The nonlinear σ-model in two dimensions

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Quantum field-theory of low dimensional systems

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Content

Classical Heisenberg model

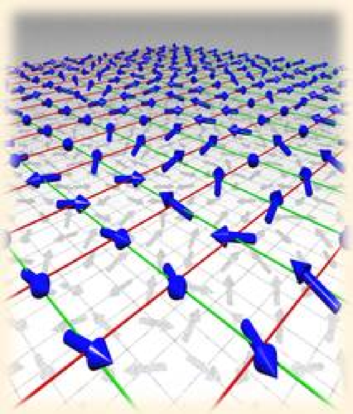
- The classical Heisenberg model is the n = 3 case of the n-vector model

$$H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

$$J > 0,\ \mathbf{S}_i \in \mathbb{R}^3, |\mathbf{S}_i| = 1$$

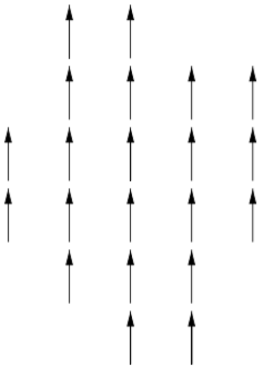
$$Z = \sum_{\{\mathbf{S}_i\}} e^{-\beta H}$$

$$= \sum_{\{\mathbf{S}_i\}} e^{\tilde{g} \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j},\ \ \tilde{g} = J\beta$$



Classical Heisenberg model

- All lattice spins aligned at $T = 0$



Classical Heisenberg model

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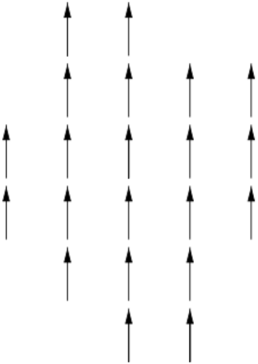
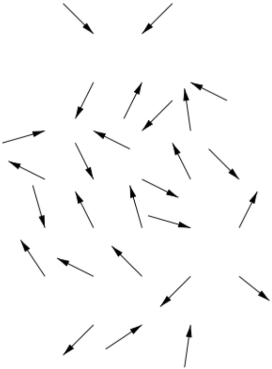
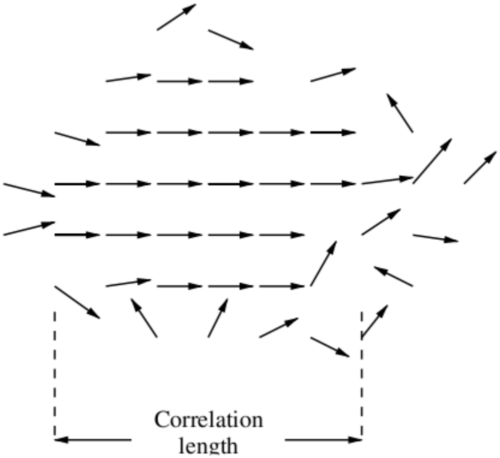


- Lattice spins randomly oriented at high temperatures



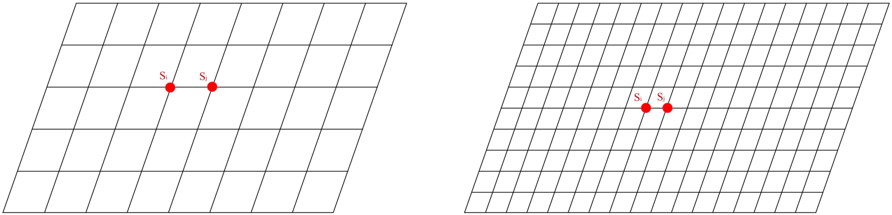
Classical Heisenberg model

- Correlated groups of spins as low temperatures are approached



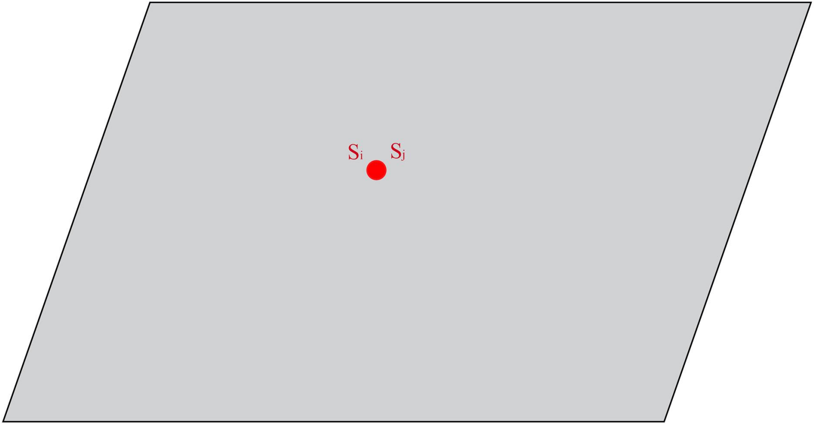
Continuum limit

- $\xi > a$



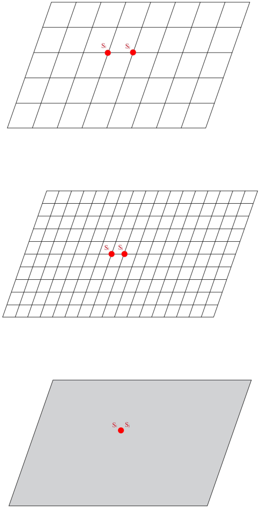
Continuum limit

- $a \to 0$, lattice $\to$ continuum



Continuum limit

- For low temperatures $\frac{\xi}{a} \to \infty$ if $a \to 0$



$$\mathbf{S}_j \approx \mathbf{S}_i + \partial_\mu \mathbf{S}_i a_\mu + \frac{1}{2}\partial_\mu \partial_\nu \mathbf{S}_i a_\mu a_\nu$$

Continuum limit

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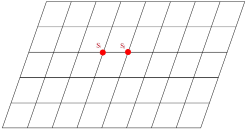

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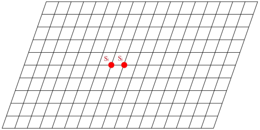
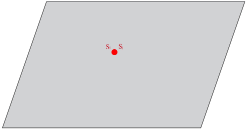


$$\begin{aligned}\mathbf{S}_j \cdot \mathbf{S}_i &\approx \mathbf{S}_i \cdot \mathbf{S}_i + \mathbf{S}_i \partial_\mu \mathbf{S}_i a_\mu + \frac{1}{2}\mathbf{S}_i \partial_\mu \partial_\nu \mathbf{S}_i a_\mu a_\nu \\ &= 1 + \frac{1}{2}\partial_\mu \left(\mathbf{S}_i \cdot \mathbf{S}_i\right) a_\mu + \frac{1}{2}\mathbf{S}_i \partial_\mu \partial_\mu \mathbf{S}_i a^2 \\ &= 1 + \frac{1}{2}\mathbf{S}_i \left(\partial_\mu\right)^2 \mathbf{S}_i a^2\end{aligned}$$



Continuum limit

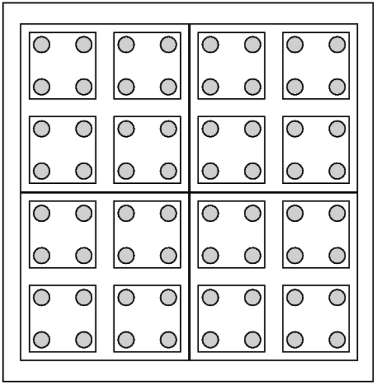
$$\begin{aligned}\sum_{\langle i,j\rangle} \mathbf{S}_i \mathbf{S}_j &\approx 2 \cdot \sum_i \frac{1}{2} \mathbf{S}_i \left(\partial_\mu\right)^2 \mathbf{S}_i a^2 \\ &= \frac{1}{a^d} \int \mathbf{S} \left(\partial_\mu\right)^2 \mathbf{S} a^2 d^d x \\ &= \frac{-1}{a^{d-2}} \int \partial_\mu \mathbf{S}\ \partial_\mu \mathbf{S}\ d^d x\end{aligned}$$

$$Z = \sum_{\{\mathbf{S}_i\}} e^{\tilde{g} \sum_{\langle i,j\rangle} \mathbf{S}_i \cdot \mathbf{S}_j} \to \int \mathcal{D}\mathcal{S}\ \delta\left(\mathbf{S}^2 - 1\right)\ e^{\frac{-\tilde{g}}{a^{d-2}} \int \partial_\mu \mathbf{S}\ \partial_\mu \mathbf{S}\ d^d x}$$

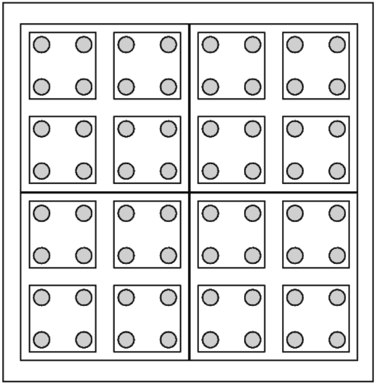
Continuum limit $\to$ nonlinear σ-model

$$S = \frac{1}{g} \int \partial_\mu \mathbf{S}\ \partial_\mu \mathbf{S}\ d^d x \qquad \frac{1}{g} = \frac{\tilde{g}}{a^{d-2}}$$

Block spin



Block spin



- Divide the solid into blocks of 2 $\times 2$ squares

$$Z = \sum_{\{\sigma\}} \exp\left(\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j\right)$$

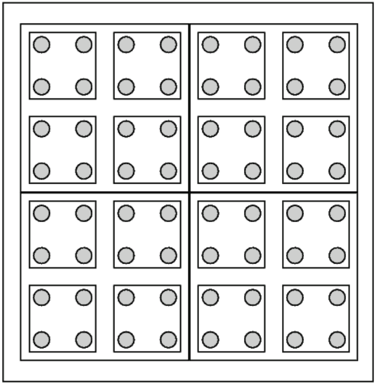
$$\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \to \sigma'_1$$

$$Z = \sum_{\{\sigma'\}} \exp\left(\sum_{\langle i,j\rangle} J'_{ij}\sigma'_i\sigma'_j\right)$$

$$\{\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4\} \to \sigma''_1$$

$$\cdots$$

Block spin



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$$\ldots.$$

RG FLow

- The change in the parameters is implemented by a certain β-function:

$$\kappa \frac{\partial g}{\partial \kappa} = \beta(g)$$

- β-function $\rightarrow$ RG flow. The values of g under the flow are called running couplings

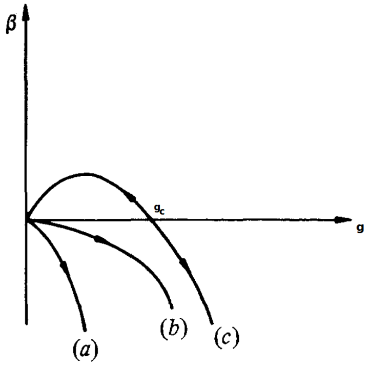
- The possible macroscopic states of a system, at a large scale, are given by a set of fixed points

RG FLow

- RG-flow equation

$$\kappa \, \frac{\partial g}{\partial \kappa} = \beta(g)$$

- small $\kappa \to$ flow in the infra-red
- no fixed point at finite temperatures for $d \leq 2$
- for $d > 2$ theory has a fixed point at $T_c \to$ phase transition



a): $d < 2$
b): $d = 2$
c): $d > 2$

Equations of motion

$$\mathcal{L} = \frac{1}{2}\partial_\mu \mathbf{\Phi} \cdot \partial^\mu \mathbf{\Phi} - U\left(\Phi(x,t)\right), \quad \mathbf{\Phi}(\mathbf{x},\mathbf{t}) = [\Phi_i(x,t); i = 1, ..., N]$$

$$\begin{aligned}(\delta S)_\Phi &= \delta \int \left[\frac{1}{2}\partial_\mu \mathbf{\Phi} \cdot \partial^\mu \mathbf{\Phi} - U\left(\mathbf{\Phi}(x,t)\right)\right] d^d x = 0 \\ &= \int \left[-\partial_\mu \partial^\mu \mathbf{\Phi} - \frac{dU}{d\mathbf{\Phi}}\right] \delta \mathbf{\Phi} d^d x = 0\end{aligned}$$

$$\Rightarrow \partial_\mu \partial^\mu \mathbf{\Phi} + \frac{dU}{d\mathbf{\Phi}} = \left(\partial_t^2 - \nabla^2\right) \mathbf{\Phi} + \frac{dU}{d\mathbf{\Phi}} = 0$$

Virial Theorem

- $U\left(\mathbf{\Phi}(x,t)\right)$ ist assumed to be non-negative and only vanishes at its absolute minima $U(\mathbf{\Phi}) \geq 0$
- A static solution $\mathbf{\Phi}(x)$ obeys

$$\nabla^2 \mathbf{\Phi} = \frac{dU}{d\mathbf{\Phi}} \tag{1}$$

- (1) extremum condition for $W[\Phi] \to \delta W[\Phi] = 0$

$$\begin{aligned} W[\Phi] &= \int \left[\frac{1}{2}\nabla_i \Phi \cdot \nabla_i \Phi + U\left(\Phi(x)\right)\right] d^d x \\ &= V_1[\Phi] + V_2[\Phi] \end{aligned}$$

Virial Theorem

- Cosider one parameter family of solutions of a static solution $\mathbf{\Phi}_1(x)$

$$\begin{aligned}\mathbf{\Phi}_\lambda &= \mathbf{\Phi}_1(\lambda x)\\ &\Rightarrow W[\Phi_\lambda] = V_1[\Phi_\lambda] + V_2[\Phi_\lambda]\\ &\qquad\qquad = \lambda^{2-d} V_1[\Phi] + \lambda^{-d} V_2[\Phi]\end{aligned}$$

- $\mathbf{\Phi}_1(x)$ is an extremum of $W[\Phi] \Rightarrow \frac{d}{d\lambda} W[\Phi_\lambda] = 0$ at $\lambda = 1$

$$(2-d)\;\; V_1[\Phi_1] = d\;\, V_2[\Phi_1]$$

- There is no non-trivial static space-dependent solution for $d \geq 3$.

The nonlinear O(3) model: The isotropic ferromagnet

- The model consists of 3 scalar fields

$$\mathbf{\Phi}(\mathbf{x},t) = \{\Phi_1(\mathbf{x},t), \Phi_2(\mathbf{x},t), \Phi_3(\mathbf{x},t)\}$$

- The dynamics of the system is given by a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sum_{\mu}\sum_{a}\left(\partial_\mu \Phi_a \partial^\mu \Phi_a\right) = \frac{1}{2}\left(\partial_\mu \mathbf{\Phi}\right)\cdot\left(\partial^\mu \mathbf{\Phi}\right)$$

- The system is subject to a constraint

$$\sum_a \Phi_a(\mathbf{x},t)^2 = \mathbf{\Phi}\cdot\mathbf{\Phi} = 1$$

The nonlinear O(3) model: The isotropic ferromagnet

- Constraint is imposed via a Lagrange multiplier

$$S[\mathbf{\Phi}] = \int d^d x \int dt \left[\frac{1}{2} (\partial_\mu \mathbf{\Phi}) \cdot (\partial^\mu \mathbf{\Phi}) + \lambda(\mathbf{x}, t) (\mathbf{\Phi} \cdot \mathbf{\Phi} - 1) \right]$$

- Obtained field equations from the Lagrangian

$$\partial_\mu \partial^\mu \mathbf{\Phi} + \lambda \mathbf{\Phi} = (\Box + \lambda) \mathbf{\Phi} = 0$$

- Lagrange mutliplier is eliminated by the given constraint

$$\lambda(\mathbf{x}, t) = \lambda \mathbf{\Phi} \cdot \mathbf{\Phi} = -\mathbf{\Phi} \Box \mathbf{\Phi}$$

The nonlinear O(3) model: The isotropic ferromagnet

- Field equations for static solutions in 2 dimensions

$$\nabla^2 \mathbf{\Phi} - \left(\mathbf{\Phi} \cdot \nabla^2 \mathbf{\Phi}\right) \mathbf{\Phi} = 0$$

- Constraint $\Rightarrow$ field configurations can be classified into homotopy sectors

The nonlinear O(3) model: What is a homotopy?

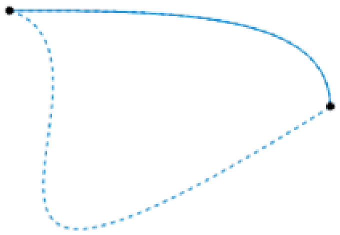
Definition

Let X, Y be topological spaces and $f, g : X \to Y$ continuous maps.

A **homotopy** from f to g is a continuous function $H : X \times [0,1] \to Y$ satisfying $H(x,0) = f(x)$ and $H(x,1) = g(x)$, for all $x \in X$.

If such a homotopy exists , we say that f is homotopic to g, and denote this by $f \simeq g$.

The nonlinear O(3) model: What is a homotopy?



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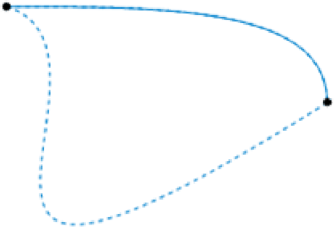
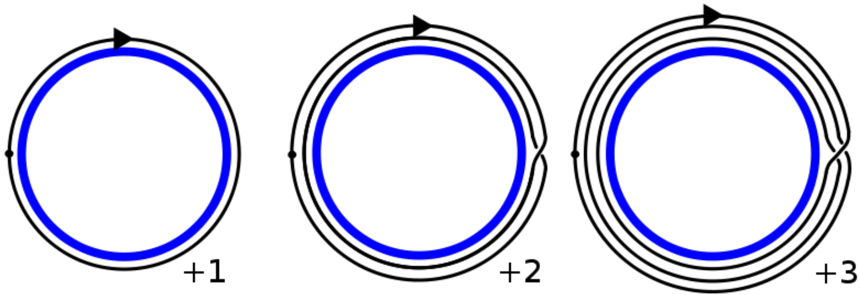


The nonlinear O(3) model: What is a homotopy?



$$S^1 \to S^1$$

The nonlinear O(3) model: Zero energy solution

- Energy of a static solution

$$E = \frac{1}{2}\int (\partial_\mu \mathbf{\Phi}) \cdot (\partial_\mu \mathbf{\Phi})\, d^2x$$

- Solutions for $E = 0$

$$\Rightarrow \partial_\mu \mathbf{\Phi}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \Rightarrow \mathbf{\Phi}(\mathbf{x}) = \mathbf{\Phi}^{(0)}$$

- O(3) symmetry $\Rightarrow$ continuous family of degenerate classical minima

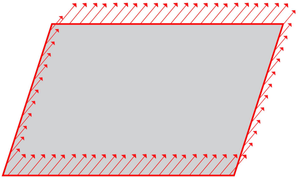
The nonlinear O(3) model: soliton solutions of finite energy

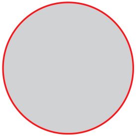
$$\frac{1}{2}\int (\partial_\mu \mathbf{\Phi})\cdot(\partial_\mu \mathbf{\Phi})\, d^2x > 0 \text{ but finite}$$

- Using polar coordinates (r, ϕ), finite energy solutions in $\mathbb{R}^2$ must satisfy the following conditions:

$$r\,||\mathbf{grad}\mathbf{\Phi}|| \to 0 \quad \text{as } r \to \infty$$

$$\lim_{r\to\infty} \mathbf{\Phi}(x) = \mathbf{\Phi}^{(0)} \quad \text{since} \quad \partial_2 \mathbf{\Phi} = \frac{1}{r}\frac{\partial \mathbf{\Phi}}{\partial \phi}$$





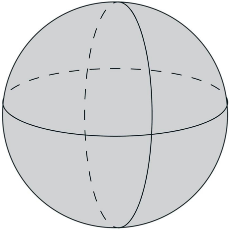
The nonlinear O(3) model: soliton solutions of finite energy

- Since $\mathbf{\Phi}(x)$ approaches the same value $\mathbf{\Phi}^{(0)}$ at all points in infinity, the plane in $\mathbb{R}^2$ can be compactified into a spherical surface $S_2^{(\text{phy})}$

- The circle at infinity is reduced to the northpole of the sphere

The nonlinear O(3) model: soliton solutions of finite energy

- Internal space $S_2^{(\text{int})}$ and coordinate space $S_2^{(\text{phy})}$ are both spherical surfaces $\Rightarrow \mathbf{\Phi} : S_2^{(\text{phy})} \to S_2^{(\text{int})}$



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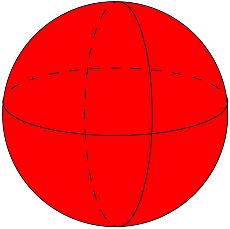


- All non-singular mappings of S_2 into S_2 can be classified into homotopy sectors.

The nonlinear O(3) model: soliton solutions of finite energy

Q can be written as an integral

$$Q = \frac{1}{8\pi} \int \epsilon_{\mu\nu} \mathbf{\Phi} \cdot (\partial_\mu \mathbf{\Phi} \times \partial_\nu \mathbf{\Phi})\, d^2x$$

- $S_2^{(\text{int})}$ can be described by $\{\xi_1, \xi_2\} \rightarrow \; dS_a^{(\text{int})} = \frac{1}{2}\epsilon_{rs}\epsilon_{abc}\frac{\partial\Phi_b}{\partial\xi_r}\frac{\partial\Phi_c}{\partial\xi_s}d^2\xi$

$$\begin{aligned}
Q &= \frac{1}{8\pi}\int \epsilon_{\mu\nu}\epsilon_{abc}\Phi_a\frac{\partial\Phi_b}{\partial x^\mu}\frac{\partial\Phi_c}{\partial x^\nu}d^2x \\
&= \frac{1}{8\pi}\int \epsilon_{\mu\nu}\epsilon_{abc}\Phi_a\frac{\partial\Phi_b}{\partial\xi_r}\frac{\partial\xi_r}{\partial x_\mu}\frac{\partial\Phi_c}{\partial\xi_s}\frac{\partial\xi_s}{\partial x_\nu}d^2x \\
Q &= \frac{1}{8\pi}\int \epsilon_{rs}\epsilon_{abc}\Phi_a\frac{\partial\Phi_b}{\partial\xi_r}\frac{\partial\Phi_c}{\partial\xi_s}d^2\xi \quad \text{as} \quad \epsilon_{rs}d^2\xi = \epsilon_{\mu\nu}\frac{\partial\xi_r}{\partial x_\mu}\frac{\partial\xi_s}{\partial x_\nu}d^2x
\end{aligned}$$

The nonlinear O(3) model: soliton solutions of finite energy

$$dS_a^{(\text{int})} = \frac{1}{2}\epsilon_{rs}\epsilon_{abc}\frac{\partial \Phi_b}{\partial \xi_r}\frac{\partial \Phi_c}{\partial \xi_s}d^2\xi \tag{2}$$

$$Q = \frac{1}{8\pi}\int \epsilon_{rs}\epsilon_{abc}\Phi_a\frac{\partial \Phi_b}{\partial \xi_r}\frac{\partial \Phi_c}{\partial \xi_s}d^2\xi \tag{3}$$

- Inserting (2) in (3) gives

$$Q = \frac{1}{4\pi}\int dS_a^{(\text{int})}\Phi_a = \frac{1}{4\pi}\int dS^{(\text{int})}$$

- Q gives the number of times the internal sphere is traversed as the coordinate space $\mathbb{R}^2$ (which is compacted into $S_2^{(\text{phy})}$) is spanned

The nonlinear O(3) model: soliton solutions of finite energy

- Homotopy classification is valid for any static field configuration with finite energy $\rightarrow$ not necessarily a solution of the field equation
- Using the following identity

$$\int \left[(\partial_\mu \mathbf{\Phi} \pm \epsilon_{\mu\nu} \mathbf{\Phi} \times \partial_\nu \mathbf{\Phi}) \cdot (\partial_\mu \mathbf{\Phi} \pm \epsilon_{\mu\sigma} \mathbf{\Phi} \times \partial_\sigma \mathbf{\Phi}) \right] d^2x \geq 0 \tag{4}$$

- Expanding (4) and using the constraint $\mathbf{\Phi} \cdot \mathbf{\Phi} = 1$ yields

$$2 \int d^2x \, (\partial_\mu \mathbf{\Phi}) \cdot (\partial_\mu \mathbf{\Phi}) \geq \pm 2 \int d^2x \epsilon_{\mu\nu} \mathbf{\Phi} \cdot (\partial_\mu \mathbf{\Phi} \times \partial_\nu \mathbf{\Phi})$$
$$E \geq 4\pi \, |Q|$$

The nonlinear O(3) model: soliton solutions of finite energy

- In any given sector Q the energy is minimised when $E = 4\pi\,|Q|$ is satisfied. This happens if and only if

$$\partial_\mu \mathbf{\Phi} = \pm \epsilon_{\mu\nu} \mathbf{\Phi} \times (\partial_\nu \mathbf{\Phi}) \quad \text{is satisfied} \tag{5}$$

- (5) can be further simplified by using a stereographic projection $\mathbf{\Phi} = \{\Phi_1, \Phi_2, \Phi_3\} \to \{\omega_1, \omega_2\}$

$$\omega_1 = \frac{2\Phi_1}{1-\Phi_3} \qquad \omega_2 = \frac{2\Phi_2}{1-\Phi_3}$$

The nonlinear O(3) model: soliton solutions of finite energy

- The stereographic projection $\{\Phi_1, \Phi_2, \Phi_3\} \to \{\omega_1, \omega_2\}$ gives us the Cauchy-Riemann equations

$$\frac{\partial \omega_1}{\partial x_1} = \pm \frac{\partial \omega_2}{\partial x_2}$$
$$\frac{\partial \omega_1}{\partial x_2} = \mp \frac{\partial \omega_2}{\partial x_1}$$

- A prototype solution for an arbitrary positive Q is given by

$$\omega(z) = \frac{[(z - z_0)]^n}{\lambda^n}, n \in \mathbb{N}, \lambda \in \mathbb{R}, z \in \mathbb{C}$$

The nonlinear O(3) model: Skyrmion solution

$$\omega_1 = \frac{2\Phi_1}{1-\Phi_3} \qquad \omega_2 = \frac{2\Phi_2}{1-\Phi_3} \qquad \omega(z) = \left(\frac{z-z_0}{\lambda}\right)^n$$

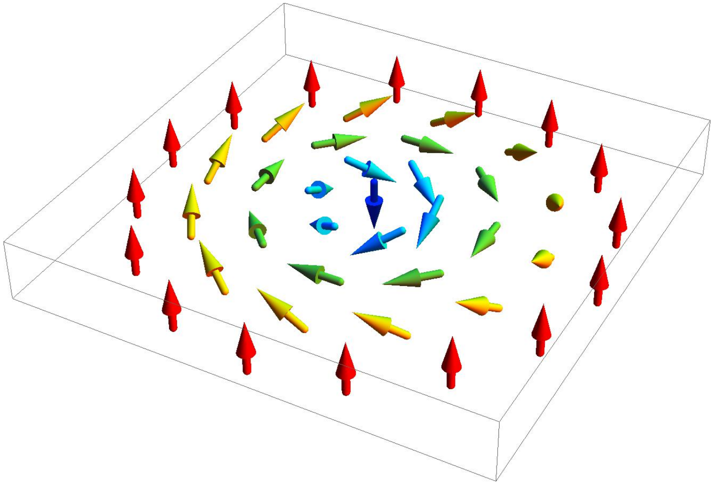
- Solution for $n = 1$ and $z_0 = 0$

$$|\omega|^2 = \omega_1^2 + \omega_2^2 = \frac{r^2}{\lambda^2} \qquad \Phi_1 = \omega_1 \frac{(1-\Phi_3)}{2} \qquad \Phi_2 = \omega_2 \frac{(1-\Phi_3)}{2}$$

- Using the constraint $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 1 \rightarrow \Phi_3 = \frac{\frac{r^2}{\lambda^2}-1}{\frac{r^2}{\lambda^2}+1}$
- Final solution $\rightarrow$ Skyrmion

$$\mathbf{\Phi} = \frac{1}{1+\left(\frac{r}{2\lambda}\right)^2}\left\{\frac{r\cos(\phi)}{\lambda}, \frac{r\sin(\phi)}{\lambda}, \left(\frac{r}{2\lambda}\right)^2 - 1\right\}$$

The nonlinear O(3) model: Skyrmion solution



Reference: `http://www.ph.utexas.edu/~karin/research-talks-and-posters.html`

Summary

- Continuum limit $\rightarrow$ nonlinear σ-model

$$S = \frac{1}{g} \int \partial_\mu \Phi_a \partial_\mu \Phi_a d^d x$$

- Constraint $\mathbf{\Phi} \cdot \mathbf{\Phi} = 1 \rightarrow \nabla^2 \mathbf{\Phi} - \left(\mathbf{\Phi} \cdot \nabla^2 \mathbf{\Phi} \right) \mathbf{\Phi} = 0$
- Finite energy solutions: $\mathbb{R}^2 \rightarrow S^2$
- Homotopy classification for static field configurations

$$Q = \frac{1}{8\pi} \int \epsilon_{rs} \epsilon_{abc} \Phi_a \frac{\partial \Phi_a}{\partial \xi_r} \frac{\partial \Phi_c}{\partial \xi_s} d^2 \xi$$

$$E \geq 4\pi \left| Q \right|$$

Summary

- Lower bound for the energy $E \geq 4\pi |Q| \Rightarrow$ first-order differential equation

$$\partial_\mu \mathbf{\Phi} = \pm \epsilon_{\mu\nu} \mathbf{\Phi} \times (\partial_\nu \mathbf{\Phi}) \tag{6}$$

- Using a stereographic projection $S_2^{(int)} \to \mathbb{R}^2 \Rightarrow \omega = \omega_1 + i\omega_2$

$$\frac{\partial \omega_1}{\partial x_1} = \pm \frac{\partial \omega_2}{\partial x_2} \qquad \frac{\partial \omega_1}{\partial x_2} = \mp \frac{\partial \omega_2}{\partial x_1}$$

- Solution to Cauchy-Riemann equations is well known $\to$ Skyrmion

$$\mathbf{\Phi} = \frac{1}{1 + \left(\frac{r}{2\lambda}\right)^2} \left\{ \frac{r\cos(\phi)}{\lambda}, \frac{r\sin(\phi)}{\lambda}, \left(\frac{r}{2\lambda}\right)^2 - 1 \right\}$$

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