The non-linear sigma model and topological excitations

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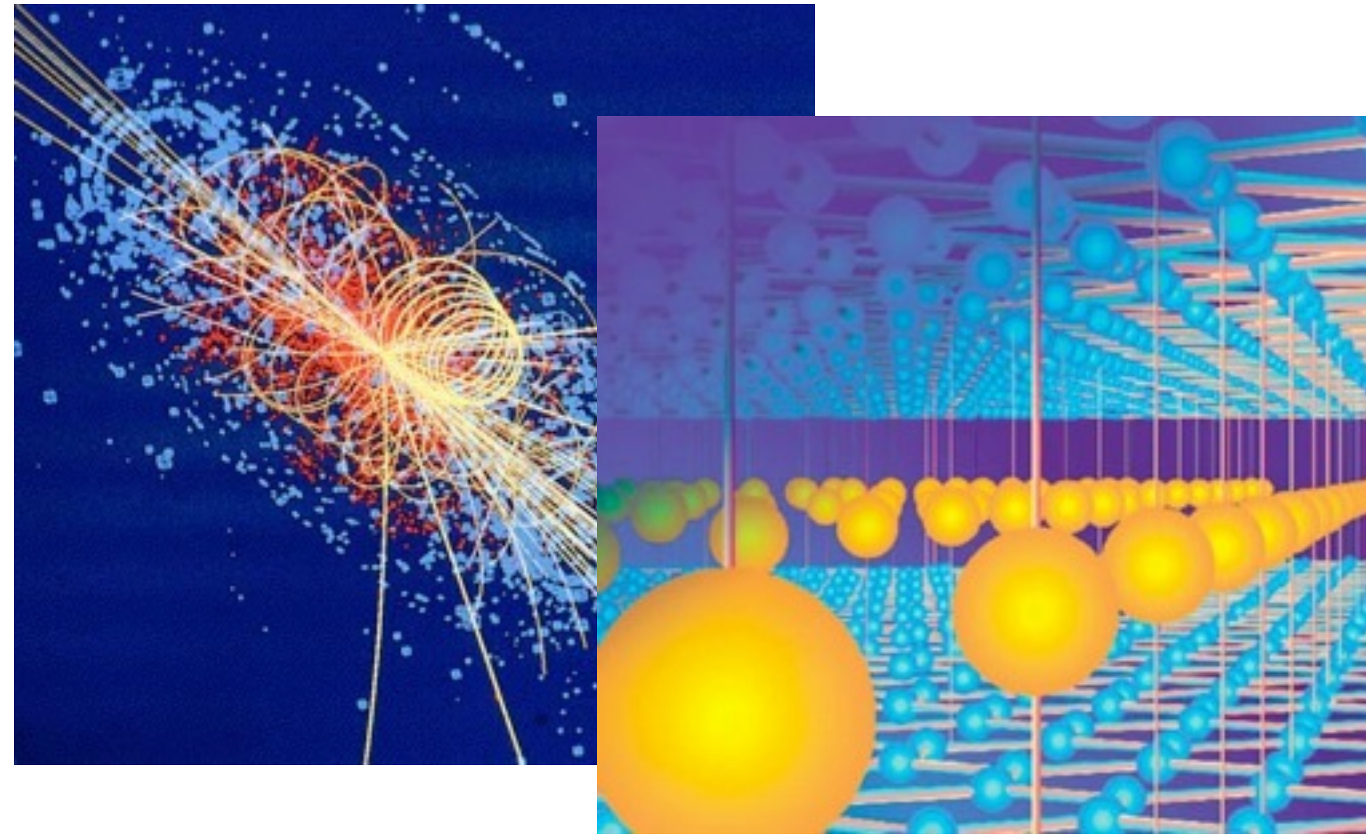
Motivation

- Field-theory in physics
  - Elementary particles
  - Statistical physics
  - Condensed matter physics



- Path-integral quantisation
  - The non-linear sigma model
  - Topological excitations

Why???

Why the non-linear sigma model?

- **Field-theory for classical ferromagnets**

  Talk by Andreas Löhle

- **Field-theory for quantum antiferromagnets**

  Talks by Marco Ströbel and Wolfgang Voesch

- **Field-theory for disordered electronic systems**

  W. Bernreuther and F. Wegner, Phys. Rev. Lett. 57, 1383 (1986)

- **Field-theory for frustrated quantum antiferromagnets**

  Deconfined quantum criticality → fractionalisation

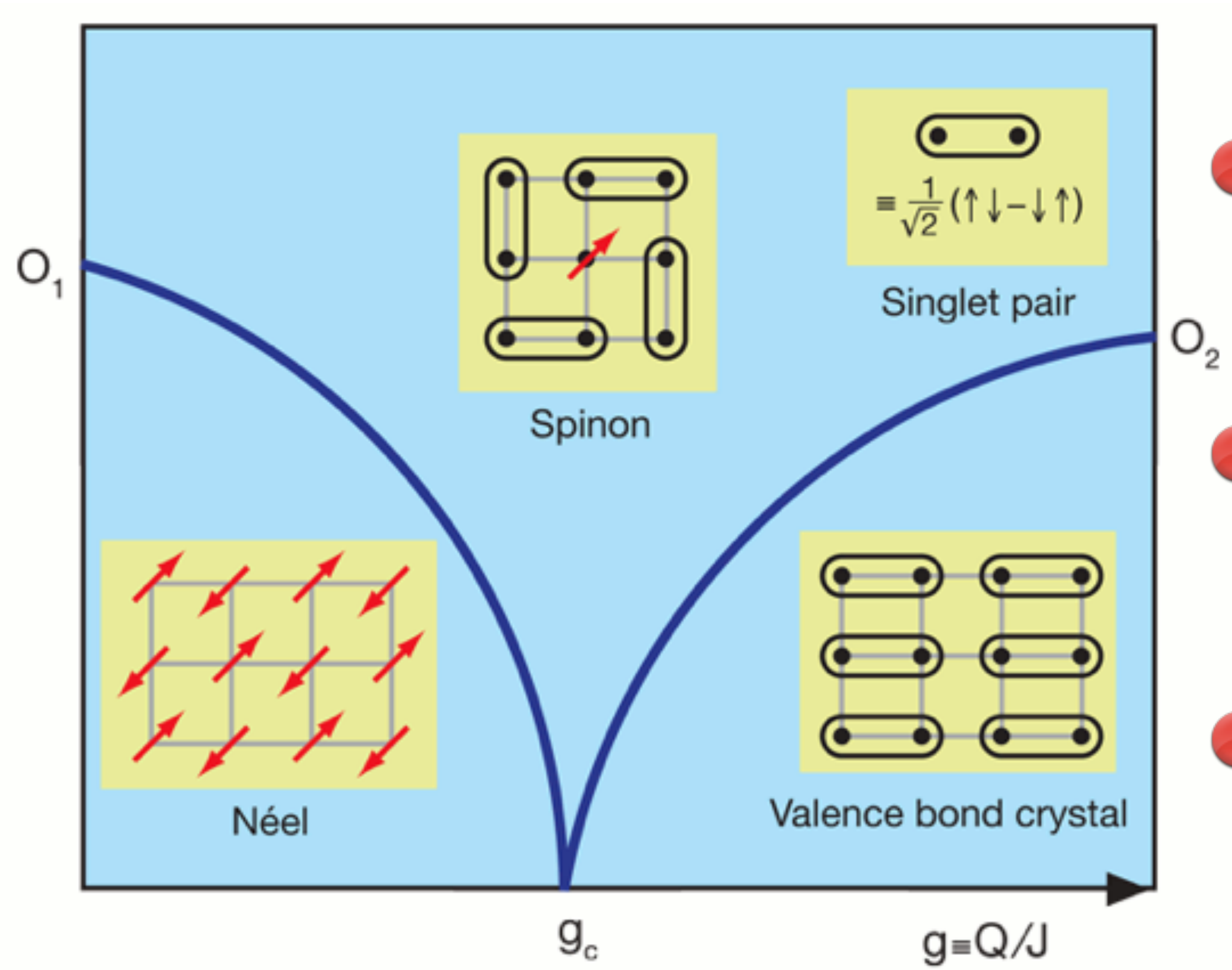
  T. Senthil, et al., Science 303, 1490 (2004)

Why the non-linear sigma model?

- **Field-theory for frustrated quantum antiferromagnets**

Deconfined quantum criticality → fractionalisation

T. Senthil, et al., Science 303, 1490 (2004)



- Continuous transition between two different broken symmetries
- Escapes the Landau-Ginzburg paradigm
- Fractionalized excitations in the quantum critical region

Why topology?

- **Field-theory for classical ferromagnets**

  Talk by Andreas Löhle

- **Field-theory for quantum antiferromagnets**

  Talks by Marco Ströbel and Wolfgang Voesch

- **Topological insulators and superconductors**

  X-.L. Qi and S-.C. Zhang, Rev. Mod. Phys. 83, 1057 (2011)

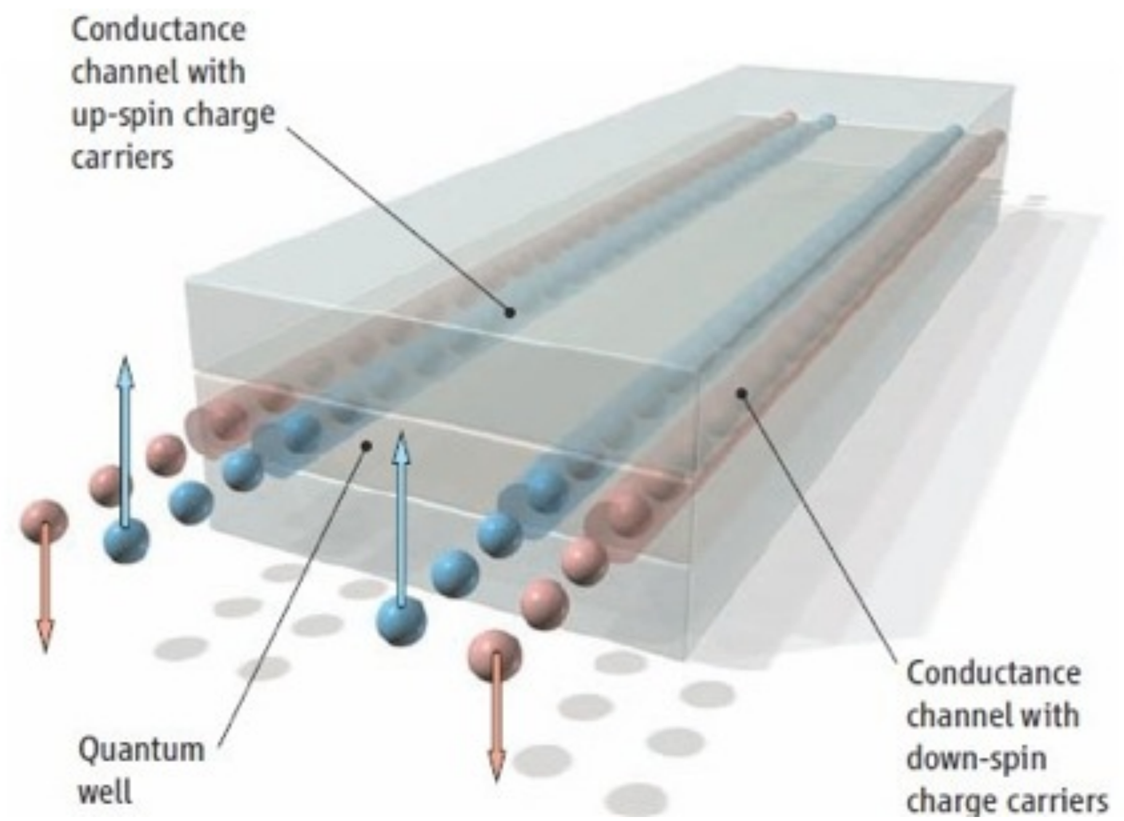
- **Topological quantum computing**

  C. Nayak, et al., Rev. Mod. Phys. 80, 1083 (2008)

Topological insulators

Insulators in bulk + spin current carrying edge states

→ **Quantum spin Hall insulator**



Experimental observation in HgTe quantum wells

M. König *et al.*, Science **318**, 766 (2007)

Topological insulators

Massless Dirac fermions in solid-state

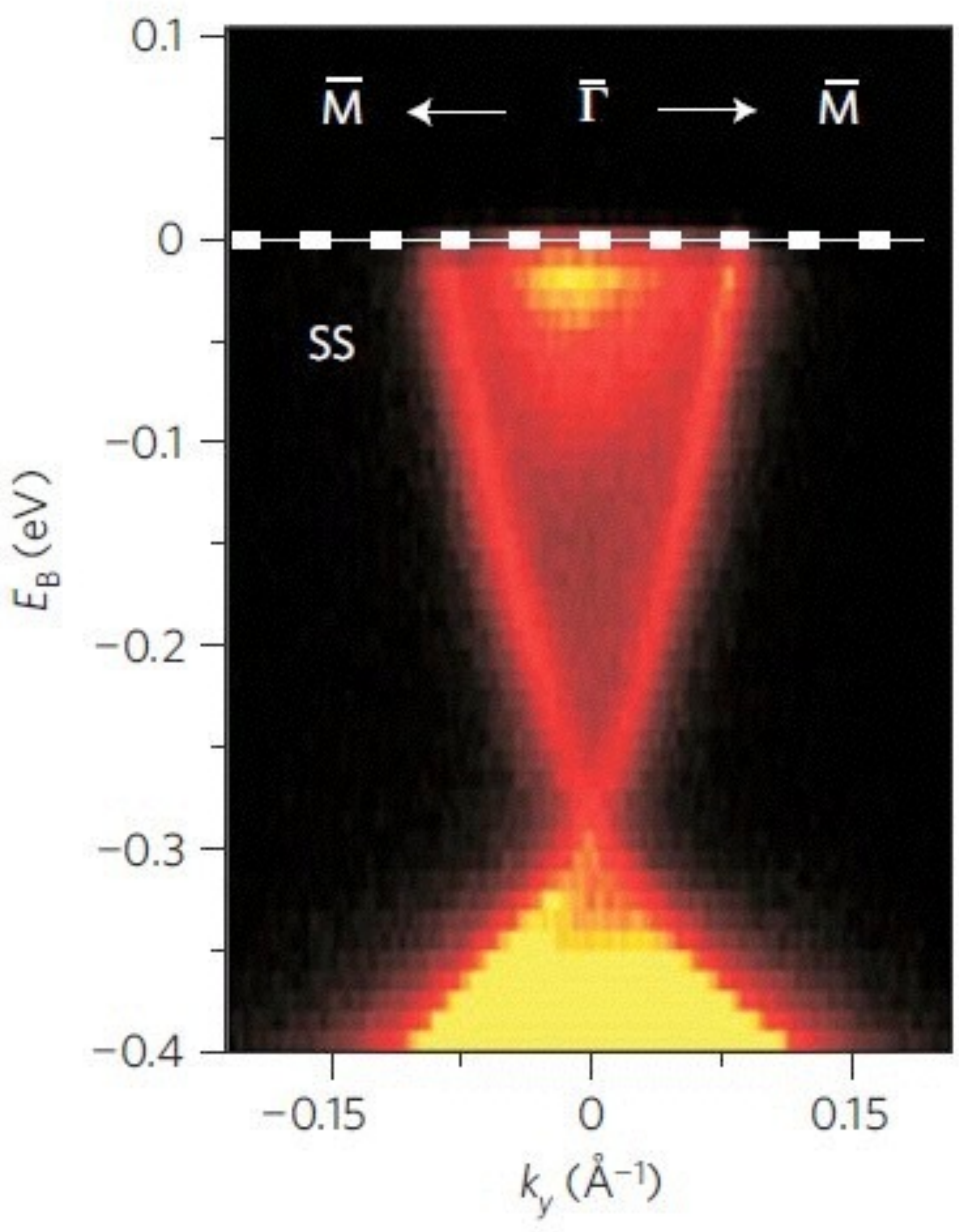
Topological insulators in three dimensions

S. Fu, C. L. Kane, E. J. Mele, PRL **98**, 106803 (2007)

Dirac cone at the surface

ARPES in Bi_2Se_3

Y. Xia *et al*, Nat. Phys. **5**, 398 (2009)



The non-linear sigma model

The non-linear sigma model

- **Action**

$$S = \frac{1}{2g} \int \mathrm{d}^d x \, \partial_\mu n^a \partial_\mu n^a \, , \qquad a = 1, 2, 3 \, , \qquad \vec{n}^2 = 1$$

- **Partition function**

$$Z = \int \mathcal{D}\vec{n}(\vec{x}) \, \delta\left(\vec{n}^2 - 1\right) \, \mathrm{e}^{-S}$$

- **Weak coupling:** g **small** → **ordered phase**

- **Strong coupling:** g **large** → **disordered phase**

The non-linear sigma model

- **Resolve the constraint** $\vec{n} = (\vec{\pi}, \sigma) \rightarrow \sigma = \sqrt{1-\vec{\pi}^2}$
- **Action**

$$S = \frac{1}{2g} \int \mathrm{d}^d x \left[\partial_\mu \pi^a \partial_\mu \pi^a + \partial_\mu \sqrt{1-\vec{\pi}^2} \partial_\mu \sqrt{1-\vec{\pi}^2} \right]$$

- **Partition function**

$$Z \propto \int \frac{\mathcal{D}\vec{\pi}}{\sqrt{1-\vec{\pi}^2}} \, \mathrm{e}^{-S}$$ → no constraint

- **Change variables** $\pi^a \rightarrow \sqrt{g}\pi^a$

$$\rightarrow \quad S = \frac{1}{2} \int \mathrm{d}^d x \left\{ \partial_\mu \pi^a \partial_\mu \pi^a + \frac{1}{g} \partial_\mu \sqrt{1-g\vec{\pi}^2} \partial_\mu \sqrt{1-g\vec{\pi}^2} \right\}$$

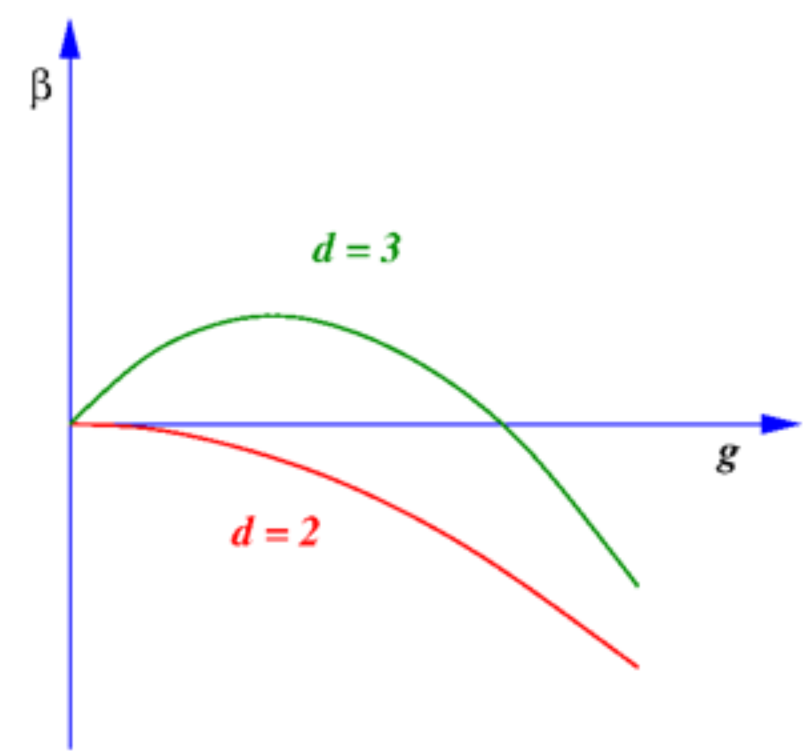
The non-linear sigma model and renormalization group

- Spontaneous symmetry breaking
  - Continuous phase transition → $\xi \to \infty$

    → scale invariance

- Renormalization group: mapping of systems at different length scales

$$\beta = \kappa \frac{\partial g}{\partial \kappa}$$



- Non-linear sigma model

$$\beta = \epsilon g - (N-2) g^2$$

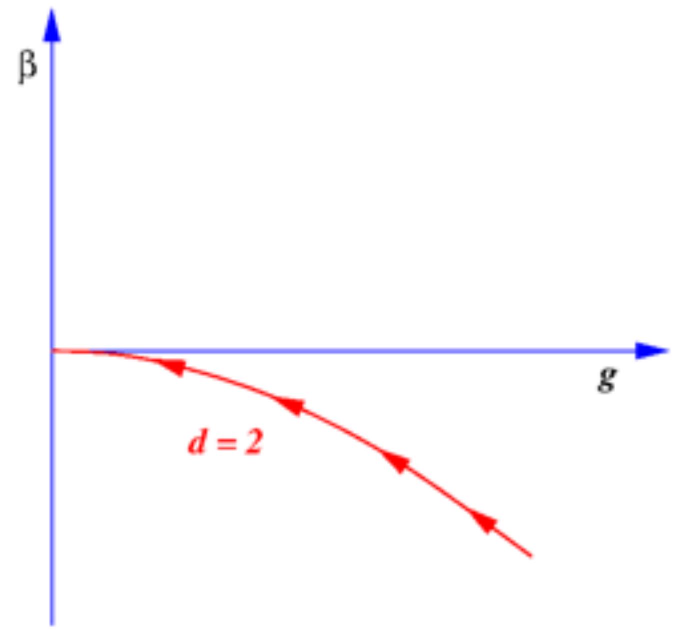
$$\epsilon = d - 2$$

N : number of components

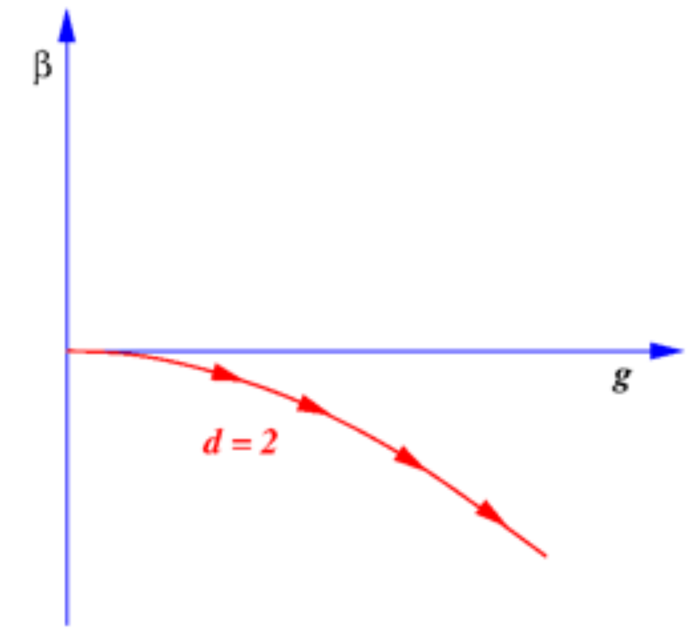
Renormalization group flow

- **d=2:** $$\beta = \kappa\frac{\partial g}{\partial\kappa} = -(N-2)g^2$$

- **UV flow:** $\kappa \longrightarrow \infty$



- **IR flow:** $\kappa \longrightarrow 0$



- **Asymptotic freedom for** $\kappa \longrightarrow \infty$

- **Spontaneous mass generation for** $\kappa \longrightarrow 0$

D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973)
H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973)

Renormalization group flow

- d>2: $\beta = \epsilon g - (N-2)g^2$ → $\beta = 0$ for $g^* = \dfrac{\epsilon}{N-2}$

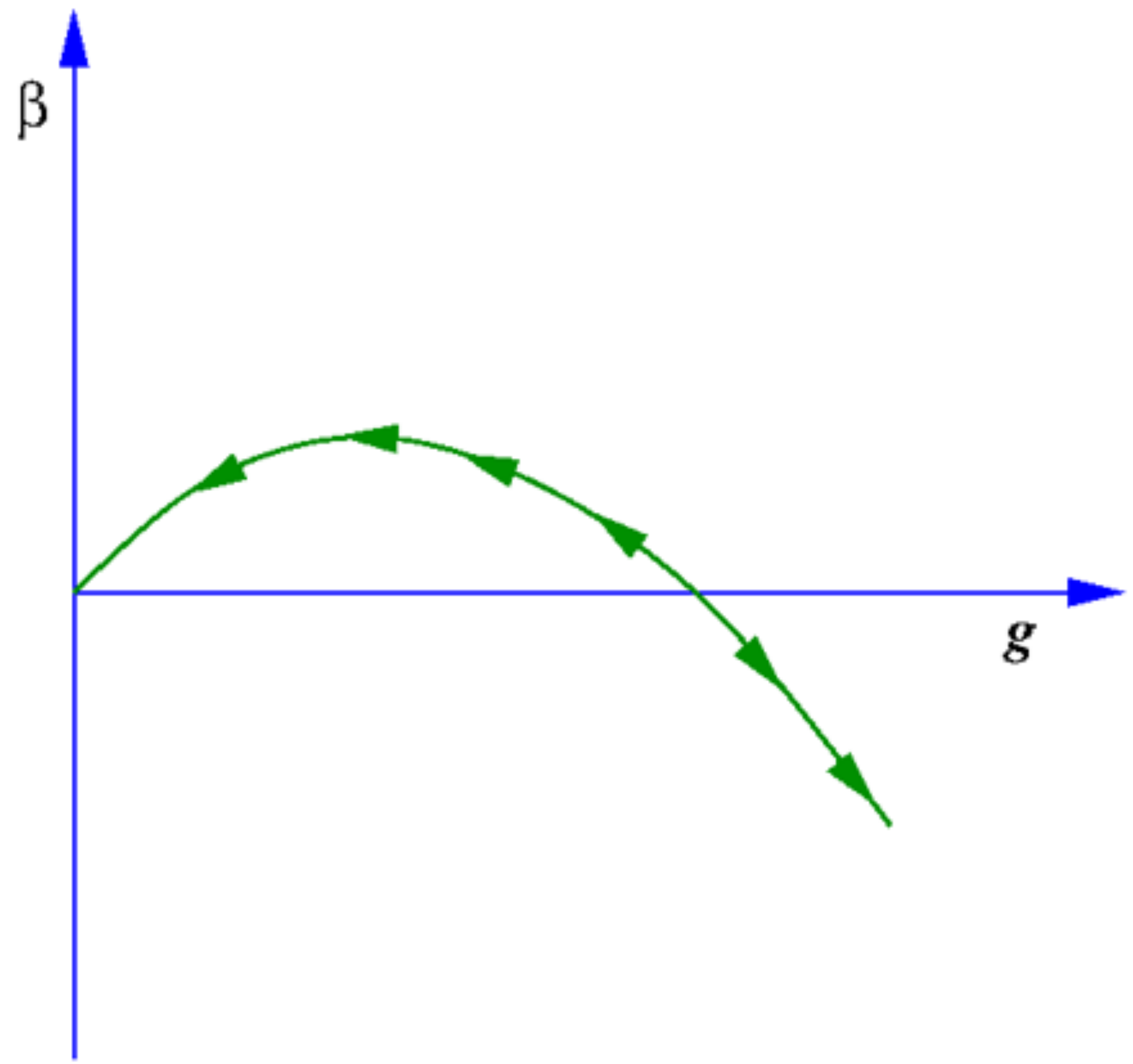
  ↓

  g^* : unstable fixed point

  ↓

  critical point

- IR flow: $\kappa \to 0$



The non-linear sigma model and topological invariants

The non-linear sigma model

- **Action**

$$S = \frac{1}{2g}\int \mathrm{d}^d x\, \partial_\mu n^a \partial_\mu n^a \,, \qquad a = 1,2,3\,, \qquad \vec{n}^2 = 1$$

- **Conserved currents:** $\partial_\mu j^\mu = 0$

$$j^\mu = \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\vec{n}\cdot(\partial_\nu \vec{n}\times\partial_\lambda \vec{n})$$

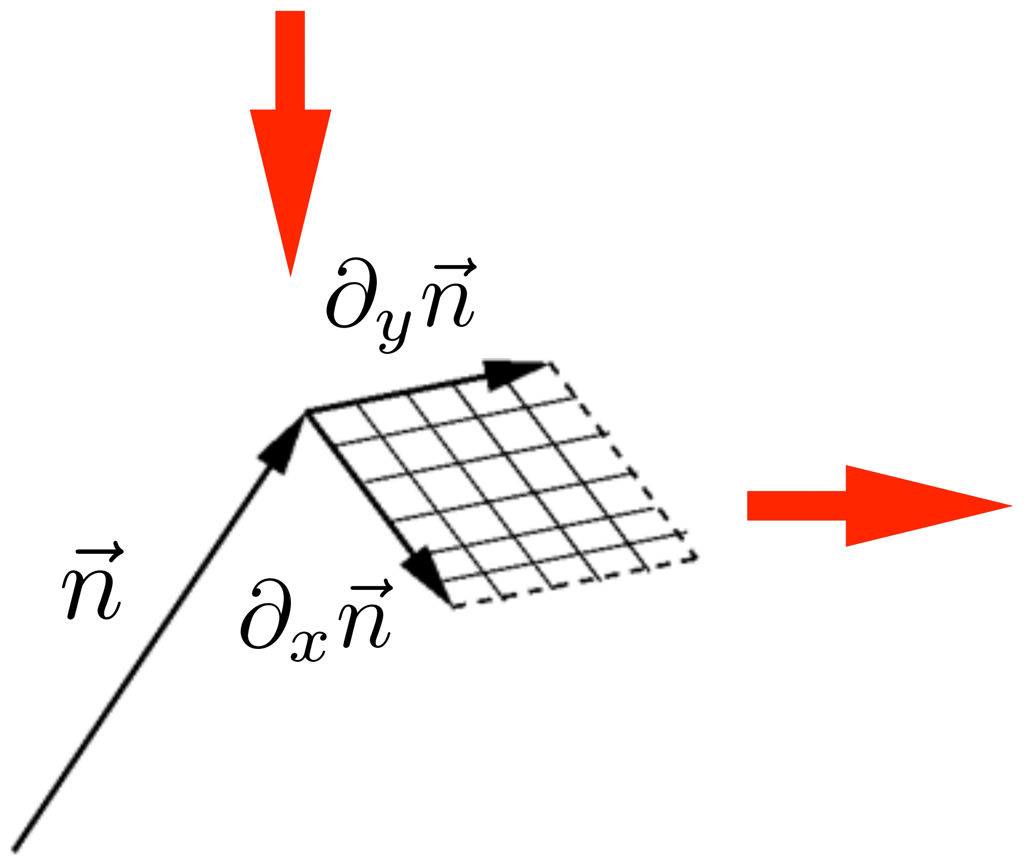
- **Total charge**

$$Q = \int \mathrm{d}^{d-1}x\, j^0 = \frac{1}{8\pi}\int \mathrm{d}^{d-1}x\, \varepsilon^{0\nu\lambda}\vec{n}\cdot(\partial_\nu \vec{n}\times\partial_\lambda \vec{n})$$

The Pontryagin number

- d=2+1

$$Q = \frac{1}{4\pi}\int \mathrm{d}^2x\, \vec{n}\cdot\left(\partial_x\vec{n}\times\partial_y\vec{n}\right)$$



surface element in internal space

$Q:$ covering of sphere in internal space while roaming space → $\pi_2\left(S^2\right)=\mathbf{Z}$

Homotopy groups

B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, Modern Geometry II, Springer (New York, 1985)

- **Def.:** A homotopy of a map $f : M \to N$ is a map

$$F : M \times I \to N \, , \quad (I = [0,1])$$

Each of the maps $f_t(x) = F(x,t)$ is said to be homotopic to the initial map f. The set of all maps homotopic to a particular map f is a homotopy class

- Homotopy group: $\pi_n(X)$ is the set of homotopy classes of maps

$$f : S^n \to X$$

Skyrmions

Skyrmions in d=2

R. Rajaraman, Solitons and Instantons, North Holland (Amsterdam, 1982)

Identity

A. A. Belavin and A. M. Polyakov, JETP Lett. 22, 245 (1975)

$$\int \mathrm{d}^2x \, (\partial_\mu \vec{n} \pm \varepsilon^{\mu\nu} \vec{n} \times \partial_\nu \vec{n})^2 \geq 0$$

$$\hookrightarrow \int \mathrm{d}^2x \, \partial_\mu \vec{n} \cdot \partial_\mu \vec{n} \geq \pm \int \mathrm{d}^2x \, \varepsilon^{\mu\nu} \vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n}) \quad \Rightarrow \quad E \geq 4\pi |Q|$$

$$\hookrightarrow E = 4\pi |Q| \Longrightarrow \partial_\mu \vec{n} = \mp \varepsilon^{\mu\nu} \vec{n} \times \partial_\nu \vec{n}$$

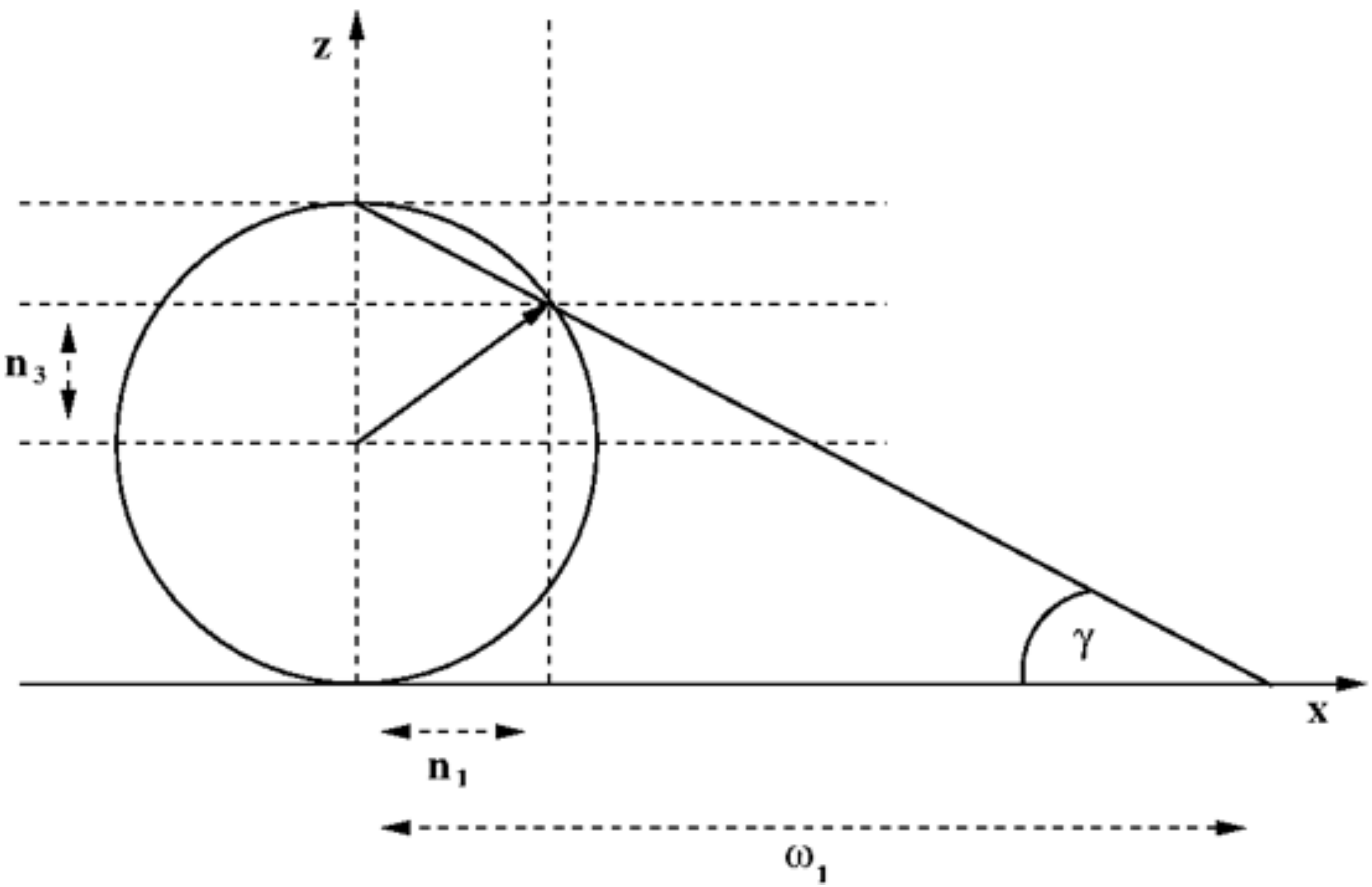
Differential equation for skyrmions

Skyrmions

Skyrmions in d=2

R. Rajaraman, Solitons and Instantons, North Holland (Amsterdam, 1982)

Stereographic projection



$$\tan\gamma = \frac{1-n_3}{n1}$$

$$\omega_1 = \frac{2n1}{1-n_3}$$

$$\omega_2 = \frac{2n_2}{1-n_3}$$

$$\partial_x\omega_1 = \mp\partial_y\omega_2$$

$$\partial_y\omega_1 = \pm\partial_x\omega_2$$

Solutions: $\omega(z) = \omega_1 + i\omega_2$ analytic functions

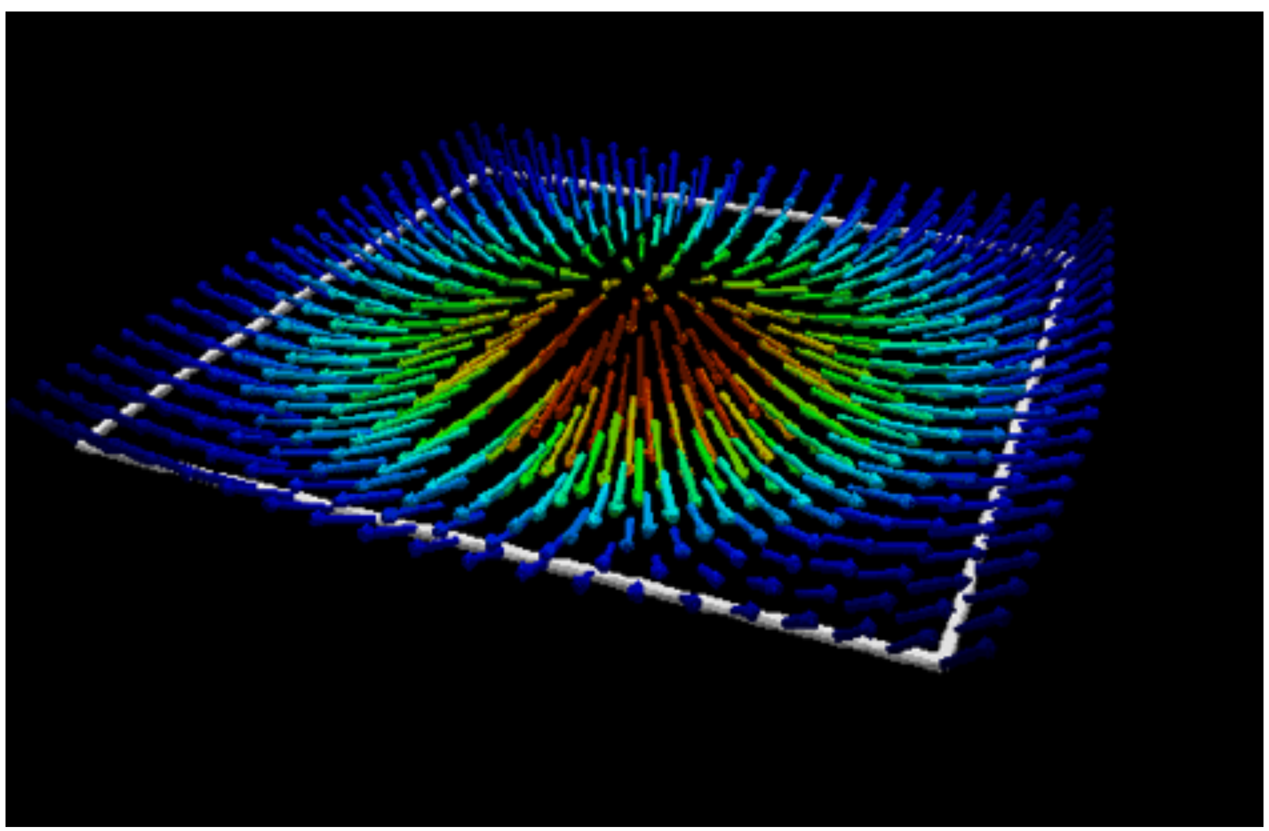
Skyrmions

Skyrmions in d=2

R. Rajaraman, Solitons and Instantons, North Holland (Amsterdam, 1982)

Analytic functions → **monomials** $\omega(z) = \left[\frac{z - z_0}{\lambda}\right]^n$

For n=1 $\vec{n} = \frac{1}{1 + \left(\frac{r}{\lambda}\right)^2}\left[\frac{r}{\lambda}\cos\varphi, \frac{r}{\lambda}\sin\varphi, \left(\frac{2}{2\lambda}\right)^2 - 1\right]$



For n general

$$E = 4\pi|n|$$

↓

$$Q = n$$

Summary

Why field-theory?

- Close to a continuous phase transition
  → Extract universal features
- Determine topological properties

Why non-linear sigma model?

- Universal properties of magnetic systems
  - d=2: spontaneous mass generation ↔ Mermin-Wagner theorem
  - d=3: spontaneous symmetry breaking
  - d=2: topological excitations → Quantum field-theory?