ_i by using **exponential mapping**:

$$g=\operatorname{\mathsf{exp}}\operatorname{\mathsf{i}}\sum_{i=1}^3 heta_j\sigma_j$$
 with $heta_j\in\mathbb{R}$

Coherent states for spins

Felix Engel

Analogue spin

Using group SU(2)

The elements g of the group SU(2) are 2×2 unitary matrices with determinant 1:

$$g \in \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \right\}, \quad \text{with } |\alpha|^2 + |\beta|^2 = 1$$

We define a subgroup $H \subset SU(2)$ of diagonal matrices:

$$h \in \left\{ egin{pmatrix} lpha & 0 \ 0 & lpha^* \end{pmatrix}
ight\}, \quad ext{with } |lpha|^2 = 1 \ \Rightarrow \ lpha = \mathrm{e}^{\mathrm{i}arphi}$$

$$g=\operatorname{\mathsf{exp}}\operatorname{\mathsf{i}}\sum_{j=1}^3 heta_j\sigma_j\quad ext{with } heta_j\in\mathbb{I}$$

Coherent states for spins

The elements g of the group SU(2) are 2×2 unitary matrices with determinant 1:

$$g \in \left\{ egin{pmatrix} lpha & eta \ -eta^* & lpha^* \end{pmatrix}
ight\}, \quad ext{with } |lpha|^2 + |eta|^2 = 1$$

We define a subgroup $H \subset SU(2)$ of diagonal matrices:

$$h \in \left\{ egin{pmatrix} lpha & 0 \ 0 & lpha^* \end{pmatrix}
ight\}, \quad ext{with } |lpha|^2 = 1 \ \Rightarrow \ lpha = \mathrm{e}^{\mathrm{i}arphi}$$

We can write an element g of the group SU(2) with the so called generators σ_i by using exponential mapping:

$$g = \exp \mathsf{i} \sum_{i=1}^3 \theta_j \sigma_j \quad \mathsf{with} \ \theta_j \in \mathbb{R}$$

The generators of the group SU(2) are the well known **pauli matrices**

Felix Engel

Analogue spin Using group

SU(2)

Coherent states for spins

Felix Engel

Motivation

CS for harmo

CS for spin

Analogue spin states

Alternative parametrization
Using group

SU(2)

Since, the operator S_z acting on a state $|S, m\rangle \in \mathsf{Hilbert}$ space does **not generate a new state** we define:

$$G/H = X = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix} \right\}, \text{ with } \alpha \in \mathbb{R}$$

The determinant of a element of X is 1:

$$\alpha^2 + \beta_1^2 + \beta_2^2 \stackrel{!}{=} 1$$
, where $\beta \equiv \beta_1 + i\beta_2$

 \Rightarrow every element of X can be **associated** with a point on a sphere S^2

We use X to construct the coherent states with generators σ_x , σ_y

$$g_X = \exp\left[\mathrm{i}\theta\left(x\frac{\sigma_X}{2} + y\frac{\sigma_Y}{2}\right)\right], \quad \text{with } g_X \in X$$

With the spherical angles θ and ϕ we can use the **parametrization**

$$x = \sin \phi$$
 $y = -\cos \phi$ $0 \le \theta \le \pi$ $0 \le \phi \le 2\pi$

Coherent states for spins

Felix Engel

Motivatio

IVIOLIVALIOII

CS for harm

CS for spir

Analogue spin states Alternative parametrization Using group

SU(2)

Since, the operator S_z acting on a state $|S, m\rangle \in \mathsf{Hilbert}$ space does **not generate a new state** we define:

$$G/H = X = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix} \right\}, \text{ with } \alpha \in \mathbb{R}$$

The determinant of a element of X is 1:

$$\alpha^2 + \beta_1^2 + \beta_2^2 \stackrel{!}{=} 1$$
, where $\beta \equiv \beta_1 + i\beta_2$

 \Rightarrow every element of X can be **associated** with a point on a sphere S^2

We use X to construct the coherent states with generators σ_x , σ_y

$$g_X = \exp\left[i\theta\left(x\frac{\sigma_x}{2} + y\frac{\sigma_y}{2}\right)\right], \quad \text{with } g_X \in X$$

With the spherical angles θ and ϕ we can use the **parametrization**:

$$x = \sin \phi$$
 $y = -\cos \phi$ $0 \le \theta \le \pi$ $0 \le \phi \le 2\pi$

Coherent states for spins

Felix Engel

Motivatio

COCCU

CS for spin

Analogue spin states Alternative parametrization Using group

SU(2)

Since, the operator S_z acting on a state $|S, m\rangle \in \text{Hilbert space does}$ not generate a new state we define:

$$G/H = X = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix} \right\}, \quad \text{with } \alpha \in \mathbb{R}$$

The determinant of a element of X is 1:

$$\alpha^2 + \beta_1^2 + \beta_2^2 \stackrel{!}{=} 1$$
, where $\beta \equiv \beta_1 + i\beta_2$

 \Rightarrow every element of X can be **associated** with a point on a sphere S^2

We use X to **construct** the coherent states with **generators** σ_x , σ_y

$$g_X = \exp\left[i\theta\left(x\frac{\sigma_x}{2} + y\frac{\sigma_y}{2}\right)\right], \quad \text{with } g_X \in X$$

With the spherical angles θ and ϕ we can use the **parametrization**:

$$x = \sin \phi$$
 $y = -\cos \phi$ $0 \le \theta \le \pi$ $0 \le \phi \le 2\pi$

Coherent states for spins

Felix Engel

Motivati

CC C .

CS for spins

Analogue spin states Alternative parametrization Using group

SU(2)

Since, the operator S_z acting on a state $|S, m\rangle \in \mathsf{Hilbert}$ space does **not generate a new state** we define:

$$G/H = X = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix} \right\}, \quad \text{with } \alpha \in \mathbb{R}$$

The determinant of a element of X is 1:

$$\alpha^2 + \beta_1^2 + \beta_2^2 \stackrel{!}{=} 1$$
, where $\beta \equiv \beta_1 + i\beta_2$

 \Rightarrow every element of X can be **associated** with a point on a sphere S^2

We use X to **construct** the coherent states with **generators** σ_x , σ_y

$$g_X = \exp\left[i\theta\left(x\frac{\sigma_X}{2} + y\frac{\sigma_y}{2}\right)\right], \quad \text{with } g_X \in X$$

With the spherical angles θ and ϕ we can use the **parametrization**:

$$x = \sin \phi$$
 $y = -\cos \phi$ $0 \le \theta \le \pi$ $0 \le \phi \le 2\pi$

Coherent states for spins

Felix Engel

NAMES OF STREET

INIOEINATIOI

CS for spins

Analogue spin states
Alternative

Using group SU(2)

Summary

Therefore we get:

$$\begin{split} g_X &= \exp\left[\mathrm{i}\theta\left(\sin\phi\frac{\sigma_x}{2} - \cos\phi\frac{\sigma_y}{2}\right)\right] \\ &= \exp\left(\xi S_+ - \xi^* S_-\right), \quad \text{where } \xi \equiv -\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\phi} \end{split}$$

With $|0\rangle \equiv |S,S\rangle$ we obtain the **coherent spin states**

$$|\theta,\phi\rangle = g_X |0\rangle = e^{i\theta(x\cdot s)} |0\rangle$$
 $s = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$ $x = \begin{pmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{pmatrix}$

Proof of **equality** of both constructions of the coherent spin state

$$\exp\left(\xi S_{+}-\xi^{*}S_{-}
ight)=\mathrm{e}^{\zeta'S_{-}}\mathrm{e}^{-\eta S_{z}}\mathrm{e}^{\zeta S_{-}}$$

$$\zeta = - anrac{ heta}{2} \mathrm{e}^{-\mathrm{i}\phi} \quad \eta = -2\ln\left(\cosrac{ heta}{2}
ight) = \ln\left(1+\left|\zeta
ight|^2
ight) \quad \zeta' = -\zeta^*$$

Coherent states for spins

Felix Engel

NA. 12

iviotivatioi

oscillator

CS for spins
Analogue spin

states
Alternative

Using group SU(2)

SU(2)

Summary

Therefore we get:

$$\begin{split} g_X &= \exp\left[\mathrm{i}\theta\left(\sin\phi\frac{\sigma_x}{2} - \cos\phi\frac{\sigma_y}{2}\right)\right] \\ &= \exp\left(\xi S_+ - \xi^* S_-\right), \quad \text{where } \xi \equiv -\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\phi} \end{split}$$

With $|0\rangle \equiv |S,S\rangle$ we obtain the **coherent spin states**

$$|\theta,\phi\rangle = g_X |0\rangle = e^{i\theta(\mathbf{x}\cdot\mathbf{S})} |0\rangle$$
 $\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$ $\mathbf{x} = \begin{pmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{pmatrix}$

Proof of **equality** of both constructions of the coherent spin states

$$\exp(\xi S_{+} - \xi^{*} S_{-}) = e^{\zeta' S_{-}} e^{-\eta S_{z}} e^{\zeta S}$$

$$\zeta = -\tan \tfrac{\theta}{2} \mathrm{e}^{-\mathrm{i}\phi} \quad \eta = -2\ln\left(\cos \tfrac{\theta}{2}\right) = \ln\left(1 + |\zeta|^2\right) \quad \zeta' = -\zeta^*$$

Coherent states for spins

Felix Engel

NAMES OF STREET

iviotivatioi

CS for spins

Analogue spin states
Alternative

Using group SU(2)

SU(2)

Therefore we get:

$$\begin{split} g_X &= \exp\left[\mathrm{i}\theta\left(\sin\phi\frac{\sigma_X}{2} - \cos\phi\frac{\sigma_y}{2}\right)\right] \\ &= \exp\left(\xi S_+ - \xi^* S_-\right), \quad \text{where } \xi \equiv -\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\phi} \end{split}$$

With $|0\rangle \equiv |S,S\rangle$ we obtain the **coherent spin states**

$$|\theta,\phi\rangle = g_X |0\rangle = e^{i\theta(\mathbf{x}\cdot\mathbf{S})} |0\rangle$$
 $\mathbf{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$ $\mathbf{x} = \begin{pmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{pmatrix}$

Proof of **equality** of both constructions of the coherent spin states

$$\exp(\xi S_{+} - \xi^{*} S_{-}) = e^{\zeta' S_{-}} e^{-\eta S_{z}} e^{\zeta S_{+}}$$

$$\zeta = -\tan \tfrac{\theta}{2} \mathrm{e}^{-\mathrm{i}\phi} \quad \eta = -2\ln\left(\cos \tfrac{\theta}{2}\right) = \ln\left(1 + \left|\zeta\right|^2\right) \quad \zeta' = -\zeta^*$$

Coherent states for spins

Felix Engel

Motivatio

CS for harmo

CS for spins Analogue spin

states Alternative parametrization

Using group SU(2)

Summary

The effect of the representation g_X acting on the state $|S,S\rangle$

$$\exp\left[i\theta(\mathbf{x}\cdot\mathbf{S})\right]|S,S\rangle = e^{\zeta'S_{-}}e^{-\eta S_{z}}\underbrace{e^{\zeta S_{+}}|S,S\rangle}_{\mathbb{I}|S,S\rangle}$$

$$= e^{\zeta'S_{-}}\sum_{n=0}^{\infty}\frac{(-\eta)^{n}}{n!}\underbrace{(S_{z})^{n}|S,S\rangle}_{=(S)^{n}|S,S\rangle}$$

$$= \exp\left(-\eta S\right)\sum_{n=0}^{2S}\frac{(\zeta')^{n}}{n!}(S_{-})^{n}|S,S\rangle$$

$$= \left(\cos\frac{\theta}{2}\right)^{2S}\sum_{n=0}^{2S}(\zeta')^{n}\left[\frac{(2S)!}{n!(2S-n)!}\right]^{1/2}|S,S-n\rangle$$

$$= |O\rangle$$

Coherent states for spins

Felix Engel

Motivatio

CC C I

CC C

Analogue spin states
Alternative

Using group SU(2)

Summary

The effect of the representation g_X acting on the state $|S,S\rangle$

$$\exp\left[i\theta(\boldsymbol{x}\cdot\boldsymbol{S})\right]|S,S\rangle = e^{\zeta'S} - e^{-\eta S_z} \underbrace{e^{\zeta S_+}|S,S\rangle}_{\mathbf{1}|S,S\rangle}$$

$$= e^{\zeta'S} - \sum_{n=0}^{\infty} \frac{(-\eta)^n}{n!} \underbrace{(S_z)^n|S,S\rangle}_{=(S)^n|S,S\rangle}$$

$$= \exp\left(-\eta S\right) \sum_{n=0}^{2S} \frac{(\zeta')^n}{n!} (S_-)^n|S,S\rangle$$

$$= \left(\cos\frac{\theta}{2}\right)^{2S} \sum_{n=0}^{2S} (\zeta')^n \left[\frac{(2S)!}{n!(2S-n)!}\right]^{1/2} |S,S-n\rangle$$

$$= |O\rangle$$

Coherent states for spins

Felix Engel

Motivotic

CC C 1

CS for spins

Analogue spin

Alternative parametrization

Using group SU(2)

Summani

The effect of the representation g_X acting on the state $|S,S\rangle$

$$\exp\left[i\theta(\mathbf{x}\cdot\mathbf{S})\right]|S,S\rangle = e^{\zeta'S_{-}}e^{-\eta S_{z}}\underbrace{e^{\zeta S_{+}}|S,S\rangle}_{1|S,S\rangle}$$

$$= e^{\zeta'S_{-}}\sum_{n=0}^{\infty}\frac{(-\eta)^{n}}{n!}\underbrace{(S_{z})^{n}|S,S\rangle}_{=(S)^{n}|S,S\rangle}$$

$$= \exp\left(-\eta S\right)\sum_{n=0}^{2S}\frac{(\zeta')^{n}}{n!}(S_{-})^{n}|S,S\rangle$$

$$= \left(\cos\frac{\theta}{2}\right)^{2S}\sum_{n=0}^{2S}(\zeta')^{n}\left[\frac{(2S)!}{n!(2S-n)!}\right]^{1/2}|S,S-n\rangle$$

$$= |O\rangle$$

Coherent states for spins

Felix Engel

Motivotic

Motivation

. . .

CS for spins
Analogue spin

Alternative

Using group

SU(2)

Summary

The effect of the representation g_X acting on the state $|S,S\rangle$

$$\exp\left[i\theta(\mathbf{x}\cdot\mathbf{S})\right]|S,S\rangle = e^{\zeta'S_{-}}e^{-\eta S_{z}}\underbrace{e^{\zeta S_{+}}|S,S\rangle}_{\mathbb{1}|S,S\rangle}$$

$$= e^{\zeta'S_{-}}\sum_{n=0}^{\infty}\frac{(-\eta)^{n}}{n!}\underbrace{(S_{z})^{n}|S,S\rangle}_{=(S)^{n}|S,S\rangle}$$

$$= \exp\left(-\eta S\right)\sum_{n=0}^{2S}\frac{(\zeta')^{n}}{n!}(S_{-})^{n}|S,S\rangle$$

$$= \left(\cos\frac{\theta}{2}\right)^{2S}\sum_{n=0}^{2S}(\zeta')^{n}\left[\frac{(2S)!}{n!(2S-n)!}\right]^{1/2}|S,S-n\rangle$$

$$= |\Omega\rangle$$

Coherent states for spins

Felix Engel

Motivotic

IVIOTIVATION

CS for spins
Analogue spin

Alternative

Using group

SU(2)

Summary

The effect of the representation g_X acting on the state $|S,S\rangle$

$$\exp\left[i\theta(\mathbf{x}\cdot\mathbf{S})\right]|S,S\rangle = e^{\zeta'S_{-}}e^{-\eta S_{z}}\underbrace{e^{\zeta S_{+}}|S,S\rangle}_{1|S,S\rangle}$$

$$= e^{\zeta'S_{-}}\sum_{n=0}^{\infty}\frac{(-\eta)^{n}}{n!}\underbrace{(S_{z})^{n}|S,S\rangle}_{=(S)^{n}|S,S\rangle}$$

$$= \exp\left(-\eta S\right)\sum_{n=0}^{2S}\frac{(\zeta')^{n}}{n!}(S_{-})^{n}|S,S\rangle$$

$$= \left(\cos\frac{\theta}{2}\right)^{2S}\sum_{n=0}^{2S}(\zeta')^{n}\left[\frac{(2S)!}{n!(2S-n)!}\right]^{1/2}|S,S-n\rangle$$

$$= |\Omega\rangle$$

Coherent states for spins

Felix Engel

Motivotic

MOLIVALION

Analogue spin

Alternative

Using group

SU(2)

Summary

The effect of the representation g_X acting on the state $|S, S\rangle$

$$\exp\left[i\theta(\mathbf{x}\cdot\mathbf{S})\right]|S,S\rangle = e^{\zeta'S_{-}}e^{-\eta S_{z}}\underbrace{e^{\zeta S_{+}}|S,S\rangle}_{1|S,S\rangle}$$

$$= e^{\zeta'S_{-}}\sum_{n=0}^{\infty}\frac{(-\eta)^{n}}{n!}\underbrace{(S_{z})^{n}|S,S\rangle}_{=(S)^{n}|S,S\rangle}$$

$$= \exp\left(-\eta S\right)\sum_{n=0}^{2S}\frac{(\zeta')^{n}}{n!}(S_{-})^{n}|S,S\rangle$$

$$= \left(\cos\frac{\theta}{2}\right)^{2S}\sum_{n=0}^{2S}(\zeta')^{n}\left[\frac{(2S)!}{n!(2S-n)!}\right]^{1/2}|S,S-n\rangle$$

$$= |\Omega\rangle$$

Coherent states for spins

Felix Engel

Motivatio

Motivation

CS for spins
Analogue spin

Alternative

Using group

SU(2)

Summary

The effect of the representation g_X acting on the state $|S,S\rangle$

$$\begin{aligned} \exp\left[\mathrm{i}\theta(\mathbf{x}\cdot\mathbf{S})\right]|S,S\rangle &= \mathrm{e}^{\zeta'S_{-}}\mathrm{e}^{-\eta S_{z}}\underbrace{\mathrm{e}^{\zeta S_{+}}|S,S\rangle}_{1|S,S\rangle} \\ &= \mathrm{e}^{\zeta'S_{-}}\sum_{n=0}^{\infty}\frac{(-\eta)^{n}}{n!}\underbrace{(S_{z})^{n}|S,S\rangle}_{=(S)^{n}|S,S\rangle} \\ &= \exp\left(-\eta S\right)\sum_{n=0}^{2S}\frac{(\zeta')^{n}}{n!}(S_{-})^{n}|S,S\rangle \\ &= \left(\cos\frac{\theta}{2}\right)^{2S}\sum_{n=0}^{2S}(\zeta')^{n}\left[\frac{(2S)!}{n!(2S-n)!}\right]^{1/2}|S,S-n\rangle \\ &= |\Omega\rangle \end{aligned}$$

Coherent states for spins

Felix Engel

Motivation

CS for harmor

CS for spins
Analogue spin states
Alternative parametrization
Using group
SU(2)

Summary

- We construct the coherent spin states in analogy to the harmonic oscillator as well as based on the properties of the group SU(2)
- We have shown an alternativ parametrization by the stereographic projection
- We calculate some relevant expectation values and work out that the expectation value of the spin operator in the coherent spin states corresponds to a vector moving on a sphere with radius S

- 1 J. M. Radcliffe, Some properties of coherent spin states, J. Phys. A: Gen. Phys., 1971, Vol 4 Printed in Great Britain
- 2 A. Perelomov, Generalized Coherent States and Their Applications, Springer (1986)

Coherent states for spins

Felix Engel

Analogue spin

Summary

- We construct the coherent spin states in analogy to the harmonic oscillator as well as based on the properties of the group SU(2)
- We have shown an alternativ parametrization by the
- We calculate some relevant expectation values and work out that

Coherent states for spins

Felix Engel

Motivation

CS for harmo

CS for spin

Analogue spin states Alternative parametrization Using group SU(2)

Summary

- We construct the coherent spin states in analogy to the harmonic oscillator as well as based on the properties of the group SU(2)
- We have shown an alternativ parametrization by the stereographic projection
- We calculate some relevant expectation values and work out that the expectation value of the spin operator in the coherent spin states corresponds to a vector moving on a sphere with radius S

- 1 J. M. Radcliffe, Some properties of coherent spin states, J. Phys. A: Gen. Phys., 1971, Vol 4 Printed in Great Britain
- A. Perelomov, Generalized Coherent States and Their Applications, Springer (1986)

Coherent states for spins

Felix Engel

Motivation

CS for harmo

CS for spir

Analogue spin states Alternative parametrization Using group SU(2)

Summary

- We construct the coherent spin states in analogy to the harmonic oscillator as well as based on the properties of the group SU(2)
- We have shown an alternativ parametrization by the stereographic projection
- We calculate some relevant expectation values and work out that the expectation value of the spin operator in the coherent spin states corresponds to a vector moving on a sphere with radius S

- 1 J. M. Radcliffe, Some properties of coherent spin states, J. Phys. A: Gen. Phys., 1971, Vol 4 Printed in Great Britain
- A. Perelomov, Generalized Coherent States and Their Applications, Springer (1986)

Coherent states for spins

Felix Engel

Motivati

CS for harmo

CS for spins

Analogue spin states Alternative parametrization Using group SU(2)

Summary

- We construct the coherent spin states in analogy to the harmonic oscillator as well as based on the properties of the group SU(2)
- We have shown an alternativ parametrization by the stereographic projection
- We calculate some relevant expectation values and work out that the expectation value of the spin operator in the coherent spin states corresponds to a vector moving on a sphere with radius S

- 1 J. M. Radcliffe, Some properties of coherent spin states, J. Phys. A: Gen. Phys., 1971, Vol 4 Printed in Great Britain
- 2 A. Perelomov, Generalized Coherent States and Their Applications, Springer (1986)