Coherent states for spins

Seminar: Quantum field-theory of low dimensional systems

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Motivation

Why do we need coherent states for spins?

- We can describe magnetic systems in which we are interested in the degree of freedom of the spins

$$H = -\sum_{n,m} J(n,m)\boldsymbol{S}_n \cdot \boldsymbol{S}_m$$

- By using the coherent states of spins we are able to construct the path-integral for them
- We obtain a more "classical-like" picture of spins by using the coherent spin states

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Coherent states for a harmonic oscillator I

A coherent state (CS) $|\alpha\rangle$ is completely defined by $\alpha \in \mathbb{C}$

$$
\begin{aligned}
|\alpha\rangle &= \pi^{-1/2} e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
&= \pi^{-1/2} e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\left(\alpha a^\dagger\right)^n}{n!} |0\rangle \\
&= \pi^{-1/2} e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle
\end{aligned}
$$

- We create the state $|n\rangle$ **from the vacuum state** $|0\rangle$ by using the creation operator
- Therefore, we end up with the **creation operator in the exponential**

Coherent states for a harmonic oscillator II

Coherent state for a harmonic oscillator

$$|\alpha\rangle = \pi^{-1/2} e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle \,, \; \alpha \in \mathbb{C}$$

- These states build an overcomplete set

$$\int_{\mathbb{C}} d\alpha \, |\alpha\rangle \langle\alpha| = \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{1}$$

- As you have already seen we can calculate quantities like the partition function

$$Z = \int \prod_{\alpha} d\Phi_\alpha^* d\Phi_\alpha e^{-\sum_\alpha \Phi_\alpha^* \Phi_\alpha} \langle \Phi | e^{-\beta(\hat{H}-\mu\hat{N})} | \Phi \rangle$$

Spin algebra

Eigenstates for operators: $S_z, \boldsymbol{S}^2$ $(m = -S, -S+1, ..., S)$

$$S_z \left|S, m\right\rangle = m \left|S, m\right\rangle \qquad \boldsymbol{S}^2 \left|S, m\right\rangle = S(S+1) \left|S, m\right\rangle$$

Creation & annihilation operator: $S_\pm \equiv S_x \pm \mathrm{i} S_y$

$$S_+ \left|S, m\right\rangle = \sqrt{(S+m+1)(S-m)} \left|S, m+1\right\rangle$$
$$S_- \left|S, m\right\rangle = \sqrt{(S-m+1)(S+m)} \left|S, m-1\right\rangle$$

Commutation relations:

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k \qquad [S_z, S_\pm] = \pm S_\pm \qquad [S_+, S_-] = 2S_z$$

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Choice of the "vacuum state" $|0\rangle$ I

Minimize the uncertainty relation

$$\langle \Delta S_x \rangle_{\psi_0} \langle \Delta S_y \rangle_{\psi_0} \geq \frac{1}{2} \left| \langle S_z \rangle_{\psi_0} \right|$$

Consider the left side:

$$\begin{aligned}
\langle \Delta S_x \rangle_{|S,m\rangle} \langle \Delta S_y \rangle_{|S,m\rangle} &= \left[\left(\langle S_x^2 \rangle - \langle S_x \rangle^2 \right) \left(\langle S_y^2 \rangle - \langle S_y \rangle^2 \right) \right]^{\frac{1}{2}} \\
&= \frac{1}{4} \left[\langle (S_+ + S_-)^2 \rangle - (\langle S_+ + S_- \rangle)^2 \right]^{\frac{1}{2}} \\
&\quad \times \left[(\langle S_+ - S_- \rangle)^2 - \langle (S_+ - S_-)^2 \rangle \right]^{\frac{1}{2}} \\
&= \frac{1}{4} \langle S, m | S_+ S_- + S_- S_+ | S, m \rangle \\
&= \frac{1}{4} \left[(S - m + 1)(S + m) + (S + m + 1)(S - m) \right]
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$\Rightarrow$ minimum at: $m = \pm S$

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$$= \frac{1}{4} \left[\left\langle (S_+ + S_-)^2 \right\rangle - \left(\langle S_+ + S_- \rangle \right)^2 \right]^{\frac{1}{2}}$$

$$\times \left[\left(\langle S_+ - S_- \rangle \right)^2 - \left\langle (S_+ - S_-)^2 \right\rangle \right]^{\frac{1}{2}}$$

$$= \frac{1}{4} \langle S, m | S_+ S_- + S_- S_+ | S, m \rangle$$

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For reasons of simplicity we use the following definition

Definition

$$S_z\,|p\rangle \equiv (S-p)\,|p\rangle\,, \quad 0 \le p \le 2S$$

Now we **construct** the state $|p\rangle$ from **"vacuum state"** $|0\rangle \equiv |S,S\rangle$ by using the **annihilation operator** p times

$$\begin{aligned}(S_-)^p\,|0\rangle &= [(S-S+1)(S-(S-1)+1)\dots(S-(S-p+1)+1)]^{1/2}\\ &\quad\times[(S+S)(S+S-1)\dots(S+S-p+1)]^{1/2}\,|S,S-p\rangle\\ &= \left[\frac{p!(2S)!}{(2S-p)!}\right]^{1/2}|p\rangle\end{aligned}$$

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Consider single particle of spin S

Analogue to the CS of a harmonic oscillator

Coherent state HO (reminder)

$$|\alpha\rangle = \pi^{-1/2} e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle \, , \, \alpha \in \mathbb{C}$$

where we create the state $|\alpha\rangle$ using the creation operator $a^\dagger$ on the vacuum state $|0\rangle$

We **introduce** a CS $|\mu\rangle$ for a single particle of spin S by:

Definition of CS for spin S

$$|\mu\rangle = \frac{1}{\sqrt{N}} \cdot e^{\mu S_-} |0\rangle \, , \, \mu \in \mathbb{C}$$

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Calculation of normalization factor N

Series expansion of the exponential function

$$|\mu\rangle = \frac{e^{\mu S_-}}{\sqrt{N}}|0\rangle = \frac{1}{\sqrt{N}}\sum_{p=0}^{2S}\frac{\mu^p}{p!}(S_-)^p|0\rangle = \frac{1}{\sqrt{N}}\sum_{p=0}^{2S}\mu^p\sqrt{\frac{(2S)!}{(2S-p)!p!}}|p\rangle$$

States $|p\rangle$ build an orthogonal basis this leads us to:

$$\langle\mu|\mu\rangle = \frac{1}{N}\sum_{p,p'=0}^{2S}\left[\left(\frac{(2S)!}{(2S-p)!p!}\right)\left(\frac{(2S)!}{(2S-p')!p'!}\right)\right]^{1/2}\mu^p(\mu^*)^{p'}\delta_{p,p'}$$

$$= \frac{1}{N}\sum_{p=0}^{2S}\frac{(2S)!}{(2S-p)!p!}|\mu|^{2p} = \frac{1}{N}\sum_{p=0}^{2S}\binom{2S}{p}\left(|\mu|^2\right)^p$$

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Completeness relation of coherent spin states I

Coherent spin states $|\mu\rangle$ build an overcomplete set of states

$$\mathbb{1} = \sum_{p=0}^{2S} |p\rangle\langle p| = \int_{\mathbb{C}} \mathrm{d}\mu \, |\mu\rangle\langle\mu| \cdot w(|\mu|^2)$$

By using the substitution $\mu = \rho \cdot e^{i\phi}$ we obtain:

$$\int_{\mathbb{C}} \mathrm{d}\mu \, |\mu\rangle\langle\mu| \cdot w(|\mu|^2) = \sum_{p=0}^{2S} |p\rangle\langle p| \frac{(2S)!}{(2S-p)!p!} \underbrace{2\pi \int_0^\infty \frac{w(\rho^2)\rho^{2p+1}}{(1+\rho^2)^{2S}} \mathrm{d}\rho}_{\equiv \mathrm{I}(S,p)}$$

Therefore, we need a weight function $w(\rho^2)$ which fulfills the condition:

$$\mathrm{I}(S,p) = \frac{(2S-p)!p!}{(2S)!}$$

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A good choice seems to be $w(\rho^2) = \frac{2S+1}{\pi}\frac{1}{(1+\rho^2)^2}$

$$\mathrm{I}(S,p) = 2\pi\int_0^\infty \frac{w(\rho^2)\rho^{2p+1}}{(1+\rho^2)^{2S}}\mathrm{d}\rho \overset{[\rho^2=\sigma]}{=} \pi\int_0^\infty \frac{w(\sigma)\sigma^p}{(1+\sigma)^{2S}}\mathrm{d}\sigma =$$

$$=(2S+1)\int_0^\infty \frac{\sigma^p\,\mathrm{d}\sigma}{(1+\sigma)^{2S+2}}$$

$$=(2S+1)\underbrace{\left[\frac{\sigma^p}{-(2S+1)(1+\sigma)^{2S+1}}\right]_0^\infty}_{=0} - \frac{(2S+1)p}{-(2S+1)}\int_0^\infty \frac{\sigma^{p-1}\mathrm{d}\sigma}{(1+\sigma)^{2S+1}}$$

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where we determine $\mathrm{I}(S,p)$ using the integration by parts p times

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$$\Rightarrow \quad \mathbb{1} = \sum_{p=0}^{2S} |p\rangle\langle p| = \frac{2S+1}{\pi}\int_{\mathbb{C}} \mathrm{d}\mu\, |\mu\rangle\langle\mu| \cdot \frac{1}{(1+|\mu|^2)^2}$$

Overlap integral of two different states $|\mu\rangle, |\nu\rangle$

Now we are able to calculate the overlap integral $\langle\nu|\mu\rangle$

$$\langle\nu|\mu\rangle = \frac{1}{\sqrt{N_\mu N_\nu}} \sum_{p,p'=0}^{2S} \left[\left(\frac{(2S)!}{(2S-p'!)p'!}\frac{(2S)!}{(2S-p!)p!}\right)\right]^{\frac{1}{2}} (\nu^*)^{p'} \mu^p \delta_{p,p'}$$

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For the absolute value we obtain:

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High spin limit I

We use the *Holstein-Primakoff transformation* to obtain the high spin limit of the coherent spin states

$$S_+ \equiv \left(\sqrt{2S - a^\dagger a}\right) a \qquad S_- \equiv a^\dagger \left(\sqrt{2S - a^\dagger a}\right)$$

$$S_z \equiv S - a^\dagger a$$

It is not difficult to see that this definition fulfills the commutation relation:

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k$$

For $S \gg 1$ we can expand S_- in a Taylor series

$$S_- \to (2S)^{1/2} a^\dagger$$

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Therefore we obtain:

$$\exp(\mu S_-) \to \exp(\alpha a^\dagger)$$

Remember a **coherent spin state**

$$|\mu\rangle = (1+|\mu|^2)^{-S} e^{\mu S_-} |0\rangle$$

In the limit $S \to \infty$ a normalized spin state $|\alpha\rangle_S$ is:

$$\lim_{S\to\infty} |\alpha\rangle_S = \underbrace{\lim_{S\to\infty} \left(1 + \frac{|\alpha|^2}{2S}\right)^{-S}}_{\to \exp(-|\alpha|^2/2)} e^{\alpha a^\dagger} |0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle$$

Except a factor $\pi^{-1/2}$, this result is equal to the **coherent states of a harmonic oscillator**

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Important expectation values I

The coherent spin states are:

$$|\mu\rangle = (1+|\mu|^2)^{-S} \sum_{p=0}^{2S} \left(\frac{(2S)!}{(2S-p)!p!} \right)^{1/2} \mu^p |p\rangle$$

Remember state $|p\rangle$ is defined as:

$$\hat{S}_z |p\rangle \equiv (S-p)|p\rangle, \quad 0 \leq p \leq 2S$$

Therefore we define an operator $\hat{p}$:

$$\hat{p} \equiv S - \hat{S}_z \quad \Rightarrow \quad \hat{p}|p\rangle = p|p\rangle$$

Expectation value of the operator $\hat{p}$ in the coherent spin states is:

$$\langle \mu | \hat{p} | \mu \rangle = (1+|\mu|^2)^{-2S} \sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} |\mu|^{2p} \cdot p$$

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Since, we can write the sum as

$$\sum_{p=0}^{2S} \frac{(2S)!}{(2S-p)!p!} \left(|\mu|^2\right)^p \cdot p = |\mu|^2 \frac{\partial}{\partial(|\mu|^2)}(1+|\mu|^2)^{2S}$$

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Same way to calculate the following expectation values:

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What we have reached so far

- In **analogy** to the coherent states of the harmonic oscillator we **construct coherent spin states**

$$|\mu\rangle = (1 + |\mu|^2)^{-S} \cdot e^{\mu S_-} |0\rangle$$

- We find a state which **minimizes** the uncertainty relation

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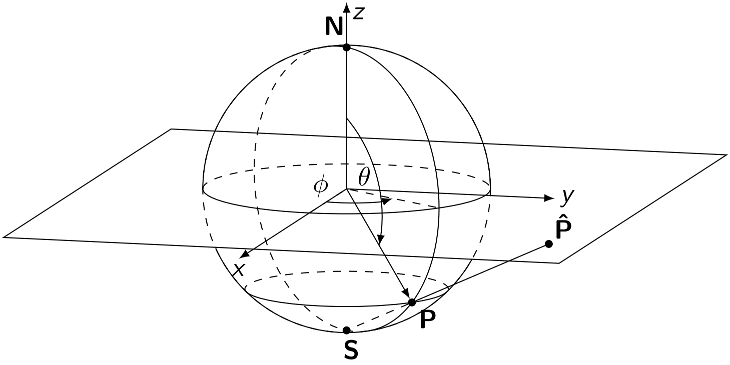
Stereographic projection I

An alternative parametrization of μ is the **stereographic projection**

$$\mu \equiv \tan\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi} \quad 0 \leq \theta < \pi \quad 0 \leq \phi < 2\pi$$



A coherent state $|\mu\rangle$ can be **associated** with a point on the **unit sphere**

Stereographic projection II

With respect to previous calculations we get the **normalized coherent spin states**

$$|\theta,\phi\rangle \equiv |\Omega\rangle = \left(\cos\frac{\theta}{2}\right)^{2S} \mathrm{e}^{\left(\tan\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\phi} S_-\right)} |0\rangle$$

with the **completeness relation**

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the **overlap integral** of the states $|\Omega\rangle, |\Omega'\rangle$

$$\langle\Omega'|\Omega\rangle = \left[\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta' + \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'\mathrm{e}^{\mathrm{i}(\phi-\phi')}\right]^{2S}$$

Further calculations give us

$$|\langle\Omega'|\Omega\rangle| = \left[\frac{1+\boldsymbol{n}\cdot\boldsymbol{n}'}{2}\right]^{S} \qquad \boldsymbol{n} \equiv (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$$

where $\boldsymbol{n}, \boldsymbol{n}'$ are **unit vectors** defined by angles (θ,ϕ), (θ',ϕ')

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Stereographic projection III

The expectation value of the spin components

By using previous results we get:

$$\langle\Omega|\hat{p}|\Omega\rangle = \frac{2S\left(\tan\frac{\theta}{2}\right)^2}{1+\left(\tan\frac{\theta}{2}\right)^2} = S\left(1-\cos\theta\right)$$

$$\langle\Omega|S_+|\Omega\rangle = S\sin\theta e^{i\phi} \qquad \langle\Omega|S_-|\Omega\rangle = S\sin\theta e^{-i\phi}$$

Thus, the expectation value is given by:

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Coherent spin states using properties of SU(2) I

The elements g of the group SU(2) are 2×2 unitary matrices with determinant 1:

$$g \in \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \right\}, \quad \text{with } |\alpha|^2 + |\beta|^2 = 1$$

We define a subgroup $H \subset \mathrm{SU}(2)$ of diagonal matrices:

$$h \in \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} \right\}, \quad \text{with } |\alpha|^2 = 1 \;\Rightarrow\; \alpha = e^{i\varphi}$$

We can write an element g of the **group SU(2)** with the so called **generators** σ_j by using **exponential mapping**:

$$g = \exp i \sum_{j=1}^{3} \theta_j \sigma_j \quad \text{with } \theta_j \in \mathbb{R}$$

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The determinant of a element of X is 1:

$$\alpha^2 + \beta_1^2 + \beta_2^2 \overset{!}{=} 1, \quad \text{where } \beta \equiv \beta_1 + i\beta_2$$

$\Rightarrow$ every element of X can be **associated** with a point on a sphere S^2

We use X to **construct** the coherent states with **generators** σ_x, σ_y

$$g_X = \exp\left[i\theta \left(x\frac{\sigma_x}{2} + y\frac{\sigma_y}{2} \right) \right], \quad \text{with } g_X \in X$$

With the spherical angles θ and ϕ we can use the **parametrization**:

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Therefore we get:

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With $|0\rangle \equiv |S, S\rangle$ we obtain the **coherent spin states**

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Proof of **equality** of both constructions of the coherent spin states

$$\exp\left(\xi S_+ - \xi^* S_-\right) = e^{\zeta' S_-}e^{-\eta S_z}e^{\zeta S_+}$$

$$\zeta = -\tan\tfrac{\theta}{2}e^{-i\phi} \quad \eta = -2\ln\left(\cos\tfrac{\theta}{2}\right) = \ln\left(1 + |\zeta|^2\right) \quad \zeta' = -\zeta^*$$

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Coherent spin states using properties of SU(2) IV

The effect of the representation g_X acting on the state $|S, S\rangle$

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$$= |\Omega\rangle$$

$\Rightarrow$ both constructions of the coherent spin states are equal

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Summary

- We construct the coherent spin states in analogy to the harmonic oscillator as well as based on the properties of the group SU(2)
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- We calculate some relevant expectation values and work out that the expectation value of the spin operator in the coherent spin states corresponds to a vector moving on a sphere with radius S

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