# Path-integrals for spin-fields

Quantum field-theory
of low dimensional systems

Jan Lotze

University of Stuttgart

27. May 2014

# Overview

# Motivation



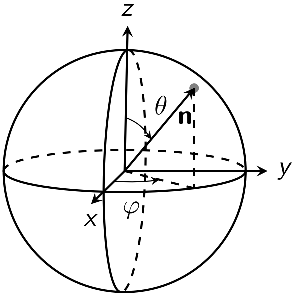
- General
  - Alternative approach to quantum mechanics
  - Path-integral starting point for classical limits
- Results
  - Quantisation of spins
  - Classical description of spins
  - Action

$\hookrightarrow$ Equations of motion

# Properties of coherent states for spins



- "Vacuum"
  $|0\rangle = |S, -S\rangle$
- Angular representation
  $$|\mathbf{n}(\theta,\varphi)\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}}(\theta,\varphi) \cdot \binom{2S}{m}^{1/2} |S, -S+m\rangle$$
  with $\mathcal{K}_{\mathrm{m}}(\theta,\varphi) = \cos^{2\mathrm{S}}\left(\frac{\theta}{2}\right) \cdot \left[-\tan\left(\frac{\theta}{2}\right)\exp(-i\varphi)\right]^{\mathrm{m}}$
- Normalisation
  $\langle \mathbf{n}|\mathbf{n}\rangle = 1$
- Resolution of identity
  $$\mathbb{1} = \frac{2S+1}{4\pi} \cdot \int_0^{\pi} \mathrm{d}\theta \sin(\theta) \int_0^{2\pi} \mathrm{d}\varphi |\mathbf{n}\rangle\langle\mathbf{n}|$$
- Differential
  $\mathrm{d}\mathbf{n} = \frac{2S+1}{4\pi} \sin(\theta) \cdot \mathrm{d}\theta \mathrm{d}\varphi$
- Expectation value
  $\langle \mathbf{n}|\hat{\boldsymbol{\mathcal{S}}}|\mathbf{n}\rangle = -\mathbf{n}S$



# Trotter slicing

- Canonical partition function

$$Z = \mathrm{Tr}\left[\mathcal{T}_\beta\ \mathrm{e}^{-\beta\mathcal{H}}\right] = \int \mathrm{d}\mathbf{n}\ \langle\mathbf{n}|\mathrm{e}^{-\beta\mathcal{H}}|\mathbf{n}\rangle$$

coherent states

"time" ordering

# Trotter slicing



- Canonical partition function

$$Z = \mathrm{Tr}\left[\mathcal{T}_\beta\, \mathrm{e}^{-\beta\mathcal{H}}\right] = \int \mathrm{d}\mathbf{n}\, \langle \mathbf{n}|\mathrm{e}^{-\beta\mathcal{H}}|\mathbf{n}\rangle$$

- Trotter slicing

$$\mathrm{e}^{-\beta\mathcal{H}} = \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}\cdot L} \underset{\text{slicing}}{\approx} \underbrace{\mathrm{e}^{-\frac{\beta}{L}\mathcal{H}} \cdot \ldots \cdot \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}}}_{L \text{ times}}$$

slicing

$\hookrightarrow$ error $\mathcal{O}\left[(\beta/L)^2\right]$

# Trotter slicing



- Canonical partition function

$$Z = \mathrm{Tr}\left[\mathcal{T}_\beta \, \mathrm{e}^{-\beta\mathcal{H}}\right] = \int \mathrm{d}\mathbf{n} \, \langle \mathbf{n}|\mathrm{e}^{-\beta\mathcal{H}}|\mathbf{n}\rangle$$

- Trotter slicing

$$\mathrm{e}^{-\beta\mathcal{H}} = \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}\cdot L} \approx \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}} \cdot \ldots \cdot \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}}$$

$$= \mathrm{e}^{-\Delta\tau\mathcal{H}} \cdot \ldots \cdot \mathrm{e}^{-\Delta\tau\mathcal{H}} = \prod_{\mathrm{k}=1}^{\mathrm{L}} \mathrm{e}^{-\Delta\tau\mathcal{H}}$$

$$\Delta\tau = \tau_{\mathrm{k}} - \tau_{\mathrm{k-1}} = \frac{\beta}{L}$$

$\hookrightarrow$ later: parametrisation $\tau = \tau(\theta, \varphi)$

- Canonical partition function

$$Z = \mathrm{Tr}\left[\mathcal{T}_\beta\, \mathrm{e}^{-\beta\mathcal{H}}\right] = \int \mathrm{d}\mathbf{n}\, \langle \mathbf{n}|\mathrm{e}^{-\beta\mathcal{H}}|\mathbf{n}\rangle$$

- Trotter slicing

$$\mathrm{e}^{-\beta\mathcal{H}} = \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}\cdot L} \approx \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}} \cdot \ldots \cdot \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}}$$

$$= \mathrm{e}^{-\Delta\tau\mathcal{H}} \cdot \ldots \cdot \mathrm{e}^{-\Delta\tau\mathcal{H}} = \prod_{k=1}^{L} \mathrm{e}^{-\Delta\tau\mathcal{H}}$$

- Insert identity operators

$$\int \mathrm{d}\mathbf{n}\, \langle \mathbf{n}|\mathrm{e}^{-\beta\mathcal{H}}|\mathbf{n}\rangle = \int \mathrm{d}\mathbf{n}\, \langle \mathbf{n}|\mathrm{e}^{-\Delta\tau\mathcal{H}} \cdot \ldots \cdot \mathrm{e}^{-\Delta\tau\mathcal{H}}|\mathbf{n}\rangle$$

$$= \int \mathrm{d}\mathbf{n}_L \langle \mathbf{n}_L|\mathrm{e}^{-\Delta\tau\mathcal{H}} \int \mathrm{d}\mathbf{n}_{L-1}|\mathbf{n}_{L-1}\rangle\langle \mathbf{n}_{L-1}|\mathrm{e}^{-\Delta\tau\mathcal{H}} \cdot \ldots$$

$$\ldots \cdot \int \mathrm{d}\mathbf{n}_1|\mathbf{n}_1\rangle\langle \mathbf{n}_1|\mathrm{e}^{-\Delta\tau\mathcal{H}}|\mathbf{n}_0\rangle$$

periodic boundary conditions $\mathbf{n}_L = \mathbf{n} = \mathbf{n}_0$

$L-1$ identity operators

# Trotter slicing



- Canonical partition function

$$Z = \mathrm{Tr}\left[\mathcal{T}_\beta \, \mathrm{e}^{-\beta\mathcal{H}}\right] = \int \mathrm{d}\mathbf{n} \, \langle \mathbf{n}|\mathrm{e}^{-\beta\mathcal{H}}|\mathbf{n}\rangle$$

- Trotter slicing

$$\mathrm{e}^{-\beta\mathcal{H}} = \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}\cdot L} \approx \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}} \cdot \ldots \cdot \mathrm{e}^{-\frac{\beta}{L}\mathcal{H}}$$
$$= \mathrm{e}^{-\Delta\tau\mathcal{H}} \cdot \ldots \cdot \mathrm{e}^{-\Delta\tau\mathcal{H}} = \prod_{\mathrm{k}=1}^{\mathrm{L}} \mathrm{e}^{-\Delta\tau\mathcal{H}}$$

- Insert identity operators

$$\int \mathrm{d}\mathbf{n} \, \langle \mathbf{n}|\mathrm{e}^{-\beta\mathcal{H}}|\mathbf{n}\rangle = \int \mathrm{d}\mathbf{n} \, \langle \mathbf{n}|\mathrm{e}^{-\Delta\tau\mathcal{H}} \cdot \ldots \cdot \mathrm{e}^{-\Delta\tau\mathcal{H}}|\mathbf{n}\rangle$$
$$= \int \mathrm{d}\mathbf{n}_\mathrm{L} \langle \mathbf{n}_\mathrm{L}|\mathrm{e}^{-\Delta\tau\mathcal{H}} \int \mathrm{d}\mathbf{n}_{\mathrm{L}\text{-}1} |\mathbf{n}_{\mathrm{L}\text{-}1}\rangle\langle\mathbf{n}_{\mathrm{L}\text{-}1}|\mathrm{e}^{-\Delta\tau\mathcal{H}} \cdot \ldots$$
$$\ldots \cdot \int \mathrm{d}\mathbf{n}_1 |\mathbf{n}_1\rangle\langle\mathbf{n}_1|\mathrm{e}^{-\Delta\tau\mathcal{H}}|\mathbf{n}_0\rangle$$

- Result

$$Z = \lim_{\mathrm{L}\to\infty} \int \left[\prod_{k=1}^{L} \mathrm{d}\mathbf{n}_\mathrm{k}\right] \left[\prod_{k=1}^{L} \langle \mathbf{n}_\mathrm{k}|\mathrm{e}^{-\Delta\tau\mathcal{H}}|\mathbf{n}_{\mathrm{k}\text{-}1}\rangle\right]$$

# Expand exponential functions



- Start
$$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} \mathrm{d}\mathbf{n}_\mathrm{k} \right] \left[ \prod_{k=1}^{L} \langle \mathbf{n}_\mathrm{k} | \mathrm{e}^{-\Delta\tau\mathcal{H}} | \mathbf{n}_\mathrm{k\text{-}1} \rangle \right]$$

# Expand exponential functions



- Start

  $$Z = \lim_{\mathrm{L}\to\infty} \int \left[\prod_{k=1}^{L} \mathrm{d}\mathbf{n}_{\mathrm{k}}\right] \left[\prod_{k=1}^{L} \langle \mathbf{n}_{\mathrm{k}} | \mathrm{e}^{-\Delta\tau\mathcal{H}} | \mathbf{n}_{\mathrm{k-1}} \rangle\right]$$

- Series expansion (SE)

  $$\mathrm{e}^{ax} = \sum_{\mathrm{k}=0}^{\infty} \frac{(ax)^{\mathbf{k}}}{k!} = 1 + ax + \mathcal{O}\left[(ax)^2\right]$$

# Expand exponential functions



- Start
$$Z = \lim_{\mathrm{L}\to\infty} \int \left[\prod_{k=1}^{L} \mathrm{d}\mathbf{n}_{\mathrm{k}}\right] \left[\prod_{k=1}^{L} \langle \mathbf{n}_{\mathrm{k}} | \mathrm{e}^{-\Delta\tau\mathcal{H}} | \mathbf{n}_{\mathrm{k\text{-}1}} \rangle\right]$$
- Series expansion (SE)
$$\mathrm{e}^{ax} = \sum_{\mathrm{k}=0}^{\infty} \frac{(ax)^{\mathbf{k}}}{k!} = 1 + ax + \mathcal{O}\left[(ax)^2\right]$$
- Matrix elements
$$\langle \mathbf{n}_{\mathrm{k}} | \mathrm{e}^{-\Delta\tau\mathcal{H}} | \mathbf{n}_{\mathrm{k\text{-}1}} \rangle \overset{(SE)}{\approx} \langle \mathbf{n}_{\mathrm{k}} | 1 - \Delta\tau\mathcal{H} | \mathbf{n}_{\mathrm{k\text{-}1}} \rangle$$

# Expand exponential functions



- Start
  $$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \left[ \prod_{k=1}^{L} \langle \mathbf{n}_k | e^{-\Delta\tau\mathcal{H}} | \mathbf{n}_{k-1} \rangle \right]$$
- Series expansion (SE)
  $$e^{ax} = \sum_{k=0}^{\infty} \frac{(ax)^k}{k!} = 1 + ax + \mathcal{O}\left[(ax)^2\right]$$
- Matrix elements
  $$\begin{aligned} \langle \mathbf{n}_k | e^{-\Delta\tau\mathcal{H}} | \mathbf{n}_{k-1} \rangle &\overset{(SE)}{\approx} \langle \mathbf{n}_k | 1 - \Delta\tau\mathcal{H} | \mathbf{n}_{k-1} \rangle \\ &= \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle - \Delta\tau \langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle \\ &= \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle \cdot \left( 1 - \Delta\tau \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle} \right) \end{aligned}$$

# Expand exponential functions



- Start
$$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \left[ \prod_{k=1}^{L} \langle \mathbf{n}_k | e^{-\Delta\tau\mathcal{H}} | \mathbf{n}_{k-1} \rangle \right]$$
- Series expansion (SE)
$$e^{ax} = \sum_{k=0}^{\infty} \frac{(ax)^k}{k!} = 1 + ax + \mathcal{O}\left[(ax)^2\right]$$
- Matrix elements
$$\begin{aligned}
\langle \mathbf{n}_k | e^{-\Delta\tau\mathcal{H}} | \mathbf{n}_{k-1} \rangle &\overset{(SE)}{\approx} \langle \mathbf{n}_k | 1 - \Delta\tau\mathcal{H} | \mathbf{n}_{k-1} \rangle \\
&= \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle - \Delta\tau \langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle \\
&= \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle \cdot \left( 1 - \Delta\tau \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle} \right) \\
&\overset{(SE)}{\approx} \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle \cdot \exp\left( -\Delta\tau \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle} \right)
\end{aligned}$$

# Discrete path integral



## Canonical partition function

$$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle \cdot \exp\left( - \sum_{k=1}^{L} \Delta\tau \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle} \right)$$

- Wick rotation: $\tau \to \frac{i}{\hbar} \cdot t$

## Propagator

$$K = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle \cdot \exp\left( - \frac{i}{\hbar} \sum_{k=1}^{L} \Delta t \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle} \right)$$

# Overlap matrix element



- Start

$$Z = \lim_{\mathrm{L}\to\infty} \int \left[\prod_{k=1}^{L} \mathrm{d}\mathbf{n}_{\mathrm{k}}\right] \left[\prod_{k=1}^{L} \langle \mathbf{n}_{\mathrm{k}} | \mathbf{n}_{\mathrm{k\text{-}1}} \rangle \cdot \exp\left(-\Delta\tau \frac{\langle \mathbf{n}_{\mathrm{k}} | \mathcal{H} | \mathbf{n}_{\mathrm{k\text{-}1}} \rangle}{\langle \mathbf{n}_{\mathrm{k}} | \mathbf{n}_{\mathrm{k\text{-}1}} \rangle}\right)\right]$$

- Overlap matrix element

$$\langle \mathbf{n}_{\mathrm{k}} | \mathbf{n}_{\mathrm{k\text{-}1}} \rangle = (1 - \langle \mathbf{n}_{\mathrm{k}} | \mathbf{n}_{\mathrm{k}} \rangle) + \langle \mathbf{n}_{\mathrm{k}} | \mathbf{n}_{\mathrm{k\text{-}1}} \rangle$$

normalisation

$$\langle \mathbf{n}_{\mathrm{k}} | \mathbf{n}_{\mathrm{k}} \rangle = 1$$

# Overlap matrix element



- Start

$$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \left[ \prod_{k=1}^{L} \langle \mathbf{n}_k | \mathbf{n}_{k\text{-}1} \rangle \cdot \exp\left( -\Delta\tau \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k\text{-}1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k\text{-}1} \rangle} \right) \right]$$

- Overlap matrix element

$$\langle \mathbf{n}_k | \mathbf{n}_{k\text{-}1} \rangle = (1 - \langle \mathbf{n}_k | \mathbf{n}_k \rangle) + \langle \mathbf{n}_k | \mathbf{n}_{k\text{-}1} \rangle$$

$$= 1 - \langle \mathbf{n}_k | \cdot ( | \mathbf{n}_k \rangle - | \mathbf{n}_{k\text{-}1} \rangle )$$

$$\overset{(SE)}{\approx} \exp\left[ - \langle \mathbf{n}_k | ( | \mathbf{n}_k \rangle - | \mathbf{n}_{k\text{-}1} \rangle ) \right]$$

# Overlap matrix element



- Start

$$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \left[ \prod_{k=1}^{L} \langle \mathbf{n}_k | \mathbf{n}_{k\text{-}1} \rangle \cdot \exp\left( -\Delta\tau \frac{\langle \mathbf{n_k} | \mathcal{H} | \mathbf{n_{k\text{-}1}} \rangle}{\langle \mathbf{n_k} | \mathbf{n_{k\text{-}1}} \rangle} \right) \right]$$

- Overlap matrix element

$$\begin{aligned} \langle \mathbf{n}_k | \mathbf{n}_{k\text{-}1} \rangle &= (1 - \langle \mathbf{n}_k | \mathbf{n}_k \rangle) + \langle \mathbf{n}_k | \mathbf{n}_{k\text{-}1} \rangle \\ &= 1 - \langle \mathbf{n}_k | \cdot (|\mathbf{n}_k\rangle - |\mathbf{n}_{k\text{-}1}\rangle) \\ &\overset{(SE)}{\approx} \exp\left[ -\langle \mathbf{n}_k | (|\mathbf{n}_k\rangle - |\mathbf{n}_{k\text{-}1}\rangle) \right] \\ &= \exp\left[ -\langle \mathbf{n}_k | \left( \frac{|\mathbf{n_k}\rangle - |\mathbf{n_{k\text{-}1}}\rangle}{\Delta\tau} \right) \cdot \Delta\tau \right] \end{aligned}$$

# Overlap matrix element



- Start

$$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \left[ \prod_{k=1}^{L} \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle \cdot \exp\left( -\Delta\tau \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle} \right) \right]$$

- Overlap matrix element

$$\begin{aligned} \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle &= (1 - \langle \mathbf{n}_k | \mathbf{n}_k \rangle) + \langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle \\ &= 1 - \langle \mathbf{n}_k | \cdot (|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle) \\ &\overset{(SE)}{\approx} \exp\left[ -\langle \mathbf{n}_k | (|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle) \right] \\ &= \exp\left[ -\langle \mathbf{n}_k | \left( \frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau} \right) \cdot \Delta\tau \right] \end{aligned}$$

- Result

$$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \exp\left[ -\sum_{k=1}^{L} \Delta\tau \left( \langle \mathbf{n}_k | \left( \frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau} \right) + \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle} \right) \right]$$

# Continuous path integral



- Start

$$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \exp \left[ - \sum_{k=1}^{L} \Delta\tau \left( \langle \mathbf{n}_k | \left( \frac{|\mathbf{n_k}\rangle - |\mathbf{n_{k-1}}\rangle}{\Delta\tau} \right) + \frac{\langle \mathbf{n_k} | \mathcal{H} | \mathbf{n_{k-1}} \rangle}{\langle \mathbf{n_k} | \mathbf{n_{k-1}} \rangle} \right) \right]$$

- Difference quotient

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

# Continuous path integral

- Start

$$Z = \lim_{L \to \infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \exp \left[ - \sum_{k=1}^{L} \Delta\tau \left( \langle \mathbf{n}_k | \left( \frac{|\mathbf{n_k}\rangle - |\mathbf{n_{k\text{-}1}}\rangle}{\Delta\tau} \right) + \frac{\langle \mathbf{n_k} | \mathcal{H} | \mathbf{n_{k\text{-}1}} \rangle}{\langle \mathbf{n_k} | \mathbf{n_{k\text{-}1}} \rangle} \right) \right]$$

- Difference quotient

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

- Continuum limit

$$L \to \infty \quad \hat{=} \quad \Delta\tau \to 0 \quad \hat{=} \quad |\mathbf{n}_k\rangle \to |\mathbf{n}_{k-1}\rangle$$

# Continuous path integral



- Start
$$Z = \lim_{L\to\infty} \int \left[\prod_{k=1}^{L} d\mathbf{n}_k\right] \exp\left[-\sum_{k=1}^{L} \Delta\tau \left(\langle \mathbf{n}_k | \left(\frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau}\right) + \frac{\langle \mathbf{n}_k|\mathcal{H}|\mathbf{n}_{k-1}\rangle}{\langle \mathbf{n}_k|\mathbf{n}_{k-1}\rangle}\right)\right]$$
- Difference quotient
$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x)-f(x-h)}{h}$$
- Continuum limit
$$L \to \infty \quad \hat{=} \quad \Delta\tau \to 0 \quad \hat{=} \quad |\mathbf{n}_k\rangle \to |\mathbf{n}_{k-1}\rangle$$
  - $\lim_{\Delta\tau\to 0} \left(\frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau}\right) = \frac{\partial |\mathbf{n}_k\rangle}{\partial\tau}$

# Continuous path integral



- Start

$$Z = \lim_{L\to\infty} \int \left[\prod_{k=1}^{L} d\mathbf{n}_k\right] \exp\left[-\sum_{k=1}^{L} \Delta\tau \left(\langle \mathbf{n}_k| \left(\frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau}\right) + \frac{\langle \mathbf{n}_k|\mathcal{H}|\mathbf{n}_{k-1}\rangle}{\langle \mathbf{n}_k|\mathbf{n}_{k-1}\rangle}\right)\right]$$

- Difference quotient

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}$$

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$$L \to \infty \quad \hat{=} \quad \Delta\tau \to 0 \quad \hat{=} \quad |\mathbf{n}_k\rangle \to |\mathbf{n}_{k-1}\rangle$$

  - $$\lim_{\Delta\tau\to 0} \left(\frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau}\right) = \frac{\partial |\mathbf{n}_k\rangle}{\partial\tau}$$
  - $$\lim_{\Delta\tau\to 0} \frac{\langle \mathbf{n}_k|\mathcal{H}|\mathbf{n}_{k-1}\rangle}{\langle \mathbf{n}_k|\mathbf{n}_{k-1}\rangle} \approx \frac{\langle \mathbf{n}_k|\mathcal{H}|\mathbf{n}_k\rangle}{\langle \mathbf{n}_k|\mathbf{n}_k\rangle} = \langle \mathbf{n}_k|\mathcal{H}|\mathbf{n}_k\rangle$$

normalisation

$\langle \mathbf{n}_k|\mathbf{n}_k\rangle = 1$

# Continuous path integral

- Start

$$Z = \lim_{L\to\infty} \int \left[\prod_{k=1}^{L} d\mathbf{n}_k\right] \exp\left[-\sum_{k=1}^{L} \Delta\tau \left(\langle \mathbf{n}_k| \left(\frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau}\right) + \frac{\langle \mathbf{n}_k|\mathcal{H}|\mathbf{n}_{k-1}\rangle}{\langle \mathbf{n}_k|\mathbf{n}_{k-1}\rangle}\right)\right]$$

- Difference quotient

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x)-f(x-h)}{h}$$

- Continuum limit

$$L \to \infty \quad \hat{=} \quad \Delta\tau \to 0 \quad \hat{=} \quad |\mathbf{n}_k\rangle \to |\mathbf{n}_{k-1}\rangle$$

  - $$\lim_{\Delta\tau\to 0} \left(\frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau}\right) = \frac{\partial |\mathbf{n}_k\rangle}{\partial \tau}$$
  - $$\lim_{\Delta\tau\to 0} \frac{\langle \mathbf{n}_k|\mathcal{H}|\mathbf{n}_{k-1}\rangle}{\langle \mathbf{n}_k|\mathbf{n}_{k-1}\rangle} \approx \frac{\langle \mathbf{n}_k|\mathcal{H}|\mathbf{n}_{k}\rangle}{\langle \mathbf{n}_k|\mathbf{n}_{k}\rangle} = \langle \mathbf{n}_k|\mathcal{H}|\mathbf{n}_k\rangle$$
  - $$\lim_{\Delta\tau\to 0} \sum_{k=1}^{L} \Delta\tau \rightsquigarrow \int_{0\cdot\Delta\tau}^{L\cdot\Delta\tau} d\tau \;\hat{=}\; \int_0^\beta d\tau$$

$$\Delta\tau = \frac{\beta}{L}$$

# Continuous path integral



- Start

$$Z = \lim_{L\to\infty} \int \left[ \prod_{k=1}^{L} d\mathbf{n}_k \right] \exp\left[ -\sum_{k=1}^{L} \Delta\tau \left( \langle \mathbf{n}_k | \left( \frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau} \right) + \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle} \right) \right]$$

- Difference quotient

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}$$

- Continuum limit

$$L \to \infty \quad \hat{=} \quad \Delta\tau \to 0 \quad \hat{=} \quad |\mathbf{n}_k\rangle \to |\mathbf{n}_{k-1}\rangle$$

  - $$\lim_{\Delta\tau\to 0} \left( \frac{|\mathbf{n}_k\rangle - |\mathbf{n}_{k-1}\rangle}{\Delta\tau} \right) = \frac{\partial |\mathbf{n}_k\rangle}{\partial\tau}$$
  - $$\lim_{\Delta\tau\to 0} \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k-1} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k-1} \rangle} \approx \frac{\langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_{k} \rangle}{\langle \mathbf{n}_k | \mathbf{n}_{k} \rangle} = \langle \mathbf{n}_k | \mathcal{H} | \mathbf{n}_k \rangle$$
  - $$\lim_{\Delta\tau\to 0} \sum_{k=1}^{L} \Delta\tau \rightsquigarrow \int_{0\cdot\Delta\tau}^{L\cdot\Delta\tau} d\tau \;\hat{=}\; \int_0^\beta d\tau$$

- Result

$$Z = \int \mathcal{D}\mathbf{n} \exp\left[ -\int_0^\beta d\tau \left( \langle \mathbf{n} | \frac{\partial}{\partial\tau} | \mathbf{n} \rangle + \langle \mathbf{n} | \mathcal{H} | \mathbf{n} \rangle \right) \right] \text{ with } \mathcal{D}\mathbf{n} = \lim_{L\to\infty} \prod_{k=1}^{L} d\mathbf{n}_k$$

# Continuous path integral



## Canonical partition function

$$Z = \int \mathcal{D}\mathbf{n} \exp\left[ - \int_0^\beta \mathrm{d}\tau \left( \langle \mathbf{n} | \frac{\partial}{\partial \tau} | \mathbf{n} \rangle + \langle \mathbf{n} | \mathcal{H} | \mathbf{n} \rangle \right) \right]$$

- Wick rotation: $\tau \rightarrow \frac{i}{\hbar} \cdot t$

## Propagator

$$K = \int \mathcal{D}\mathbf{n} \exp\left[ \frac{i}{\hbar} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \left( \langle \mathbf{n} | i\hbar \frac{\partial}{\partial t} | \mathbf{n} \rangle - \langle \mathbf{n} | \mathcal{H} | \mathbf{n} \rangle \right) \right]$$

- Multiple spins: $|\mathbf{n}\rangle = \bigotimes_{\mathrm{i}=1}^{N} |\mathbf{n}_\mathrm{i}\rangle$

# Identification as a phase I



- Objective
  $\langle \mathbf{n} | \frac{\partial}{\partial \tau} | \mathbf{n} \rangle$
- Angular representation
  $$|\mathbf{n}(\theta,\varphi)\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}}(\theta,\varphi) \cdot \binom{2S}{m}^{1/2} |S, -S+m\rangle$$
  with $\mathcal{K}_{\mathrm{m}}(\theta,\varphi) = \cos^{2\mathrm{S}}\left(\frac{\theta}{2}\right) \cdot \left[-\tan\left(\frac{\theta}{2}\right)\exp(-i\varphi)\right]^{\mathrm{m}}$

# Identification as a phase I



- Objective
  $\langle \mathbf{n} | \frac{\partial}{\partial \tau} | \mathbf{n} \rangle$
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  $$|\mathbf{n}(\theta,\varphi)\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}}(\theta,\varphi) \cdot \binom{2S}{m}^{1/2} |S, -S+m\rangle$$
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- Derivative
  $$\frac{\partial}{\partial \tau}|\mathbf{n}(\theta,\varphi)\rangle = \frac{\partial|\mathbf{n}\rangle}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \frac{\partial|\mathbf{n}\rangle}{\partial\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\partial|\mathbf{n}\rangle}{\partial\theta}\dot{\theta} + \frac{\partial|\mathbf{n}\rangle}{\partial\varphi}\dot{\varphi} = ①$$

# Identification as a phase I



- Objective
  $\langle \mathbf{n} | \frac{\partial}{\partial \tau} | \mathbf{n} \rangle$
- Angular representation
  $$|\mathbf{n}(\theta,\varphi)\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}}(\theta,\varphi) \cdot \binom{2S}{m}^{1/2} |S, -S+m\rangle$$
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  $\frac{\partial}{\partial \tau}|\mathbf{n}(\theta,\varphi)\rangle = \frac{\partial|\mathbf{n}\rangle}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \frac{\partial|\mathbf{n}\rangle}{\partial\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\partial|\mathbf{n}\rangle}{\partial\theta}\dot{\theta} + \frac{\partial|\mathbf{n}\rangle}{\partial\varphi}\dot{\varphi} = (1)$
  - Auxiliary calculation
    $\frac{\partial\mathcal{K}_{\mathbf{m}}}{\partial\theta} = \mathcal{K}_{\mathrm{m}} \cdot \left[-S\tan\left(\frac{\theta}{2}\right) + m\cdot\left(2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^{-1}\right]$
    $\frac{\partial\mathcal{K}_{\mathbf{m}}}{\partial\varphi} = -im\cdot\mathcal{K}_{\mathrm{m}}$

# Identification as a phase I



- Objective
  $\langle \mathbf{n} | \frac{\partial}{\partial \tau} | \mathbf{n} \rangle$
- Angular representation
  $$|\mathbf{n}(\theta,\varphi)\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}}(\theta,\varphi) \cdot \binom{2S}{m}^{1/2} |S, -S+m\rangle$$
  with $\mathcal{K}_{\mathrm{m}}(\theta,\varphi) = \cos^{2\mathrm{S}}\left(\frac{\theta}{2}\right) \cdot \left[-\tan\left(\frac{\theta}{2}\right)\exp(-i\varphi)\right]^{\mathrm{m}}$
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  $\frac{\partial}{\partial \tau}|\mathbf{n}(\theta,\varphi)\rangle = \frac{\partial|\mathbf{n}\rangle}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \frac{\partial|\mathbf{n}\rangle}{\partial\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\partial|\mathbf{n}\rangle}{\partial\theta}\dot{\theta} + \frac{\partial|\mathbf{n}\rangle}{\partial\varphi}\dot{\varphi} = ①$
  - Auxiliary calculation
    $\frac{\partial\mathcal{K}_{\mathbf{m}}}{\partial\theta} = \mathcal{K}_{\mathrm{m}} \cdot \left[-S\tan\left(\frac{\theta}{2}\right) + m \cdot \left(2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^{-1}\right]$
    $\frac{\partial\mathcal{K}_{\mathbf{m}}}{\partial\varphi} = -im \cdot \mathcal{K}_{\mathrm{m}}$

  $$① = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}} \cdot \left[\mathcal{A}(\theta,\dot{\theta}) + m \cdot \mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right] \cdot \binom{2S}{m}^{1/2} |S, -S+m\rangle$$

# Identification as a phase II



- Start
$$\frac{\partial}{\partial\tau}\left|\mathbf{n}\right\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}} \cdot \left[\mathcal{A}(\theta,\dot{\theta}) + m \cdot \mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right] \cdot \binom{2S}{m}^{1/2} \left|S,-S+m\right\rangle$$

# Identification as a phase II



- Start
  $$\frac{\partial}{\partial\tau}\left|\mathbf{n}\right\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}} \cdot \left[\mathcal{A}(\theta,\dot\theta) + m\cdot\mathcal{B}(\theta,\dot\theta,\dot\varphi)\right]\cdot \binom{2S}{m}^{1/2} \left|S,-S+m\right\rangle$$
- Overlap
  $$\langle S',-S'+m'|S,-S+m\rangle = \delta_{\mathrm{S',S}}\delta_{\mathrm{m',m}}$$

# Identification as a phase II



- Start
$$\frac{\partial}{\partial\tau}|\mathbf{n}\rangle = \sum_{\mathrm{m}=0}^{2S}\mathcal{K}_{\mathrm{m}}\cdot\left[\mathcal{A}(\theta,\dot\theta)+m\cdot\mathcal{B}(\theta,\dot\theta,\dot\varphi)\right]\cdot\binom{2S}{m}^{1/2}|S,-S+m\rangle$$
- Overlap
$$\langle S',-S'+m'|S,-S+m\rangle = \delta_{\mathrm{S}',\mathrm{S}}\delta_{\mathrm{m}',\mathrm{m}}$$
- Matrix element
$$\langle\mathbf{n}|\tfrac{\partial}{\partial\tau}|\mathbf{n}\rangle = \sum_{\mathrm{m}=0}^{2S}|\mathcal{K}_{\mathrm{m}}|^2\cdot\left[\mathcal{A}(\theta,\dot\theta)+m\cdot\mathcal{B}(\theta,\dot\theta,\dot\varphi)\right]\cdot\binom{2S}{m}$$
$$= \langle\mathbf{n}|\mathbf{n}\rangle\,\mathcal{A}(\theta,\dot\theta)+\mathcal{B}(\theta,\dot\theta,\dot\varphi)\cdot\sum_{\mathrm{m}=0}^{2S}|\mathcal{K}_{\mathrm{m}}|^2\cdot m\binom{2S}{m}$$
$$|\mathbf{n}(\theta,\varphi)\rangle = \sum_{\mathrm{m}=0}^{2S}\mathcal{K}_{\mathrm{m}}(\theta,\varphi)\cdot\binom{2S}{m}^{1/2}|S,-S+m\rangle$$

# Identification as a phase II



- Start
$$\frac{\partial}{\partial\tau}|\mathbf{n}\rangle = \sum_{\mathrm{m}=0}^{2S}\mathcal{K}_{\mathrm{m}}\cdot\left[\mathcal{A}(\theta,\dot{\theta}) + m\cdot\mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right]\cdot\binom{2S}{m}^{1/2}|S,-S+m\rangle$$
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$$\langle S',-S'+m'|S,-S+m\rangle = \delta_{\mathrm{S}',\mathrm{S}}\delta_{\mathrm{m}',\mathrm{m}}$$
- Matrix element
$$\langle\mathbf{n}|\tfrac{\partial}{\partial\tau}|\mathbf{n}\rangle = \sum_{\mathrm{m}=0}^{2S}|\mathcal{K}_{\mathrm{m}}|^2\cdot\left[\mathcal{A}(\theta,\dot{\theta}) + m\cdot\mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right]\cdot\binom{2S}{m}$$
$$= \mathcal{A}(\theta,\dot{\theta}) + \mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\cdot\sum_{\mathrm{m}=0}^{2S}|\mathcal{K}_{\mathrm{m}}|^2\cdot m\binom{2S}{m} = ②$$

# Identification as a phase II



- Start

$$\frac{\partial}{\partial\tau}|\mathbf{n}\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}} \cdot \left[\mathcal{A}(\theta,\dot{\theta}) + m\cdot\mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right] \cdot \binom{2S}{m}^{1/2} |S,-S+m\rangle$$

- Overlap

$$\langle S', -S'+m' | S, -S+m\rangle = \delta_{\mathrm{S}',\mathrm{S}}\delta_{\mathrm{m}',\mathrm{m}}$$

- Matrix element

$$\langle\mathbf{n}|\frac{\partial}{\partial\tau}|\mathbf{n}\rangle = \sum_{\mathrm{m}=0}^{2S} |\mathcal{K}_{\mathrm{m}}|^2 \cdot \left[\mathcal{A}(\theta,\dot{\theta}) + m\cdot\mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right] \cdot \binom{2S}{m}$$

$$= \mathcal{A}(\theta,\dot{\theta}) + \mathcal{B}(\theta,\dot{\theta},\dot{\varphi}) \cdot \sum_{\mathrm{m}=0}^{2S} |\mathcal{K}_{\mathrm{m}}|^2 \cdot m\binom{2S}{m} = ②$$

- Auxiliary calculation

$$\sum_{\mathrm{m}=0}^{2S} |\mathcal{K}_{\mathrm{m}}|^2 \cdot m\binom{2S}{m} = 2S \cdot \sum_{\mathrm{n}=0}^{2S-1} |\mathcal{K}_{\mathrm{n}+1}|^2 \cdot \binom{2S-1}{n}$$

$(m-1) \to n$

$m \in [1, 2S] \to n \in [0, 2S-1]$

# Identification as a phase II



- Start
$$\frac{\partial}{\partial\tau}|\mathbf{n}\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}} \cdot \left[\mathcal{A}(\theta,\dot\theta) + m\cdot\mathcal{B}(\theta,\dot\theta,\dot\varphi)\right]\cdot\binom{2S}{m}^{1/2}|S,-S+m\rangle$$
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$$\langle S',-S'+m'|S,-S+m\rangle = \delta_{\mathrm{S',S}}\delta_{\mathrm{m',m}}$$
- Matrix element
$$\langle\mathbf{n}|\frac{\partial}{\partial\tau}|\mathbf{n}\rangle = \sum_{\mathrm{m}=0}^{2S}|\mathcal{K}_{\mathrm{m}}|^2\cdot\left[\mathcal{A}(\theta,\dot\theta) + m\cdot\mathcal{B}(\theta,\dot\theta,\dot\varphi)\right]\cdot\binom{2S}{m}$$
$$= \mathcal{A}(\theta,\dot\theta) + \mathcal{B}(\theta,\dot\theta,\dot\varphi)\cdot\sum_{\mathrm{m}=0}^{2S}|\mathcal{K}_{\mathrm{m}}|^2\cdot m\binom{2S}{m} = ②$$
- Auxiliary calculation
$$\sum_{\mathrm{m}=0}^{2S}|\mathcal{K}_{\mathrm{m}}|^2\cdot m\binom{2S}{m} = 2S\cdot\sum_{\mathrm{n}=0}^{2S-1}|\mathcal{K}_{\mathrm{n+1}}|^2\cdot\binom{2S-1}{n}$$
$$= 2S\cdot\cos^{4\mathrm{S}}\!\left(\tfrac{\theta}{2}\right)\sum_{\mathrm{n}=0}^{2S-1}\tan^2\!\left(\tfrac{\theta}{2}\right)^{\mathrm{n+1}}\cdot\binom{2S-1}{n}$$
$$\mathcal{K}_{\mathrm{n}}(\theta,\varphi) = 2S\cdot\cos^{2\mathrm{S}}\!\left(\tfrac{\theta}{2}\right)\cdot\left[-\tan\left(\tfrac{\theta}{2}\right)\exp(-i\varphi)\right]^{\mathrm{n}}$$

# Identification as a phase II



- Start

$$\frac{\partial}{\partial\tau}\left|\mathbf{n}\right\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}} \cdot \left[\mathcal{A}(\theta,\dot{\theta}) + m\cdot\mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right] \cdot \binom{2S}{m}^{1/2} \left|S,-S+m\right\rangle$$

- Overlap

$$\left\langle S',-S'+m'\middle|S,-S+m\right\rangle = \delta_{\mathrm{S}',\mathrm{S}}\delta_{\mathrm{m}',\mathrm{m}}$$

- Matrix element

$$\left\langle\mathbf{n}\middle|\frac{\partial}{\partial\tau}\middle|\mathbf{n}\right\rangle = \sum_{\mathrm{m}=0}^{2S} |\mathcal{K}_{\mathrm{m}}|^2 \cdot \left[\mathcal{A}(\theta,\dot{\theta}) + m\cdot\mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right] \cdot \binom{2S}{m}$$

$$= \mathcal{A}(\theta,\dot{\theta}) + \mathcal{B}(\theta,\dot{\theta},\dot{\varphi}) \cdot \sum_{\mathrm{m}=0}^{2S} |\mathcal{K}_{\mathrm{m}}|^2 \cdot m\binom{2S}{m} = ②$$

- Auxiliary calculation

$$\sum_{\mathrm{m}=0}^{2S} |\mathcal{K}_{\mathrm{m}}|^2 \cdot m\binom{2S}{m} = 2S\cdot\sum_{\mathrm{n}=0}^{2S-1} |\mathcal{K}_{\mathrm{n+1}}|^2 \cdot \binom{2S-1}{n}$$

$$= 2S\cdot\cos^{4\mathrm{S}}\left(\tfrac{\theta}{2}\right)\sum_{\mathrm{n}=0}^{2S-1}\tan^2\left(\tfrac{\theta}{2}\right)^{\mathrm{n+1}}\cdot\binom{2S-1}{n}$$

$$= 2S\cdot\cos^{4\mathrm{S}}\left(\tfrac{\theta}{2}\right)\cdot\tan^2\left(\tfrac{\theta}{2}\right)\left[1+\tan^2\left(\tfrac{\theta}{2}\right)\right]^{\mathrm{2S\text{-}1}}$$

$$(a+b)^n = \sum_{\mathrm{k}=0}^{n}\binom{n}{k} a^{\mathrm{n\text{-}k}} b^{\mathrm{k}}$$

# Identification as a phase II



- Start

$$\frac{\partial}{\partial\tau}\left|\mathbf{n}\right\rangle = \sum_{\mathrm{m}=0}^{2S} \mathcal{K}_{\mathrm{m}} \cdot \left[\mathcal{A}(\theta,\dot{\theta}) + m\cdot\mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right] \cdot \binom{2S}{m}^{1/2} \left|S,-S+m\right\rangle$$

- Overlap

$$\left\langle S',-S'+m'\middle|S,-S+m\right\rangle = \delta_{\mathrm{S',S}}\delta_{\mathrm{m',m}}$$

- Matrix element

$$\left\langle\mathbf{n}\right|\frac{\partial}{\partial\tau}\left|\mathbf{n}\right\rangle = \sum_{\mathrm{m}=0}^{2S} |\mathcal{K}_{\mathrm{m}}|^2 \cdot \left[\mathcal{A}(\theta,\dot{\theta}) + m\cdot\mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\right] \cdot \binom{2S}{m}$$

$$= \mathcal{A}(\theta,\dot{\theta}) + \mathcal{B}(\theta,\dot{\theta},\dot{\varphi}) \cdot \sum_{\mathrm{m}=0}^{2S} |\mathcal{K}_{\mathrm{m}}|^2 \cdot m \binom{2S}{m} = ②$$

- Auxiliary calculation

$$\sum_{\mathrm{m}=0}^{2S} |\mathcal{K}_{\mathrm{m}}|^2 \cdot m\binom{2S}{m} = 2S \cdot \sum_{\mathrm{n}=0}^{2S-1} |\mathcal{K}_{\mathrm{n+1}}|^2 \cdot \binom{2S-1}{n}$$

$$= 2S\cdot\cos^{4\mathrm{S}}\left(\tfrac{\theta}{2}\right) \sum_{\mathrm{n}=0}^{2S-1} \tan^2\left(\tfrac{\theta}{2}\right)^{\mathrm{n+1}} \cdot \binom{2S-1}{n}$$

$$= 2S\cdot\cos^{4\mathrm{S}}\left(\tfrac{\theta}{2}\right)\cdot\tan^2\left(\tfrac{\theta}{2}\right)\left[1+\tan^2\left(\tfrac{\theta}{2}\right)\right]^{\text{2S-1}}$$

$$② = \mathcal{A}(\theta,\dot{\theta}) + 2S\cdot\mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\cdot\cos^{4\mathrm{S}}\left(\tfrac{\theta}{2}\right)\cdot\tan^2\left(\tfrac{\theta}{2}\right)\left[1+\tan^2\left(\tfrac{\theta}{2}\right)\right]^{\text{2S-1}}$$

# Identification as a phase III



- Start
$$\langle \mathbf{n}|\frac{\partial}{\partial\tau}|\mathbf{n}\rangle = \mathcal{A}(\theta,\dot{\theta}) + \mathcal{B}(\theta,\dot{\theta},\dot{\varphi})\cdot 2S\cos^{4S}\left(\frac{\theta}{2}\right)\tan^2\left(\frac{\theta}{2}\right)\left[1+\tan^2\left(\frac{\theta}{2}\right)\right]^{2S-1}$$

# Identification as a phase III



- Start

$$\begin{aligned}\langle \mathbf{n}|\tfrac{\partial}{\partial\tau}|\mathbf{n}\rangle &= \mathcal{A}(\theta,\dot\theta)+\mathcal{B}(\theta,\dot\theta,\dot\varphi)\cdot 2S\cos^{4S}\!\big(\tfrac{\theta}{2}\big)\tan^2\!\big(\tfrac{\theta}{2}\big)\Big[1+\tan^2\!\big(\tfrac{\theta}{2}\big)\Big]^{2S-1}\\ &= \mathcal{A}(\theta,\dot\theta)+\mathcal{B}(\theta,\dot\theta,\dot\varphi)\cdot 2S\sin^2\!\big(\tfrac{\theta}{2}\big)\end{aligned}$$

# Identification as a phase III



- Start

$$\begin{aligned}
\langle \mathbf{n}|\tfrac{\partial}{\partial\tau}|\mathbf{n}\rangle &= \mathcal{A}(\theta,\dot\theta) + \mathcal{B}(\theta,\dot\theta,\dot\varphi)\cdot 2S\cos^{4S}\!\left(\tfrac{\theta}{2}\right)\tan^2\!\left(\tfrac{\theta}{2}\right)\left[1+\tan^2\!\left(\tfrac{\theta}{2}\right)\right]^{2S\text{-}1}\\
&= \mathcal{A}(\theta,\dot\theta) + \mathcal{B}(\theta,\dot\theta,\dot\varphi)\cdot 2S\sin^2\!\left(\tfrac{\theta}{2}\right)\\
&= -S\tan\left(\tfrac{\theta}{2}\right)\dot\theta + \left\{\left[2\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)\right]^{-1}\dot\theta - i\dot\varphi\right\}\cdot 2S\sin^2\!\left(\tfrac{\theta}{2}\right)
\end{aligned}$$

# Identification as a phase III



- Start

$$
\begin{aligned}
\langle \mathbf{n}|\tfrac{\partial}{\partial\tau}|\mathbf{n}\rangle &= \mathcal{A}(\theta,\dot{\theta}) + \mathcal{B}(\theta,\dot{\theta},\dot{\varphi}) \cdot 2S\cos^{4S}\!\left(\tfrac{\theta}{2}\right)\tan^2\!\left(\tfrac{\theta}{2}\right)\left[1+\tan^2\!\left(\tfrac{\theta}{2}\right)\right]^{2S\text{-}1} \\
&= \mathcal{A}(\theta,\dot{\theta}) + \mathcal{B}(\theta,\dot{\theta},\dot{\varphi}) \cdot 2S\sin^2\!\left(\tfrac{\theta}{2}\right) \\
&= -S\tan\left(\tfrac{\theta}{2}\right)\dot{\theta} + \left\{\left[2\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)\right]^{-1}\dot{\theta} - i\dot{\varphi}\right\} \cdot 2S\sin^2\!\left(\tfrac{\theta}{2}\right) \\
&= -iS\left[1-\cos(\theta)\right]\dot{\varphi}
\end{aligned}
$$

# Identification as a phase III



- Start

$$
\begin{aligned}
\langle \mathbf{n}|\frac{\partial}{\partial\tau}|\mathbf{n}\rangle &= \mathcal{A}(\theta,\dot\theta) + \mathcal{B}(\theta,\dot\theta,\dot\varphi)\cdot 2S\cos^{4S}\left(\tfrac{\theta}{2}\right)\tan^2\left(\tfrac{\theta}{2}\right)\left[1+\tan^2\left(\tfrac{\theta}{2}\right)\right]^{2S\text{-}1} \\
&= \mathcal{A}(\theta,\dot\theta) + \mathcal{B}(\theta,\dot\theta,\dot\varphi)\cdot 2S\sin^2\left(\tfrac{\theta}{2}\right) \\
&= -S\tan\left(\tfrac{\theta}{2}\right)\dot\theta + \left\{\left[2\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)\right]^{-1}\dot\theta - i\dot\varphi\right\}\cdot 2S\sin^2\left(\tfrac{\theta}{2}\right) \\
&= -iS\left[1-\cos(\theta)\right]\dot\varphi
\end{aligned}
$$

**Canonical partition function**

$$
Z = \int \mathcal{D}\mathbf{n}\exp\left[-\int_0^\beta d\tau\,\langle\mathbf{n}|\mathcal{H}|\mathbf{n}\rangle - iS\left[1-\cos(\theta)\right]\dot\varphi\right]
$$

- Wick rotation: $\tau \rightarrow \frac{i}{\hbar}\cdot t$

**Propagator**

$$
K = \int \mathcal{D}\mathbf{n}\exp\left[\frac{i}{\hbar}\int_{t_\mathrm{i}}^{t_\mathrm{f}} dt\,\hbar S\left[1-\cos(\theta)\right]\dot\varphi - \langle\mathbf{n}|\mathcal{H}|\mathbf{n}\rangle\right]
$$

# Magnetic monopole I



- Objective

$$\int_0^\beta \mathrm{d}\tau \, [1 - \cos(\theta)] \, \dot{\varphi}$$

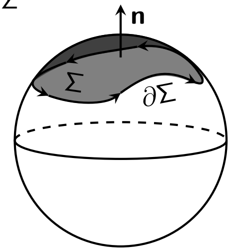
# Magnetic monopole I



- Objective

$$\int_0^\beta \mathrm{d}\tau \, [1 - \cos(\theta)] \, \dot{\varphi}$$

- Reminder
  - Periodic boundary conditions: $\mathbf{n}_\mathrm{L} = \mathbf{n} = \mathbf{n}_0$
  - $\hookrightarrow$ Line integral along loop $\partial\Sigma$ enclosing surface $\Sigma$
  - Parametrisation: $\theta(\tau), \varphi(\tau)$
  - $\hookrightarrow$ Loop $\partial\Sigma$ located on a sphere

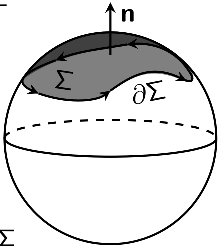


# Magnetic monopole I

- Objective

$$\int_0^\beta \mathrm{d}\tau \, [1 - \cos(\theta)] \, \dot{\varphi}$$

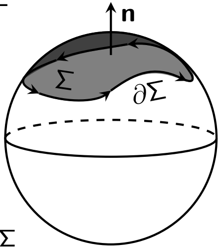
- Reminder
  - Periodic boundary conditions: $\mathbf{n}_\mathrm{L} = \mathbf{n} = \mathbf{n}_0$
  - $\hookrightarrow$ Line integral along loop $\partial\Sigma$ enclosing surface $\Sigma$
  - Parametrisation: $\theta(\tau), \varphi(\tau)$
  - $\hookrightarrow$ Loop $\partial\Sigma$ located on a sphere
- Stokes' theorem: $\oint\limits_{\partial\Sigma} \mathbf{A} \cdot \mathrm{d}\mathbf{x} = \iint\limits_{\Sigma} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n} \mathrm{d}\Sigma$




- Requirements
  - $\Sigma$: orientable; piecewise regular surfaces
  - $\partial\Sigma$: piecewise smooth curves; overall closed
  - Parametrisation: $\Sigma$ to the left, while passing $\partial\Sigma$
  - Orientation: pass surface normal $\mathbf{n}$ counterclockwise
  - $\mathbf{A}$: continuous; differentiable

# Magnetic monopole I

- Objective
  $$\int_0^\beta \mathrm{d}\tau \, [1 - \cos(\theta)] \, \dot{\varphi}$$
- Reminder
  - Periodic boundary conditions: $\mathbf{n}_\mathrm{L} = \mathbf{n} = \mathbf{n}_0$
  
  $\hookrightarrow$ Line integral along loop $\partial\Sigma$ enclosing surface $\Sigma$
  - Parametrisation: $\theta(\tau), \varphi(\tau)$
  
  $\hookrightarrow$ Loop $\partial\Sigma$ located on a sphere
- Stokes' theorem: $\oint_{\partial\Sigma} \mathbf{A} \cdot \mathrm{d}\mathbf{x} = \iint_\Sigma (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n} \mathrm{d}\Sigma$
- Requirements
  - $\Sigma$: orientable; piecewise regular surfaces
  - $\partial\Sigma$: piecewise smooth curves; overall closed
  - Parametrisation: $\Sigma$ to the left, while passing $\partial\Sigma$
  - Orientation: pass surface normal $\mathbf{n}$ counterclockwise
  - $\mathbf{A}$: continuous; differentiable




$\hookrightarrow$ New objective: determine $\mathbf{A}$

# Magnetic monopole II



- Objective

$$\underbrace{\mathbf{A}_{\mathrm{N/S}}}_{\text{label}} \text{ such, that } \oint_{\underbrace{\partial\Sigma_{\mathbf{S/N}}}_{\text{loop around south/north pole}}} \mathbf{A}_{\mathrm{N/S}} \cdot \mathrm{d}\mathbf{x} = \oint_{\partial\Sigma_{\mathbf{S/N}}} [1 - \cos(\theta)]\, \dot{\varphi}\, \mathrm{d}\tau$$

# Magnetic monopole II

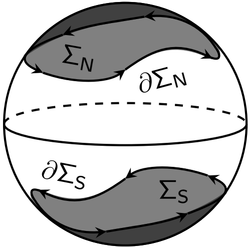


- Objective

  $\mathbf{A}_{N/S}$ such, that $\oint\limits_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{N/S}\cdot \mathrm{d}\mathbf{x} = \oint\limits_{\partial\Sigma_{\mathbf{S/N}}} [1-\cos(\theta)]\,\dot{\varphi}\,\mathrm{d}\tau$

- $\left.\begin{aligned} \mathbf{A}_N &= -\frac{1+\cos(\theta)}{\sin(\theta)}\mathbf{e}_\varphi \\ \mathbf{A}_S &= \frac{1-\cos(\theta)}{\sin(\theta)}\mathbf{e}_\varphi \end{aligned}\right\}$ with $\mathbf{e}_\varphi = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}$

- $\mathrm{d}x \ \hat{=}\ \mathrm{d}\mathbf{n}$ with $\mathbf{n} = \begin{pmatrix} \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}$



# Magnetic monopole II



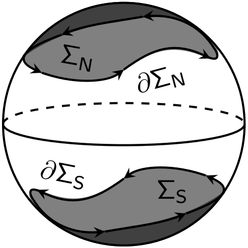
- Objective

  $\mathbf{A}_{N/S}$ such, that $\displaystyle\oint_{\partial\Sigma_{S/N}} \mathbf{A}_{N/S}\cdot d\mathbf{x} = \oint_{\partial\Sigma_{S/N}} [1-\cos(\theta)]\,\dot{\varphi}\,d\tau$

- $$\left.\begin{aligned}\mathbf{A}_N &= -\frac{1+\cos(\theta)}{\sin(\theta)}\mathbf{e}_\varphi \\ \mathbf{A}_S &= \frac{1-\cos(\theta)}{\sin(\theta)}\mathbf{e}_\varphi\end{aligned}\right\}\text{ with } \mathbf{e}_\varphi = \begin{pmatrix}-\sin(\varphi)\\ \cos(\varphi)\\ 0\end{pmatrix}$$

- $d\mathbf{x} \ \hat{=}\ d\mathbf{n}$ with $\mathbf{n} = \begin{pmatrix}\cos(\varphi)\sin(\theta)\\ \sin(\varphi)\sin(\theta)\\ \cos(\theta)\end{pmatrix}$
- Proof

  $$\oint_{\partial\Sigma_{S/N}} \mathbf{A}_{N/S}\cdot d\mathbf{n} = \oint_{\partial\Sigma_{S/N}} \mathbf{A}_{N/S}\cdot \frac{\partial \mathbf{n}}{\partial\tau}\,d\tau = ③$$



# Magnetic monopole II



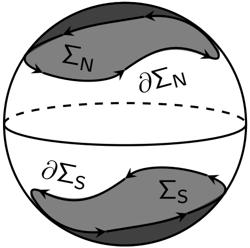
- Objective

  $\mathbf{A}_{\mathrm{N/S}}$ such, that $\displaystyle\oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \mathrm{d}\mathbf{x} = \oint_{\partial\Sigma_{\mathbf{S/N}}} [1 - \cos(\theta)]\, \dot{\varphi}\, \mathrm{d}\tau$

- $$\left.\begin{aligned} \mathbf{A}_{\mathrm{N}} &= -\frac{1+\cos(\theta)}{\sin(\theta)}\mathbf{e}_{\varphi} \\ \mathbf{A}_{\mathrm{S}} &= \frac{1-\cos(\theta)}{\sin(\theta)}\mathbf{e}_{\varphi} \end{aligned}\right\} \text{with } \mathbf{e}_{\varphi} = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}$$

- $\mathrm{d}\mathbf{x} \;\hat{=}\; \mathrm{d}\mathbf{n}$ with $\mathbf{n} = \begin{pmatrix} \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}$
- Proof

  $$\oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \mathrm{d}\mathbf{n} = \oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \frac{\partial \mathbf{n}}{\partial \tau}\, \mathrm{d}\tau = ③$$




- Auxiliary calculation

  $$\frac{\partial \mathbf{n}}{\partial \tau} = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} \sin(\theta)\dot{\varphi} + \begin{pmatrix} \cos(\varphi)\cos(\theta) \\ \sin(\varphi)\cos(\theta) \\ -\sin(\theta) \end{pmatrix} \dot{\theta} = \mathbf{e}_{\varphi} \sin(\theta)\dot{\varphi} + \mathbf{e}_{\theta}\dot{\theta}$$

# Magnetic monopole II



- Objective

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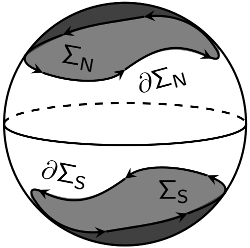
- Proof

  $$\oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathsf{N/S}} \cdot \mathrm{d}\mathbf{n} = \oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathsf{N/S}} \cdot \frac{\partial \mathbf{n}}{\partial \tau}\,\mathrm{d}\tau = ③$$




- Auxiliary calculation

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  $$③ = \oint_{\partial\Sigma_{\mathbf{S/N}}} \mp [1 \pm \cos(\theta)] \cdot \dot{\varphi}\,\mathrm{d}\tau$$

# Magnetic monopole II



- Objective

$\mathbf{A}_{\mathrm{N/S}}$ such, that $\displaystyle\oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \mathrm{d}\mathbf{x} = \oint_{\partial\Sigma_{\mathbf{S/N}}} [1 - \cos(\theta)]\, \dot{\varphi}\, \mathrm{d}\tau$

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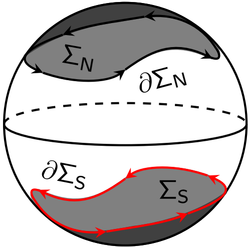
- $\mathrm{d}\mathbf{x} \;\hat{=}\; \mathrm{d}\mathbf{n}$ with $\mathbf{n} = \begin{pmatrix} \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}$
- Proof

$$\oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \mathrm{d}\mathbf{n} = \oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \frac{\partial \mathbf{n}}{\partial \tau}\, \mathrm{d}\tau = ③$$




- Auxiliary calculation

$$\frac{\partial \mathbf{n}}{\partial \tau} = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} \sin(\theta)\dot{\varphi} + \begin{pmatrix} \cos(\varphi)\cos(\theta) \\ \sin(\varphi)\cos(\theta) \\ -\sin(\theta) \end{pmatrix} \dot{\theta} = \mathbf{e}_{\varphi} \sin(\theta)\dot{\varphi} + \mathbf{e}_{\theta}\dot{\theta}$$

$$③ = \oint_{\partial\Sigma_{\mathbf{S/N}}} \mp [1 \pm \cos(\theta)] \cdot \dot{\varphi}\, \mathrm{d}\tau$$

# Magnetic monopole II

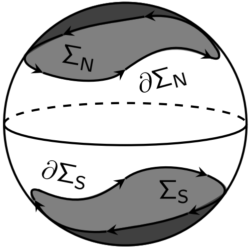
- Objective

  $\mathbf{A}_{\mathrm{N/S}}$ such, that $\displaystyle\oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \mathrm{d}\mathbf{x} = \oint_{\partial\Sigma_{\mathbf{S/N}}} [1-\cos(\theta)]\,\dot{\varphi}\,\mathrm{d}\tau$

- $$\left.\begin{aligned} \mathbf{A}_{\mathrm{N}} &= -\frac{1+\cos(\theta)}{\sin(\theta)}\mathbf{e}_{\varphi} \\ \mathbf{A}_{\mathrm{S}} &= \quad \frac{1-\cos(\theta)}{\sin(\theta)}\mathbf{e}_{\varphi} \end{aligned}\right\} \text{ with } \mathbf{e}_{\varphi} = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}$$

- $\mathrm{d}\mathbf{x} \;\hat{=}\; \mathrm{d}\mathbf{n}$ with $\mathbf{n} = \begin{pmatrix} \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}$
- Proof

  $$\oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \mathrm{d}\mathbf{n} = \oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \frac{\partial\mathbf{n}}{\partial\tau}\,\mathrm{d}\tau = ③$$




- Auxiliary calculation

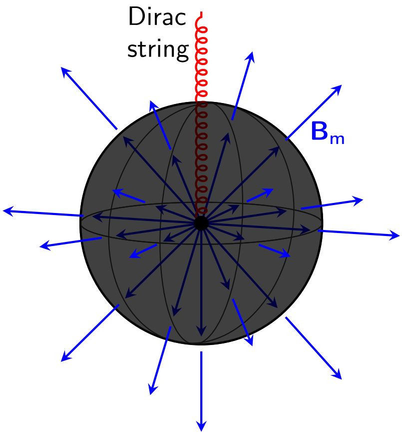
  $$\frac{\partial\mathbf{n}}{\partial\tau} = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}\sin(\theta)\dot{\varphi} + \begin{pmatrix} \cos(\varphi)\cos(\theta) \\ \sin(\varphi)\cos(\theta) \\ -\sin(\theta) \end{pmatrix}\dot{\theta} = \mathbf{e}_{\varphi}\sin(\theta)\dot{\varphi} + \mathbf{e}_{\theta}\dot{\theta}$$

  $$③ = \oint_{\partial\Sigma_{\mathbf{S/N}}} \mp[1\pm\cos(\theta)]\cdot\dot{\varphi}\,\mathrm{d}\tau \underset{\substack{\varphi\to-\varphi \\ \theta\to\theta+\pi}}{=} \oint_{\partial\Sigma_{\mathbf{S'/N}}} [1-\cos(\theta)]\cdot\dot{\varphi}\,\mathrm{d}\tau$$

# Magnetic monopole III



- Generalisation
  $\mathbf{A}_{\mathrm{N/S}} = \mp\frac{1\pm\cos(\theta)}{r\cdot\sin(\theta)}\mathbf{e}_\varphi$
- Singularities
  $\lim_{\theta\to 0}\mathbf{A}_{\mathrm{N}} \to \infty$
  $\lim_{\theta\to \pi}\mathbf{A}_{\mathrm{S}} \to \infty$
- Gauge transformation
  $\mathbf{A}_{\mathrm{S}} = \mathbf{A}_{\mathrm{N}} + 2\boldsymbol{\nabla}_{\mathrm{sc}}\varphi$
- Curl
  $\mathbf{B}_{\mathrm{m}} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r^3}\mathbf{r}$
- Action
  $\mathcal{S}_{\mathrm{Berry}} = \hbar S \oint_{\partial\Sigma_{\mathrm{S/N}}} \mathbf{A}_{\mathrm{N/S}}\cdot\dot{\mathbf{n}}\,\mathrm{d}\tau$
- Interpretation
  - $\mathbf{A}$: vector potential of a magnetic (Dirac) monopole
  - $\mathcal{S}_{\mathrm{Berry}}$: spin = massless particle with "charge" $\hbar S$ feeling $\mathbf{A}$,
    **as if** it would move within $\mathbf{B}_{\mathrm{m}}$
    (c.f. $\mathcal{L}(\mathbf{x},\mathbf{v},t) \propto q\dot{\mathbf{x}}\cdot\mathbf{A}$)



# Quantisation of spins



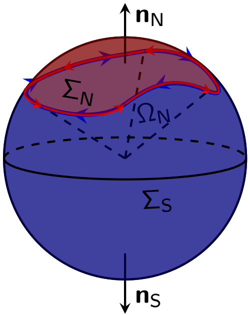
- Stokes' theorem: $\oint\limits_{\partial\Sigma} \mathbf{A} \cdot \mathrm{d}\mathbf{x} = \iint\limits_{\Sigma} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n} \mathrm{d}\Sigma$

# Quantisation of spins



- Stokes' theorem: $\oint_{\partial\Sigma} \mathbf{A} \cdot \mathrm{d}\mathbf{x} = \iint_{\Sigma} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n} \mathrm{d}\Sigma$

$\hookrightarrow$ Here: $\oint_{\partial\Sigma_{\mathrm{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \mathrm{d}\mathbf{n} = \iint_{\Sigma_{\mathrm{S/N}}} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{e}_{\mathrm{r}} \mathrm{d}\Sigma_{\mathrm{S/N}}$



# Quantisation of spins



- Stokes' theorem: $\oint\limits_{\partial\Sigma} \mathbf{A}\cdot \mathrm{d}\mathbf{x} = \iint\limits_{\Sigma} (\boldsymbol{\nabla}\times\mathbf{A})\cdot \mathbf{n}\mathrm{d}\Sigma$

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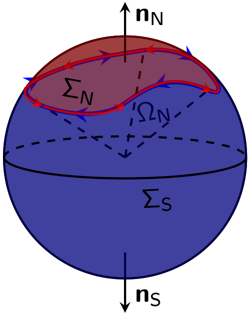
$$= \iint\limits_{\Sigma_{\mathbf{S/N}}} \mathbf{B}_{\mathrm{m}}\cdot \mathbf{e}_{\mathrm{r}}\mathrm{d}\Sigma_{\mathrm{S/N}} = \iint\limits_{\Sigma_{\mathbf{S/N}}} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{e}_{\mathrm{r}}\cdot r^2\sin(\theta)\mathrm{d}\theta\mathrm{d}\varphi$$



# Quantisation of spins

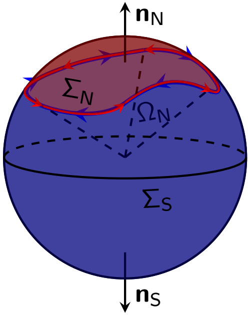


- Stokes' theorem: $\oint_{\partial\Sigma} \mathbf{A} \cdot \mathrm{d}\mathbf{x} = \iint_{\Sigma} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n} \mathrm{d}\Sigma$

$\hookrightarrow$ Here: $\oint_{\partial\Sigma_{\mathrm{S/N}}} \mathbf{A}_{\mathrm{N/S}} \cdot \mathrm{d}\mathbf{n} = \iint_{\Sigma_{\mathrm{S/N}}} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{e}_{\mathrm{r}} \mathrm{d}\Sigma_{\mathrm{S/N}}$

$$= \iint_{\Sigma_{\mathrm{S/N}}} \mathbf{B}_{\mathrm{m}} \cdot \mathbf{e}_{\mathrm{r}} \mathrm{d}\Sigma_{\mathrm{S/N}} = \iint_{\Sigma_{\mathrm{S/N}}} \frac{1}{r^3} \mathbf{r} \cdot \mathbf{e}_{\mathrm{r}} \cdot r^2 \sin(\theta) \mathrm{d}\theta \mathrm{d}\varphi$$

$$= \pm \Omega_{\mathrm{N/S}}$$



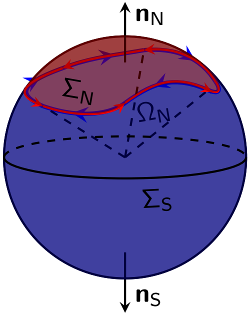
- Stokes' theorem: $\oint_{\partial\Sigma} \mathbf{A}\cdot\mathrm{d}\mathbf{x} = \iint_{\Sigma} (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{n}\mathrm{d}\Sigma$

$\hookrightarrow$ Here:

$$\oint_{\partial\Sigma_{\mathbf{S/N}}} \mathbf{A}_{\mathrm{N/S}}\cdot\mathrm{d}\mathbf{n} = \iint_{\Sigma_{\mathbf{S/N}}} (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{e}_{\mathrm{r}}\mathrm{d}\Sigma_{\mathrm{S/N}}$$
$$= \iint_{\Sigma_{\mathbf{S/N}}} \mathbf{B}_{\mathrm{m}}\cdot\mathbf{e}_{\mathrm{r}}\mathrm{d}\Sigma_{\mathrm{S/N}} = \iint_{\Sigma_{\mathbf{S/N}}} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{e}_{\mathrm{r}}\cdot r^2\sin(\theta)\mathrm{d}\theta\mathrm{d}\varphi$$
$$= \pm\Omega_{\mathrm{N/S}}$$

**Canonical partition function**

$$Z = \int\mathcal{D}\mathbf{n}\exp\left[\pm iS\Omega_{\mathrm{N/S}} - \int_0^\beta \mathrm{d}\tau\,\langle\mathbf{n}|\mathcal{H}|\mathbf{n}\rangle\right]$$



# Quantisation of spins



- Stokes' theorem: $\oint_{\partial\Sigma} \mathbf{A}\cdot \mathrm{d}\mathbf{x} = \iint_{\Sigma} (\boldsymbol{\nabla}\times\mathbf{A})\cdot \mathbf{n}\mathrm{d}\Sigma$

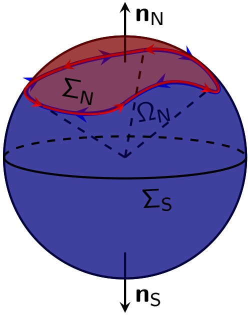
$\hookrightarrow$ Here: $\oint_{\partial\Sigma_{S/N}} \mathbf{A}_{N/S}\cdot \mathrm{d}\mathbf{n} = \iint_{\Sigma_{S/N}} (\boldsymbol{\nabla}\times\mathbf{A})\cdot \mathbf{e}_r \mathrm{d}\Sigma_{S/N}$

$$= \iint_{\Sigma_{S/N}} \mathbf{B}_m \cdot \mathbf{e}_r \mathrm{d}\Sigma_{S/N} = \iint_{\Sigma_{S/N}} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{e}_r \cdot r^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\varphi$$

$$= \pm\Omega_{N/S}$$

**Canonical partition function**

$$Z = \int \mathcal{D}\mathbf{n} \exp\left[\pm iS\Omega_{N/S} - \int_0^\beta \mathrm{d}\tau \, \langle \mathbf{n}|\mathcal{H}|\mathbf{n}\rangle\right]$$




- Definiteness

$$\exp(iS\Omega_N) \stackrel{!}{=} \exp(-iS\Omega_S)$$
$$\Leftrightarrow \exp[iS(\Omega_N + \Omega_S)] = 1$$

# Quantisation of spins



- Stokes' theorem: $\oint\limits_{\partial\Sigma} \mathbf{A}\cdot \mathrm{d}\mathbf{x} = \iint\limits_{\Sigma} (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{n}\mathrm{d}\Sigma$

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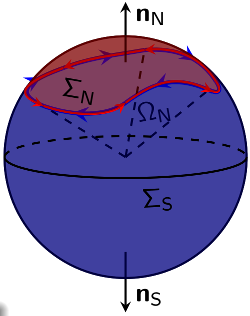
$$= \pm\Omega_{\mathrm{N/S}}$$

**Canonical partition function**

$$Z = \int \mathcal{D}\mathbf{n}\exp\left[\pm iS\Omega_{\mathrm{N/S}} - \int_0^\beta \mathrm{d}\tau\,\langle\mathbf{n}|\mathcal{H}|\mathbf{n}\rangle\right]$$




- Definiteness

$$\exp(iS\Omega_{\mathrm{N}}) \overset{!}{=} \exp(-iS\Omega_{\mathrm{S}})$$
$$\Leftrightarrow \exp[iS(\Omega_{\mathrm{N}}+\Omega_{\mathrm{S}})] = 1$$

$\hookrightarrow$ Since $\Omega_{\mathrm{N}}+\Omega_{\mathrm{S}} = 4\pi$:

**Quantisation**

$$S = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

# Results



**Canonical partition function**

$$Z = \int \mathcal{D}\mathbf{n} \exp\left[\pm iS\Omega_{\mathrm{N/S}} - \int_0^\beta \mathrm{d}\tau \, \langle \mathbf{n}|\mathcal{H}|\mathbf{n}\rangle\right]$$

- Wick rotation: $\tau \to \frac{i}{\hbar} \cdot t$

**Propagator**

$$K = \int \mathcal{D}\mathbf{n} \exp\left[\pm iS\Omega_{\mathrm{N/S}} - \frac{i}{\hbar} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \, \langle \mathbf{n}|\mathcal{H}|\mathbf{n}\rangle\right]$$

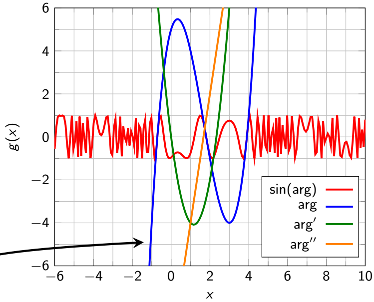
- Berry phase: $\Phi_{\mathrm{Berry}} = \Omega$
- Quantisation: $S = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

# Stationary phase approximation



- Similar methods
  - Stationary phase approximation
  - Saddle-point approximation
- Principle
  - Objective: $\int_{\mathcal{C}} \mathrm{d}z\, f(z) \mathrm{e}^{\lambda \cdot g(z)}$

    with $\mathcal{C} \in \mathbb{C}$: contour, $z \in \mathbb{C}$,
    $f(z), g(z)$: analytic, $\lambda \in \mathbb{R}$




$$\text{arg} = x^3 - 5x^2 + 3x + 5$$

  - Approximation: Neglect fast oscillations due to $\mathfrak{Im}(g(z))$

  $\hookrightarrow$ Find "stationary" points
  - Approach: "Shift" $\mathcal{C}$ onto saddle points $z_0$
    $$\left( \frac{\partial}{\partial z} g(z_0), \ldots, g^{(\mathrm{k})}(z_0) = 0, k \geq 2 \right)$$

  $\hookrightarrow$ Taylor expansion: $g(z) = g(z_0) + \frac{g^{(\mathbf{k})}(z_0)}{k!} \cdot (z - z_0)^k + \mathcal{O}\left((z - z_0)^{k+1}\right)$
- Popular case: $k = 2$ (c.f. Gaussian integral)
- Generalisation: Method of steepest descent

# Functional derivative



- Apply stationary phase approximation to path integral

$\hookrightarrow$ Function $g(x) \rightsquigarrow$ Functional $G[g(x)]$

$\hookrightarrow$ Derivative $\frac{\mathrm{d}}{\mathrm{d}x} g(x) \rightsquigarrow$ Functional Derivative $\frac{\delta}{\delta g} G[g(x)]$

- Functional derivative

$$\left| G[g+h] - G[g] - \int_{\mathcal{D}} \mathrm{d}x \frac{\delta G[g]}{\delta g}(x) h(x) \right| = o(\|h\|)$$

with $\mathcal{D}$: domain of the functions involved

- Example: $G[g] = \int_{\mathbb{R}} \mathrm{d}x\, f(x) g(x)$

$$G[g+h] = G[g] + G[h] \Leftrightarrow G[g+h] - G[g] - G[h] = 0$$
$$\Rightarrow \int \mathrm{d}x\, f(x) \frac{\delta G[g]}{\delta g}(x) \stackrel{!}{=} \int \mathrm{d}x\, f(x) h(x)$$
$$\Rightarrow \text{Comparison: } \frac{\delta G[g]}{\delta g} = h$$

# Large S expansion



- Functional derivative
  $$\left| G[g+h] - G[g] - \int \mathrm{d}x \frac{\delta G[g]}{\delta g}(x) h(x) \right| = o(\|h\|)$$
- Action
  $$\mathcal{S}[\mathbf{n}, \dot{\mathbf{n}}] = -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \hbar S\, \mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) \cdot \dot{\mathbf{n}} - H(\mathbf{n}, \dot{\mathbf{n}}, t)$$
  with $\mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) = \mp \frac{\mathbf{e_z} \times \mathbf{n}}{1 \mp \mathbf{e_z} \cdot \mathbf{n}} \mathbf{e}_\varphi$, $H(\mathbf{n}, \dot{\mathbf{n}}, t) = \langle \mathbf{n} | \mathcal{H} | \mathbf{n} \rangle$

# Large S expansion



- Functional derivative

$$\left| G[g+h] - G[g] - \int \mathrm{d}x \frac{\delta G[g]}{\delta g}(x) h(x) \right| = o(\|h\|)$$

- Action

$$\mathcal{S}[\mathbf{n}, \dot{\mathbf{n}}] = -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \hbar S\, \mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) \cdot \dot{\mathbf{n}} - H(\mathbf{n}, \dot{\mathbf{n}}, t)$$

with $\mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) = \mp \frac{\mathbf{e_z} \times \mathbf{n}}{1 \mp \mathbf{e_z} \cdot \mathbf{n}} \mathbf{e}_\varphi,\ H(\mathbf{n}, \dot{\mathbf{n}}, t) = \langle \mathbf{n} | \mathcal{H} | \mathbf{n} \rangle$

$$\hookrightarrow\ \mathcal{S}[\mathbf{n} + \boldsymbol{\eta}, \dot{\mathbf{n}} + \dot{\boldsymbol{\eta}}] = -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \hbar S\, \mathbf{A}_{\mathrm{N/S}}(\mathbf{n} + \boldsymbol{\eta}) \cdot (\dot{\mathbf{n}} + \dot{\boldsymbol{\eta}}) - H(\mathbf{n} + \boldsymbol{\eta}, \dot{\mathbf{n}} + \dot{\boldsymbol{\eta}}, t)$$

# Large S expansion



- Functional derivative

$$\left| G[g+h] - G[g] - \int \mathrm{d}x \, \frac{\delta G[g]}{\delta g}(x) h(x) \right| = o(\|h\|)$$

- Action

$$\mathcal{S}[\mathbf{n}, \dot{\mathbf{n}}] = -\int_{t_\mathbf{i}}^{t_\mathbf{f}} \mathrm{d}t \, \hbar S \, \mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) \cdot \dot{\mathbf{n}} - H(\mathbf{n}, \dot{\mathbf{n}}, t)$$

with $\mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) = \mp \frac{\mathbf{e_z} \times \mathbf{n}}{1 \mp \mathbf{e_z} \cdot \mathbf{n}} \mathbf{e}_\varphi$, $H(\mathbf{n}, \dot{\mathbf{n}}, t) = \langle \mathbf{n} | \mathcal{H} | \mathbf{n} \rangle$

$$\hookrightarrow \; \mathcal{S}[\mathbf{n}+\boldsymbol{\eta}, \dot{\mathbf{n}}+\dot{\boldsymbol{\eta}}] = -\int_{t_\mathbf{i}}^{t_\mathbf{f}} \mathrm{d}t \, \hbar S \, \mathbf{A}_{\mathrm{N/S}}(\mathbf{n}+\boldsymbol{\eta}) \cdot (\dot{\mathbf{n}}+\dot{\boldsymbol{\eta}}) - H(\mathbf{n}+\boldsymbol{\eta}, \dot{\mathbf{n}}+\dot{\boldsymbol{\eta}}, t)$$

$$\to -\int_{t_\mathbf{i}}^{t_\mathbf{f}} \mathrm{d}t \, \hbar S \, A^\alpha_{\mathrm{N/S}}(\mathbf{n}) \cdot \left( \dot{n}^\alpha + \dot{\eta}^\alpha \right) + \hbar S \, \partial_{n^\alpha} A^\beta_{\mathrm{N/S}}(\mathbf{n}) \cdot \eta^\alpha \dot{n}^\beta - H(\mathbf{n}, \dot{\mathbf{n}}, t) - \partial_{n^\alpha} H(\mathbf{n}, \dot{\mathbf{n}}, t) \cdot \eta^\alpha - \partial_{\dot{n}^\alpha} H(\mathbf{n}, \dot{\mathbf{n}}, t) \cdot \dot{\eta}^\alpha + \mathcal{O}(\eta^2)$$

Taylor expansion

# Large S expansion



- Functional derivative
  $$\left| G[g+h] - G[g] - \int \mathrm{d}x \, \frac{\delta G[g]}{\delta g}(x) h(x) \right| = o(\|h\|)$$
- Action
  $$\mathcal{S}[\mathbf{n}, \dot{\mathbf{n}}] = -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \, \hbar S \, \mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) \cdot \dot{\mathbf{n}} - H(\mathbf{n}, \dot{\mathbf{n}}, t)$$
  with $\mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) = \mp \frac{\mathbf{e_z} \times \mathbf{n}}{1 \mp \mathbf{e_z} \cdot \mathbf{n}} \mathbf{e}_\varphi$, $H(\mathbf{n}, \dot{\mathbf{n}}, t) = \langle \mathbf{n} | \mathcal{H} | \mathbf{n} \rangle$

$$\hookrightarrow \; \mathcal{S}[\mathbf{n}+\boldsymbol{\eta}, \dot{\mathbf{n}}+\dot{\boldsymbol{\eta}}] = -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \, \hbar S \, \mathbf{A}_{\mathrm{N/S}}(\mathbf{n}+\boldsymbol{\eta}) \cdot (\dot{\mathbf{n}}+\dot{\boldsymbol{\eta}}) - H(\mathbf{n}+\boldsymbol{\eta}, \dot{\mathbf{n}}+\dot{\boldsymbol{\eta}}, t)$$

$$\begin{aligned} \to -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \, & \hbar S \, A^{\alpha}_{\mathrm{N/S}}(\mathbf{n}) \cdot \left( \dot{n}^{\alpha} + \dot{\eta}^{\alpha} \right) \\ & + \hbar S \, \partial_{n^\alpha} A^{\beta}_{\mathrm{N/S}}(\mathbf{n}) \cdot \eta^{\alpha} \dot{n}^{\beta} - H(\mathbf{n}, \dot{\mathbf{n}}, t) \\ & - \partial_{n^\alpha} H(\mathbf{n}, \dot{\mathbf{n}}, t) \cdot \eta^{\alpha} - \partial_{\dot{n}^\alpha} H(\mathbf{n}, \dot{\mathbf{n}}, t) \cdot \dot{\eta}^{\alpha} \\ & + \mathcal{O}(\eta^2) \end{aligned}$$

Partial integration

with $\boldsymbol{\eta}(t_\mathrm{i}) = \mathbf{0}$, $\boldsymbol{\eta}(t_\mathrm{f}) = \mathbf{0}$

$$\begin{aligned} = S[\mathbf{n}, \dot{\mathbf{n}}] - \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \, \hbar S & \left( -\dot{A}^{\alpha}_{\mathrm{N/S}}(\mathbf{n}) + \partial_{n^\alpha} A^{\beta}_{\mathrm{N/S}}(\mathbf{n}) \dot{n}^{\beta} \right) \eta^{\alpha} \\ & - \left( \partial_{n^\alpha} H(\mathbf{n}, \dot{\mathbf{n}}, t) - \partial_{\dot{n}^\alpha} \dot{H}(\mathbf{n}, \dot{\mathbf{n}}, t) \right) \eta^{\alpha} \\ & + \mathcal{O}\left(\eta^2\right) \end{aligned}$$

# Large S expansion



- Functional derivative
  $$\left| G[g+h] - G[g] - \int \mathrm{d}x \frac{\delta G[g]}{\delta g}(x) h(x) \right| = o(\|h\|)$$
- Action
  $$\mathcal{S}[\mathbf{n}, \dot{\mathbf{n}}] = -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \hbar S\, \mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) \cdot \dot{\mathbf{n}} - H(\mathbf{n}, \dot{\mathbf{n}}, t)$$
  with $\mathbf{A}_{\mathrm{N/S}}(\mathbf{n}) = \mp \frac{\mathbf{e_z} \times \mathbf{n}}{1 \mp \mathbf{e_z} \cdot \mathbf{n}} \mathbf{e}_\varphi$, $H(\mathbf{n}, \dot{\mathbf{n}}, t) = \langle \mathbf{n} | \mathcal{H} | \mathbf{n} \rangle$

$$\begin{aligned}
\hookrightarrow\ \mathcal{S}[\mathbf{n}+\boldsymbol{\eta}, \dot{\mathbf{n}}+\dot{\boldsymbol{\eta}}] &= -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \hbar S\, \mathbf{A}_{\mathrm{N/S}}(\mathbf{n}+\boldsymbol{\eta}) \cdot (\dot{\mathbf{n}}+\dot{\boldsymbol{\eta}}) \\
&\qquad - H(\mathbf{n}+\boldsymbol{\eta}, \dot{\mathbf{n}}+\dot{\boldsymbol{\eta}}, t) \\
&\to -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \hbar S\, A^\alpha_{\mathrm{N/S}}(\mathbf{n}) \cdot \left(\dot{n^\alpha} + \dot{\eta^\alpha}\right) \\
&\qquad + \hbar S\, \partial_{n^\alpha} A^\beta_{\mathrm{N/S}}(\mathbf{n}) \cdot \eta^\alpha \dot{n}^\beta - H(\mathbf{n}, \dot{\mathbf{n}}, t) \\
&\qquad - \partial_{n^\alpha} H(\mathbf{n}, \dot{\mathbf{n}}, t) \cdot \eta^\alpha - \partial_{\dot{n}^\alpha} H(\mathbf{n}, \dot{\mathbf{n}}, t) \cdot \dot{\eta}^\alpha \\
&\qquad + \mathcal{O}(\eta^2) \\
&= S[\mathbf{n}, \dot{\mathbf{n}}] - \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \hbar S \left( -\dot{A}^\alpha_{\mathrm{N/S}}(\mathbf{n}) + \partial_{n^\alpha} A^\beta_{\mathrm{N/S}}(\mathbf{n}) \dot{n}^\beta \right) \eta^\alpha \\
&\qquad - \left( \partial_{n^\alpha} H(\mathbf{n}, \dot{\mathbf{n}}, t) - \partial_{\dot{n}^\alpha} \dot{H}(\mathbf{n}, \dot{\mathbf{n}}, t) \right) \eta^\alpha \\
&\qquad + \mathcal{O}\left(\eta^2\right)
\end{aligned}$$

$\hookrightarrow\ \frac{\delta}{\delta \mathbf{n}} \mathcal{S}$

# Equation of motion & Example: Bloch equations



- Stationarity

$$\frac{\delta}{\delta \mathbf{n}}\mathcal{S} = -\hbar S\left(-\dot{A}^{\alpha}_{\mathrm{N/S}}(\mathbf{n}) + \partial_{n^\alpha} A^{\beta}_{\mathrm{N/S}}(\mathbf{n})\dot{n}^{\beta}\right) + \left(\partial_{n^\alpha} H(\mathbf{n}, \dot{\mathbf{n}}, t) - \partial_{\dot{n}^\alpha} \dot{H}(\mathbf{n}, \dot{\mathbf{n}}, t)\right) \stackrel{!}{=} 0$$

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- Apply $\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}$ and $\dot{A}^{\alpha}_{\mathrm{N/S}}(\mathbf{n}) = \partial_{n^\beta} A^{\alpha}_{\mathrm{N/S}}(\mathbf{n})\dot{n}^{\beta}$
$\hookrightarrow -\dot{A}^{\alpha}_{\mathrm{N/S}}(\mathbf{n}) + \partial_{n^\alpha} A^{\beta}_{\mathrm{N/S}}(\mathbf{n})\dot{n}^{\beta} \to \dot{\mathbf{n}} \times \left(\boldsymbol{\nabla} \times \mathbf{A}_{\mathrm{N/S}}(\mathbf{n})\right)$
- From before: $\boldsymbol{\nabla} \times \mathbf{A}_{\mathrm{N/S}} = \mathbf{B}_{\mathrm{m}}\big|_{r=1} = \frac{1}{r^3}\mathbf{r}\Big|_{r=1} = \mathbf{n}$

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- Result
$$-\hbar S \cdot (\dot{\mathbf{n}} \times \mathbf{n}) = -\boldsymbol{\nabla}_{\mathbf{n}} H(\mathbf{n},\dot{\mathbf{n}},t) + \boldsymbol{\nabla}_{\dot{\mathbf{n}}}\dot{H}(\mathbf{n},\dot{\mathbf{n}},t)$$

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**Equation of motion**

$$\hbar S \cdot \dot{\mathbf{n}} = \mathbf{n} \times \left[\boldsymbol{\nabla}_{\mathbf{n}} H(\mathbf{n},\dot{\mathbf{n}},t) - \boldsymbol{\nabla}_{\dot{\mathbf{n}}}\dot{H}(\mathbf{n},\dot{\mathbf{n}},t)\right]$$

# Equation of motion & Example: Bloch equations

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$$\frac{\delta}{\delta \mathbf{n}}\mathcal{S} = -\hbar S\left(-\dot{A}^{\alpha}_{\mathrm{N/S}}(\mathbf{n}) + \partial_{n^\alpha} A^{\beta}_{\mathrm{N/S}}(\mathbf{n})\dot{n}^{\beta}\right) + \left(\partial_{n^\alpha} H(\mathbf{n},\dot{\mathbf{n}},t) - \partial_{\dot{n}^\alpha}\dot{H}(\mathbf{n},\dot{\mathbf{n}},t)\right) \overset{!}{=} 0$$

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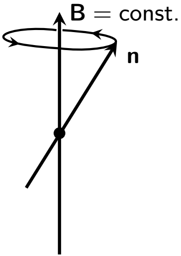


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- Example: Bloch equations
  - Hamiltonian: $\mathcal{H} = -\gamma \mathbf{B}\cdot\hat{\boldsymbol{\mathcal{S}}} \to H(\mathbf{n},\dot{\mathbf{n}},t) = \gamma\hbar S\mathbf{B}\cdot\mathbf{n}$

$\hookrightarrow \dot{\mathbf{n}} = \gamma \cdot (\mathbf{n} \times \mathbf{B})$

# Adiabatic time evolution & Berry phase



- Adiabatic theorem
  If the Hamiltonian $\mathcal{H}(t)$ governing the time evolution of a system changes **slow enough**, the system remains in its *n*-th eigenstate $|\psi_{\mathsf{n}}\rangle$, where *n* denotes the quantum number.

# Adiabatic time evolution & Berry phase



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- Principle
  - Slow changing parameter $\mathbf{R}(t)$
  - At time $t$: $\mathcal{H}(\mathbf{R}(t))\,|\psi_n(\mathbf{R}(t))\rangle = E_n(\mathbf{R}(t))\,|\psi_n(\mathbf{R}(t))\rangle$
  - Objective: Time evolution $|\psi_n(\mathbf{R}(t_0), t_0; t)\rangle$

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  - Adiabatic change: $|\psi_n(\mathbf{R}(t_0), t_0; t)\rangle \propto |\psi_n(\mathbf{R}(t))\rangle$
  - Educated guess:
    $$|\psi_n(\mathbf{R}(t_0), t_0; t)\rangle = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}t'\, E_n(\mathbf{R}(t'))\right] e^{i\Phi_n(t)}\,|\psi_n(\mathbf{R}(t))\rangle$$

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  $\hookrightarrow$ Insert $|\psi_n(\mathbf{R}(t_0), t_0; t)\rangle$:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\,|\psi_n(\mathbf{R}(t_0), t_0; t)\rangle &= \left(E_n(\mathbf{R}(t)) - \hbar\dot{\Phi}_n(t)\right)|\psi_n(\mathbf{R}(t_0), t_0; t)\rangle \\ &\quad + i\hbar\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, E_n(\mathbf{R}(t'))\right] e^{i\Phi_n(t)}\,|\dot{\psi}_n(\mathbf{R}(t))\rangle \end{aligned}$$

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$\hookrightarrow$ $\dot{\Phi}_\mathrm{n}(t) = i\,\langle\psi_\mathrm{n}(\mathbf{R}(t))|\frac{\partial}{\partial t}|\psi_\mathrm{n}(\mathbf{R}(t))\rangle \in \mathbb{R}$

# Example: Adiabatic evolution of a spin



- Objective: $\Phi = \int_0^\beta \mathrm{d}\tau\, \mathfrak{Im}[\langle\psi(t)|\frac{\partial}{\partial\tau}|\psi(t)\rangle]$
  with $|\psi(t=0)\rangle$: ground state
- Hamiltonian: $\mathcal{H}(\tau) = -\kappa(\tau)\mathbf{n}(\tau)\cdot\hat{\boldsymbol{\mathcal{S}}}$ with $\kappa(\tau)>0$, $\hat{\boldsymbol{\mathcal{S}}} = \frac{\hbar}{2}S\boldsymbol{\sigma}$,
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$\hookrightarrow$ Eigenenergies: $E_{\mathbf{n}} = \mp\frac{\hbar}{2}\kappa(\tau) \rightsquigarrow$ Ground state: $|\mathbf{n}+\rangle$

$\hookrightarrow$ Spectral representation:

$$\mathcal{U} = \mathbb{1}\left(\exp\left[\frac{i}{2}\kappa(\tau)St\right]|\mathbf{n}+\rangle\langle\mathbf{n}+| + \exp\left[-\frac{i}{2}\kappa(\tau)St\right]|\mathbf{n}-\rangle\langle\mathbf{n}-|\right)$$

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$\hookrightarrow |\psi(t)\rangle = \mathcal{U}\,|\mathbf{n}+\rangle = \exp\left[\frac{i}{2}\kappa(\tau)St\right]|\mathbf{n}+\rangle$

$\hookrightarrow$ $\frac{\partial}{\partial\tau}|\psi(t)\rangle = \exp[\frac{i}{2}\kappa(\tau)St]\cdot\left[\frac{i}{2}\frac{\partial\kappa}{\partial\tau}St\,|\mathbf{n}+\rangle - \frac{\dot\theta}{2}|\mathbf{n}-\rangle + i\dot\varphi\sin\left(\frac{\theta}{2}\right)e^{i\varphi}\mathbf{e_z}\right]$

# Example: Adiabatic evolution of a spin



- Objective: $\Phi = \int_0^\beta \mathrm{d}\tau\, \mathfrak{Im}[\langle\psi(t)|\frac{\partial}{\partial\tau}|\psi(t)\rangle]$
  with $|\psi(t=0)\rangle$: ground state
- Hamiltonian: $\mathcal{H}(\tau) = -\kappa(\tau)\mathbf{n}(\tau)\cdot\hat{\boldsymbol{\mathcal{S}}}$ with $\kappa(\tau)>0$, $\hat{\boldsymbol{\mathcal{S}}} = \frac{\hbar}{2}S\boldsymbol{\sigma}$,
  parametrisation $\tau\in[0,\beta]$

$\hookrightarrow$ Operator method: $|\psi(t)\rangle = \mathcal{U}(t)\,|\psi(0)\rangle \rightsquigarrow i\hbar\frac{\partial}{\partial t}\mathcal{U}(t) = \mathcal{H}(\tau)\mathcal{U}(t)$

- Adiabatic approximation: $\mathcal{H}(\tau(t)) \approx \mathcal{H}(\tau)$

$\hookrightarrow$ Solution: $\mathcal{U} = \mathbb{1}\exp\left[-\frac{i}{\hbar}\mathcal{H}(\tau)t\right] = \mathbb{1}\exp\left[\frac{i}{2}\kappa(\tau)S\sigma_{\mathbf{n}}t\right]$

- Properties of $\sigma_{\mathbf{n}}$
  - Eigenvalues: $\sigma_{\mathbf{n}\pm} = \pm 1$
  - Eigenvectors: $|\mathbf{n}+\rangle = \begin{pmatrix}\cos(\frac{\theta}{2})\\ \sin(\frac{\theta}{2})e^{i\varphi}\end{pmatrix}$, $|\mathbf{n}-\rangle = \begin{pmatrix}\sin(\frac{\theta}{2})\\ -\cos(\frac{\theta}{2})e^{i\varphi}\end{pmatrix}$

$\hookrightarrow$ Eigenenergies: $E_{\mathbf{n}} = \mp\frac{\hbar}{2}\kappa(\tau) \rightsquigarrow$ Ground state: $|\mathbf{n}+\rangle$

$\hookrightarrow$ Spectral representation:

$$\mathcal{U} = \mathbb{1}\left(\exp\left[\frac{i}{2}\kappa(\tau)St\right]|\mathbf{n}+\rangle\langle\mathbf{n}+| + \exp\left[-\frac{i}{2}\kappa(\tau)St\right]|\mathbf{n}-\rangle\langle\mathbf{n}-|\right)$$

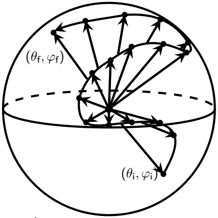
$\hookrightarrow |\psi(t)\rangle = \mathcal{U}\,|\mathbf{n}+\rangle = \exp\left[\frac{i}{2}\kappa(\tau)St\right]|\mathbf{n}+\rangle$

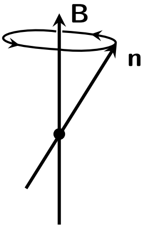
$\hookrightarrow$ $\frac{\partial}{\partial\tau}|\psi(t)\rangle = \exp[\frac{i}{2}\kappa(\tau)St]\cdot\left[\frac{i}{2}\frac{\partial\kappa}{\partial\tau}St\,|\mathbf{n}+\rangle - \frac{\dot\theta}{2}\,|\mathbf{n}-\rangle + i\dot\varphi\sin\left(\frac{\theta}{2}\right)e^{i\varphi}\mathbf{e_z}\right]$

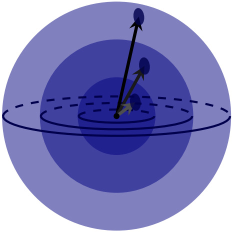
- Adiabatic approximation: $\frac{\partial\kappa}{\partial\tau}\to 0 \rightsquigarrow \Phi = iS\int_0^\beta \mathrm{d}\tau\,[1-\cos(\theta)]\dot\varphi$

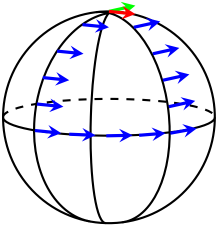
# Conclusion

- Path-integral in discrete and continuous form
  ↪ Canonical partition function and Propagator
- Berry phase: Solid angle enclosed by path
  ↪ Quantisation of spins: $S = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$
- Equation of motion for spins without constraints
  ↪ Bloch equations
  ↪ Extension via constraints (c.f. Lagrange multiplier)
- Adiabatic time evolution: Approximative description
  ↪ Berry phase





# Outlook & Applications



- Path-integral
  - Canonical partition function

  $\hookrightarrow$ Thermodynamic properties
  - Propagator

  $\hookrightarrow$ Transition probability
  - Equations of motion

  $\hookrightarrow$ Description of single spins
  - Continuum limit

  $\hookrightarrow$ Description of systems of spins: Ferromagnets, Antiferromagnets, Spin waves
- Berry phase: Realisation of a controlled phase shift gate in NMR
  - Suitable phase shift:
    Controlled phase shift gate $\rightarrow$ Part of a cNOT-gate

  $\hookrightarrow$ Important part of a quantum computer
  - Geometric nature of phase: Phase resilient to errors

# References & Further reading



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# Addendum: Quantum Ferromagnet



- Hamiltonian: $\mathcal{H} = -|J| \sum_{\langle i,j \rangle} \hat{\boldsymbol{\mathcal{S}}}_i \cdot \hat{\boldsymbol{\mathcal{S}}}_j$

$\hookrightarrow$ With $|\mathbf{n}\rangle = \bigotimes_{\mathrm{i}=1}^{N} |\mathbf{n}_{\mathrm{i}}\rangle$: $H(\mathbf{n}) = \langle \mathbf{n}|\mathcal{H}|\mathbf{n}\rangle = -|J|S^2 \sum_{\langle i,j \rangle} \mathbf{n}_i \cdot \mathbf{n}_j$

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- Continuum limit
  - Normal vector: $\mathbf{n}_\mathrm{i} \cdot \mathbf{n}_\mathrm{j} \overset{\mathbf{n}^2 = 1}{=} \frac{1}{2} \cdot \left[\mathbf{n}_\mathrm{i}^2 - 2\mathbf{n}_\mathrm{i} \cdot \mathbf{n}_\mathrm{j} + \mathbf{n}_\mathrm{j}^2 - 2\right]$

$$= \frac{1}{2} \cdot [\mathbf{n}_\mathrm{i} - \mathbf{n}_\mathrm{j}]^2 - 1$$

$\hookrightarrow$ $H(\mathbf{n}) = -\frac{|J|}{2} S^2 \sum_{\langle i,j \rangle} [\mathbf{n}_\mathrm{i} - \mathbf{n}_\mathrm{j}]^2 - \sum_{\langle i,j \rangle} 1$

# Addendum: Quantum Ferromagnet

- Motivation
- Recapitulation
- Path-integral
- Free spin-field
- Large S expansion
- Adiabatic evolution
- Conclusion
- **Applications**

- Hamiltonian: $\mathcal{H} = -|J| \sum_{\langle i,j \rangle} \hat{\boldsymbol{\mathcal{S}}}_i \cdot \hat{\boldsymbol{\mathcal{S}}}_j$

$\hookrightarrow$ With $|\mathbf{n}\rangle = \bigotimes_{\mathrm{i}=1}^{N} |\mathbf{n}_\mathrm{i}\rangle$: $H(\mathbf{n}) = \langle \mathbf{n}|\mathcal{H}|\mathbf{n}\rangle = -|J|S^2 \sum_{\langle i,j \rangle} \mathbf{n}_i \cdot \mathbf{n}_j$

- Continuum limit
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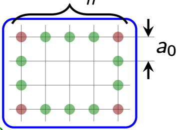
    $= \frac{1}{2} \cdot [\mathbf{n}_\mathrm{i} - \mathbf{n}_\mathrm{j}]^2 - 1$




$\hookrightarrow$ $H(\mathbf{n}) = -\frac{|J|}{2} S^2 \sum_{\langle i,j \rangle} [\mathbf{n}_\mathrm{i} - \mathbf{n}_\mathrm{j}]^2 - \sum_{\langle i,j \rangle} 1$

  - Cubic lattice in 2D: $\sum_{\langle i,j \rangle} 1 = N^2 - 4 \cdot 2 - 4 \cdot (N-2)$
    $= N(N-4) = \mathrm{const.}$
  - Spatial allocation: $H(\mathbf{n}) = -\frac{|J|}{2} S^2 \sum_{\mathbf{r}} [\mathbf{n}(\mathbf{r}) - \mathbf{n}(\mathbf{r} - \Delta\mathbf{r})]^2 + \mathrm{const.}$
  - Limit: $H(\mathbf{n}) \to H(\mathbf{n}(\mathbf{r})) = -\frac{|J|}{2} S^2 \int \frac{\mathrm{d}^d r}{a_0^{d-2}} [\nabla_\mathbf{r} \mathbf{n}(\mathbf{r})]^2 + \mathrm{const.}$

$\hookrightarrow$ Action: $\mathcal{S}[\mathbf{n}(\mathbf{r})] = -\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \hbar S \left[\int \frac{\mathrm{d}^d r}{a_0^d} \mathbf{A}_{\mathrm{N/S}}(\mathbf{n}(\mathbf{r})) \cdot \dot{\mathbf{n}}(\mathbf{r})\right] - H(\mathbf{n}(\mathbf{r}))$

# Addendum: Quantum Ferromagnet

- Hamiltonian: $\mathcal{H} = -|J| \sum_{\langle i,j \rangle} \hat{\boldsymbol{\mathcal{S}}}_i \cdot \hat{\boldsymbol{\mathcal{S}}}_j$

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- Continuum limit
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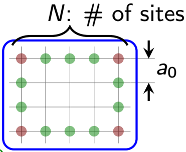


  - Spatial allocation: $H(\mathbf{n}) = -\frac{|J|}{2} S^2 \sum_{\mathbf{r}} [\mathbf{n}(\mathbf{r}) - \mathbf{n}(\mathbf{r} - \Delta\mathbf{r})]^2 + \text{const.}$
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- Large S expansion
  - Equation of motion: $\hbar S \cdot \dot{\mathbf{n}}(\mathbf{r}) = \mathbf{n}(\mathbf{r}) \times [\nabla_\mathbf{n} H(\mathbf{n}(\mathbf{r}))]$
  - With $\nabla_\mathbf{n} [\nabla_\mathbf{r} \mathbf{n}(\mathbf{r})]^2 = 2[\nabla_\mathbf{r} \cdot \nabla_\mathbf{r} \mathbf{n}] \cdot \nabla_\mathbf{n} \mathbf{r} = 2 \triangle_\mathbf{r} \mathbf{n}(\mathbf{r})$

  $\hookrightarrow$ $\hbar S \cdot \dot{\mathbf{n}}(\mathbf{r}) = |J| S^2 a_0^2 \cdot \mathbf{n}(\mathbf{r}) \times \triangle_\mathbf{r} \mathbf{n}(\mathbf{r})$