Field-theory of the quantum Heisenberg antiferromagnet in two dimensions

Quantum field-theory of low dimensional systems

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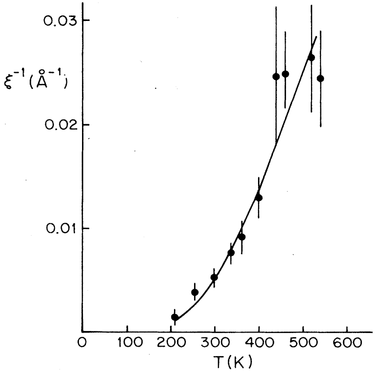
Motivation

- Two-Dimensional Antiferromagnetic Quantum Spin-Fluid in single crystal of La_2CuO_4
  - Bragg-scattering
  - correlation length ξ
- high temperature superconductivity



G. Shirane, Y. Endoh, R. J. Birgeneau, M. Kastner, M. A. Kastner, Y. Hidaka, M. Oda, M. Suzuki, and T. Murakami, *Two-Dimensional Antiferromagnetic Quantum Spin-Fluid State in* La_2CuO_4: Phys. Rev. Lett. 59, pp. 1613-1616, Oct 5 1987

Chakravarty S., Halperin B., Nelson D.: *Two-dimensional quantum Heisenberg antiferromagnet at low temperatures.* PR B , vol. 39, pp. 2344-2371, Feb 1 1989.

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Recapitulation - Spin coherent states

- start with spin-states, the eigenstates to S^2 and S_z:

$$|s, m\rangle \qquad \text{with} \qquad m = -s, ..., s$$

- choose fundamental state $|\psi\rangle = |s, s\rangle$, with $m = s$

- introduce the spin coherent state

$$|\mathbf{n}\rangle = \mathrm{e}^{\mathrm{i}\theta \mathbf{m}\mathbf{S}}|\psi\rangle$$

with $\mathbf{m} = (\sin\phi, -\cos\phi, 0)$ and $0 \leq \theta \leq \pi$: vector on unit sphere

Recapitulation - Partition function

- partition function on single-spin system

$$Z = \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right) = \int \mathrm{d}\mathbf{n}\ \langle\mathbf{n}|\mathrm{e}^{-\beta H}|\mathbf{n}\rangle$$

- with time-slicing

$$Z = \int \mathcal{D}\mathbf{n}\ \mathrm{e}^{-\mathcal{S}}$$

$$\mathcal{S} = \underbrace{\int \mathrm{d}\tau \langle\mathbf{n}|\frac{\mathrm{d}}{\mathrm{d}\tau}|\mathbf{n}\rangle}_{\mathcal{S}_B} + \underbrace{\int \mathrm{d}\tau \langle\mathbf{n}|H|\mathbf{n}\rangle}_{\mathcal{S}_H}$$

  - $\mathcal{S}_B$: the Berry-Phase
  - $\mathcal{S}_H$: the action depending on the system's Hamiltonian

Heisenberg antiferromagnet in d=1

- partition function of the system

$$Z = \int \mathcal{D}\mathbf{n} \exp\left[\underbrace{-\mathrm{i}s\sum_i \int \mathrm{d}\tau\,(\mathbf{A}\partial_\tau \mathbf{n}_i)}_{\mathcal{S}_B} - \underbrace{\int \mathrm{d}\tau \langle \mathbf{n}|H|\mathbf{n}\rangle}_{\mathcal{S}_H}\right]$$

with $H = J\sum_{<i,j>} \mathbf{s}_i \mathbf{s}_j$

- Lagrangian

$$\mathcal{L} = \underbrace{\frac{1}{16a^2J}(\partial_\tau \mathbf{n})^2 + s^2\frac{J}{2}(\partial_x \mathbf{n})^2}_{\mathcal{L}_\sigma} - \underbrace{\mathrm{i}\frac{s}{4}\epsilon_{\mu,\nu}\mathbf{n}(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})}_{\mathcal{L}_{\mathrm{topol}}}$$

Heisenberg antiferromagnet in d=2

- Heisenberg Hamiltonian:

$$\hat{H} = J \sum_{<i,j>} \hat{\mathbf{s}}_i \hat{\mathbf{s}}_j$$

  with $J > 0$

- partition function:

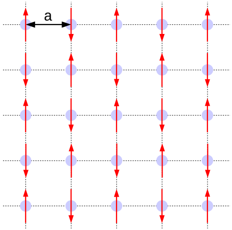
$$Z = \int \mathcal{D}\mathbf{n} \; \mathrm{e}^{-\mathcal{S}}$$

- action:

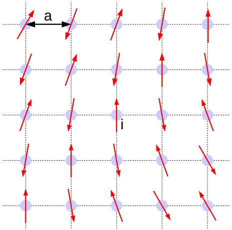
$$\mathcal{S} = \mathrm{i} \sum_{<i,j>} \int \mathrm{d}^2x \mathrm{d}\tau \mathbf{A}_i(\mathbf{x},\tau) \cdot \partial_\tau \mathbf{s}_i(\mathbf{x},\tau) + J \sum_{<i,j>} \mathbf{s}_i \mathbf{s}_j$$

Néel state

- interested in solving the Heisenberg antiferromagnet in d=2; expectations:
  - short range correlation
  - short range antiferromagnetic order
- Néel state is classical ground state
  - derivation from Néel state
- idea
  - modify Néel state, cubic lattice
  - short and long length scale fluctuations
  - → eliminate short wavelength fluctuations



Néel state

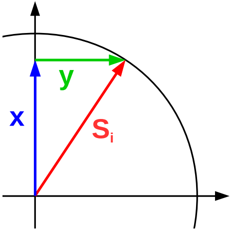


staggered fluctuations

Haldane's mapping

- Haldane mapped effective long-wavelength action into nonlinear sigma model with $d = 2 + 1$
- separate short and long length scale fluctuations
- $\mathbf{L}_i$ small fluctuation

$$\mathbf{s}_i = \underbrace{(-1)^i s \sqrt{1 - \frac{a^2 L_i^2}{s^2}}\mathbf{n}_i}_{x} + \underbrace{a\mathbf{L}_i}_{y}$$



$$|\mathbf{s}_i|^2 = (-1)^{2i} s^2 \left(1 - \frac{a^2 L_i^2}{s^2}\right) \mathbf{n}_i^2 + a^2 \mathbf{L}_i^2 = s^2$$

with the constraints:

- $\mathbf{n}_i^2 = 1$
- $a|\mathbf{L}_i| \ll s$
- $\mathbf{L}_i \cdot \mathbf{n}_i = 0$

$$\xrightarrow{\text{later}} Z = \int \mathcal{D}\mathbf{n}\ \delta(\mathbf{n}^2 - 1)\delta(\mathbf{L}\mathbf{n})\mathrm{e}^{-\mathcal{S}}$$

Kinetic term

- evaluate kinetic term of

$$\mathcal{S} = \mathrm{i} \sum_{<i,j>} \int \mathrm{d}^2x \mathrm{d}\tau \mathbf{A}_i(\mathbf{x},\tau) \cdot \partial_\tau \mathbf{s}_i(\mathbf{x},\tau) + J_H \sum_{<i,j>} \mathbf{s}_i \mathbf{s}_j$$

- evaluate $\mathbf{A}_i(\mathbf{x},\tau)$ and $\partial_\tau \mathbf{s}_i(\mathbf{x},\tau)$ separately
- for simplicity neglect position indication i ($X_i \to X$)
- Start with $\mathbf{A}(\mathbf{x},\tau) = \mathbf{A}(\mathbf{n}(\mathbf{x},\tau), \mathbf{L}(\mathbf{x},\tau))$
  reminder: $\mathbf{s} = (-1)^i s \sqrt{1 - \frac{a^2L^2}{s^2}}\mathbf{n} + a\mathbf{L}$

$$A_\mu(\mathbf{n},\mathbf{L}) = A_\mu\left((-1)^i \frac{\mathbf{s}}{s}\right) = A_\mu\left(\sqrt{1 - \frac{a^2L^2}{s^2}}\mathbf{n} + (-1)^i \frac{a\mathbf{L}}{s}\right)$$

- first order series expansion around $\sqrt{1 - \frac{a^2L^2}{s^2}}\mathbf{n}$

Kinetic term - first part

- first order expansion in a for the point

$$\sqrt{1-\frac{a^2L^2}{s^2}}\mathbf{n} = \left(1-\frac{a^2L^2}{2s^2}\right)\mathbf{n} + \mathcal{O}(a^3) \approx \mathbf{n}$$

- expansion around the point

$$\begin{aligned}
A_\mu(\mathbf{n},\mathbf{L}) =& A_\mu\left((-1)^i\frac{\mathbf{s}}{s}\right) = A_\mu\left(\sqrt{1-\frac{a^2L^2}{s^2}}\mathbf{n} + (-1)^i\frac{a\mathbf{L}}{s}\right) \\
\approx& A_\mu\left(\mathbf{n}\right) + \partial_\nu A_\mu\left(\mathbf{n}\right)\cdot\left[(-1)^i\frac{aL_\nu}{s}\right] + \\
&+\frac{1}{2}\partial_\lambda\partial_\nu A_\mu\left(\mathbf{n}\right)\cdot\left[(-1)^i\frac{aL_\nu}{s}\right]\left[(-1)^i\frac{aL_\lambda}{s}\right] + \dots \\
\approx& A_\mu\left(\mathbf{n}\right) + \partial_\nu A_\mu\left(\mathbf{n}\right)\cdot\left[(-1)^i\frac{aL_\nu}{s}\right]
\end{aligned}$$

Kinetic term - second part

- continue with derivation of second part $\partial_\tau \mathbf{s}_i(\mathbf{n}_i, \mathbf{L}_i)$

$$\begin{aligned}
\partial_\tau s_i^\mu(\mathbf{n}, \mathbf{L}) =& \partial_\tau \left[(-1)^i s \sqrt{1 - \frac{a^2 L(\tau)^2}{2s^2}} n^\mu(\tau) + a L^\mu(\tau) \right] \\
\approx& \partial_\tau \left[(-1)^i s \left(1 - \frac{a^2 L(\tau)^2}{2s^2} \right) n^\mu(\tau) + a L^\mu(\tau) + \mathcal{O}(a^2) \right] \\
\approx& (-1)^i s \cdot 1 \cdot \partial_\tau n^\mu + a \partial_\tau L^\mu + \mathcal{O}(a^2) \\
\approx& (-1)^i s \partial_\tau n^\mu + a \partial_\tau L^\mu
\end{aligned}$$

Kinetic part - evaluation of the whole expression

- now evaluate the whole expression

$$A_\mu \cdot \partial_\tau s_i^\mu \approx \left[A_\mu \left(\mathbf{n} \right) + \partial_\nu A_\mu \left(\mathbf{n} \right) \cdot \left[(-1)^i \frac{a L_\nu}{s} \right] \right] \cdot \\ \cdot \left[(-1)^i s \partial_\tau n^\mu + a \partial_\tau L^\mu \right] \\ \approx (-1)^i s A_\mu(\mathbf{n}) \partial_\tau n \mu + a \partial_\nu A_\mu(\mathbf{n}) L^\nu \partial_\tau n^\mu + a A_\mu(\mathbf{n}) \partial_\tau L^\mu + \mathcal{O}(a^2)$$

- next step: evaluate $\partial_\nu A_\mu(\mathbf{n}) L^\nu \partial_\tau n^\mu$

Kinetic term - further calculations

- auxiliary calculation for $\partial_\nu A_\mu(\mathbf{n}) L^\nu \partial_\tau n^\mu$
  - constraint for the vector potential $\nabla \times \mathbf{A} = \mathbf{n}$

$$\epsilon_{\alpha\beta\gamma}\partial_\beta A_\gamma = n_\alpha$$
$$\epsilon_{\mu\nu\alpha}\ \epsilon_{\alpha\beta\gamma}\partial_\beta A_\gamma = \epsilon_{\mu\nu\alpha}\ n_\alpha$$

  - use identity $\epsilon_{\mu\nu\alpha}\ \epsilon_{\alpha\beta\gamma} = \delta_{\mu\beta}\delta_{\nu\gamma} - \delta_{\mu\gamma}\delta_{\nu\beta}$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = \epsilon_{\mu\nu\alpha}\ n_\alpha$$
$$\partial_\nu A_\mu = \epsilon_{\nu\mu\alpha}\ n_\alpha + \partial_\mu A_\nu$$

- plug this in the result in $\partial_\nu A_\mu(\mathbf{n}) L^\nu \partial_\tau n^\mu$

$$\partial_\nu A_\mu(\mathbf{n}) L^\nu \partial_\tau n^\mu = \epsilon_{\nu\mu\alpha}\ n_\alpha L^\nu \partial_\tau n^\mu + \partial_\mu A_\nu L^\nu \partial_\tau n^\mu$$

- now have a look at the last part

$$\partial_\mu A_\nu L^\nu \partial_\tau n^\mu = L^\nu \frac{\partial A_\nu}{\partial n^\mu}\frac{\partial n^\nu}{\partial \tau} = L^\nu \partial_\tau A_\nu$$

Kinetic term - the result

- now plug it back in $A_\mu \cdot \partial_\tau s_i^\mu$

$$
\begin{aligned}
A_\mu \cdot \partial_\tau s_i^\mu \approx & (-1)^i s A_\mu \partial_\tau n\mu + a\partial_\nu A_\mu L^\nu \partial_\tau n^\mu + aA_\mu \partial_\tau L^\mu \\
\approx & (-1)^i s A_\mu \partial_\tau n\mu + a\epsilon_{\nu\mu\alpha}\ n_\alpha L^\nu \partial_\tau n^\mu + \\
& + \underbrace{aL^\nu \partial_\tau A_\nu + aA_\mu \partial_\tau L^\mu} \\
= & (-1)^i s A_\mu \partial_\tau n\mu + a\epsilon_{\nu\mu\alpha}\ n_\alpha L^\nu \partial_\tau n^\mu + a\partial_\tau (A_\mu L^\mu)
\end{aligned}
$$

- all components of the kinetic term

$$
\mathbf{A}\partial_\tau \mathbf{s} = (-1)^i s \mathbf{A}\partial_\tau \mathbf{n} - a\mathbf{L}\cdot(\mathbf{n}\times\partial_\tau \mathbf{n}) + a\,\underbrace{\partial_\tau(\mathbf{AL})}_{=0}
$$

- the total time derivation

$$
\mathcal{S} \propto \int_0^\beta \mathrm{d}\tau\ \partial_\tau(\mathbf{AL}) = \mathbf{AL}\Big|_0^\beta \overset{\text{closed}}{\underset{\text{loop}}{=}} 0
$$

Interaction term

- reminder:
  $\mathbf{s}_i = (-1)^i s\sqrt{1-\frac{a^2L_i^2}{s^2}}\mathbf{n}_i + a\mathbf{L}_i \approx (-1)^i s\left(1-\frac{a^2L_i^2}{2s^2}\right)\mathbf{n}_i + a\mathbf{L}_i$
- calculating the interaction, neglecting $\mathcal{O}(a^3)$:

$$\mathbf{s}_i\mathbf{s}_j = \left[(-1)^i s\left(1-\frac{a^2L_i^2}{2s^2}\right)\mathbf{n}_i + a\mathbf{L}_i\right]\left[(-1)^j s\left(1-\frac{a^2L_j^2}{2s^2}\right)\mathbf{n}_j + a\mathbf{L}_j\right]$$

$$\approx (-1)^{i+j}s^2\mathbf{n}_i\mathbf{n}_j - (-1)^{i+j}s^2\mathbf{n}_i\mathbf{n}_j\frac{a^2L_j^2}{2s^2} - (-1)^{i+j}s^2\mathbf{n}_i\mathbf{n}_j\frac{a^2L_i^2}{2s^2} + \\ + a^2\mathbf{L}_i\mathbf{L}_j + \mathcal{O}(a^4)$$

- used $\mathbf{nL} = 0$

$$\mathbf{s}_i\mathbf{s}_j \approx (-1)^{i+j}s^2\underbrace{\mathbf{n}_i\mathbf{n}_j}_{\psi} + \underbrace{a^2\left[\mathbf{L}_i\mathbf{L}_j - \frac{(-1)^{i+j}}{2}\mathbf{n}_i\mathbf{n}_j\left(L_i^2+L_j^2\right)\right]}_{\zeta}$$

Evaluation of ψ - part I

- evaluate first bit

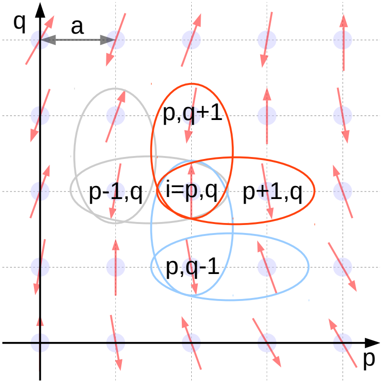
$$\psi = \mathbf{n}_i \mathbf{n}_j + 1 - \frac{1}{2} - \frac{1}{2}$$

- with $\mathbf{n}_i^2 = 1$ and $\mathbf{n}_j^2 = 1$

$$\begin{aligned}\psi =& \mathbf{n}_i \mathbf{n}_j + 1 - \frac{\mathbf{n}_i^2}{2} - \frac{\mathbf{n}_j^2}{2} \\ =& 1 - \frac{1}{2}(\mathbf{n}_i - \mathbf{n}_j)^2\end{aligned}$$

- evaluate $\mathbf{n}_i - \mathbf{n}_j$
- summation over nearest neighbor

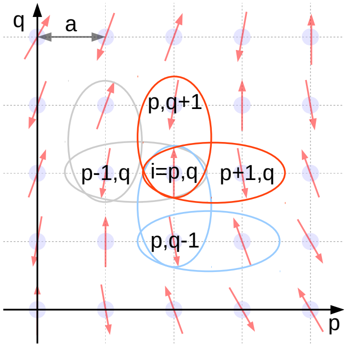
$$i \rightarrow \{p, q\}$$
$$j \rightarrow \{\{p+1, q\}, \{p, q+1\}\}$$



Evaluation of ψ - part II

- summation over nearest neighbors leads to

$$\begin{aligned}
\mathbf{n}_i - \mathbf{n}_j =& (\mathbf{n}_{p,q} - \mathbf{n}_{p+1,q}) + (\mathbf{n}_{p,q} - \mathbf{n}_{p,q+1}) \\
=& a\left[\frac{\mathbf{n}_{p,q} - \mathbf{n}_{p+1,q}}{a} + \frac{\mathbf{n}_{p,q} - \mathbf{n}_{p,q+1}}{a}\right] \\
=& a\left[\partial_x \mathbf{n}(\mathbf{x}_{p,q}) + \partial_y \mathbf{n}(\mathbf{x}_{p,q})\right] \\
=& a\left[\partial_x \mathbf{n}_i + \partial_y \mathbf{n}_i\right] \\
=& a \nabla \mathbf{n}_i
\end{aligned}$$



- $(\mathbf{n}_i - \mathbf{n}_j)^2 = a^2(\nabla \mathbf{n}_i)^2$
- finally

$$\psi = \mathbf{n}_i \mathbf{n}_j = 1 - \frac{a^2}{2}(\nabla \mathbf{n}_i)^2$$

Evaluation of ζ

- evaluate of the term:

$$\zeta = a^2 \left[\mathbf{L}_i \mathbf{L}_j - \frac{(-1)^{i+j}}{2} \mathbf{n}_i \mathbf{n}_j \left(L_i^2 + L_j^2 \right) \right]$$

- reminder: $\mathbf{n}_i \mathbf{n}_j = 1 - \frac{a^2}{2} (\nabla \mathbf{n}_i)^2$
- for nearest neighbour $(-1)^{i+j} = -1$

$$\begin{aligned} \zeta =& a^2 \mathbf{L}_i \mathbf{L}_j + \frac{a^2}{2} (L_i^2 + L_j^2) + \mathcal{O}(a^4) \\ \approx& \frac{a^2}{2} (\mathbf{L}_i + \mathbf{L}_j)^2 \end{aligned}$$

Evaluation of the interaction term

- now ψ and ζ combined

$$\mathbf{s}_i\mathbf{s}_j \approx (-1)^{i+j}s^2 - \frac{(-1)^{i+j}}{2}s^2a^2(\nabla\mathbf{n}_i)^2 + \frac{a^2}{2}(\mathbf{L}_i+\mathbf{L}_j)^2$$

- again $(-1)^{i+j} = -1$ for nearest neighbours

The Hamiltonian

$$\begin{aligned} H =& J\sum_{<i,j>}\mathbf{s}_i\mathbf{s}_j \\ =& \underbrace{-J\sum_{<i,j>}s^2}_{=\text{ classical energy}} + \frac{Ja^2}{2}\sum_{<i,j>}\left[s^2(\nabla\mathbf{n}_i)^2 + (\mathbf{L}_i+\mathbf{L}_j)^2\right]\end{aligned}$$

continuum limit for the interaction term

- continuum limit for interaction term of the action
  - $a \to 0$
  - $\mathbf{L}_j \to \mathbf{L}_i = \mathbf{L}$
  - $\sum_{<i,j>} a^2 \to \int \mathrm{d}^2 x$
- continuum limit in the action

$$\mathcal{S}_H = -J \sum_{<i,j>} \int_0^\beta \mathrm{d}\tau s^2 + $$

$$+ \int \mathrm{d}\tau \sum_{<i,j>} a^2 \frac{Js^2}{2} \left(s^2 (\nabla \mathbf{n}(\mathbf{x}))^2 + (\mathbf{L}_i + \mathbf{L}_j)^2 \right)$$

$$\mathcal{S}_H \to \underbrace{-J \int \mathrm{d}^2 x \int_0^\beta \mathrm{d}\tau \ s^2}_{=0} + \int \mathrm{d}\tau \int \mathrm{d}^2 x \left(\frac{Js^2}{2} (\nabla \mathbf{n}(\mathbf{x}))^2 + J(4\mathbf{L}^2) \right)$$

Temporary Total Action

$$\mathcal{S} = \underbrace{\mathrm{i} \sum_{<i,j>} (-1)^{i+j} s \int \mathrm{d}\tau \mathbf{A}_i(\mathbf{n}_i) \partial_\tau \mathbf{n}_i}_{=\mathcal{S}_B} - \frac{i}{a} \int \mathrm{d}^2 x \mathrm{d}\tau \mathbf{L} \cdot (\mathbf{n} \times \partial_\tau \mathbf{n}) + \underbrace{+ \frac{Js^2}{2} \int \mathrm{d}^2 x \mathrm{d}\tau (\nabla \mathbf{n})^2 + 4J \int \mathrm{d}^2 x \mathrm{d}\tau \mathbf{L}^2}_{\mathcal{S}_H}$$

- now: evaluate Berry-phase $\mathcal{S}_B$

Mathematical insertion - part I

$f(x)$ in $[a, c]$ with N steps x_i and $x_i = b + i \cdot a$ and $a = \frac{c-b}{N}$ then for $N \to \infty$ and $a \to 0$

$$\sum_{i=1}^{N} a f(x_i) \to \int_b^c \mathrm{d}x f(x)$$

later we will need $\tilde{N} = N/2$ and $\tilde{a} = 2a$. Thus

$$\sum_{i=1}^{N/2} 2a f(x_i) \to \int_b^c \mathrm{d}x f(x)$$

Mathematical insertion - part II

$\mathcal{S}_B = \mathrm{i} \sum_{<i,j>} (-1)^{i+j} s \int \mathrm{d}\tau \mathbf{A}_i(\mathbf{n}_i) \partial_\tau \mathbf{n}_i$

- $M, N \to \infty$ and $a \to 0$

$$\sum_{<i,j>} (-1)^{i+j} \mathbf{x}_i = \sum_{p=1}^{M} \sum_{q=1}^{N} (-1)^{p+q} \mathbf{x}_{p,q} = \sum_{p=1}^{M} (-1)^p \sum_{q=1}^{N/2} (\mathbf{x}_{p,2q} - \mathbf{x}_{p,2q-1})$$

$$= \sum_{p=1}^{M} \frac{(-1)^p}{2} \sum_{q=1}^{N/2} 2a \underbrace{\frac{\mathbf{x}_{p,2q} - \mathbf{x}_{p,2q-1}}{a}}_{\partial_q \mathbf{x}_{p,q}} = \sum_{p=1}^{M} \frac{(-1)^p}{2} \int \mathrm{d}q \; \partial_q \mathbf{x}_p(q)$$

$$= \frac{1}{4} \sum_{p=1}^{M/2} 2a \int \mathrm{d}q \; \partial_q \frac{\mathbf{x}_{2p}(q) - \mathbf{x}_{2p-1}(q)}{a}$$

$$= \frac{1}{4} \int \mathrm{d}p \; \mathrm{d}q \; \partial_p \partial_q \mathbf{x}(p,q)$$

Mathematical insertion - part III

- we will need

$$\partial_x(\mathbf{A}\cdot\partial_\tau\mathbf{n}) = \partial_x\mathbf{A}\cdot\partial_\tau\mathbf{n} + \mathbf{A}\cdot\partial_x\partial_\tau\mathbf{n} = \Phi$$

- with $\nabla\times\mathbf{A} = \mathbf{n}$ and $\partial_\mu A_\nu - \partial_\nu A_\mu = \epsilon_{\mu\nu\alpha}n_\alpha$

$$\partial_x\mathbf{A} = \frac{\partial}{\partial n_\nu}A_\mu\cdot\partial_x n_\nu = \partial_\nu A_\mu\cdot\partial_x n_\nu = \epsilon_{\nu\mu\alpha}n_\alpha\partial_x n_\nu + \partial_\mu A_\nu\partial_x n_\nu$$

$$\Phi = \epsilon_{\nu\mu\alpha}n_\alpha\partial_x n_\nu\partial_\tau n_\mu + \underbrace{\partial_\mu A_\nu\partial_x n_\nu\partial_\tau n_\mu + A_\mu\partial_\tau\partial_x n_\mu}_{=\partial_\tau(A_\mu\partial_x n_\mu)=0}$$

$$= -\frac{1}{2}\epsilon_{\mu\nu}\ \epsilon_{abc}n_a\partial_\mu n_b\partial_\nu n_c$$

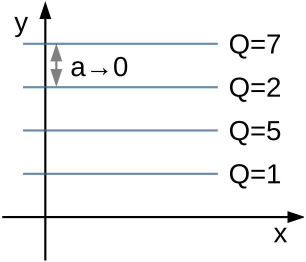
- all components

$$\partial_x(\mathbf{A}\cdot\partial_\tau\mathbf{n}) = -\frac{1}{2}\epsilon_{\mu\nu}\mathbf{n}(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n})$$

Calculation of the Berry-phase

$\sum_{<i,j>}(-1)^{i+j}\mathbf{x}_i = \frac{1}{4}\int \mathrm{d}p\,\mathrm{d}q\ \partial_p\partial_q\mathbf{x}(p,q)$ and $\partial_x(\mathbf{A}\cdot\partial_\tau\mathbf{n}) = -\frac{1}{2}\epsilon_{\mu\nu}\mathbf{n}(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n})$

$$\begin{aligned}\mathcal{S}_B =& \mathrm{i}\sum_{<i,j>}(-1)^{i+j}s\int \mathrm{d}\tau\ \underbrace{\mathbf{A}_i(\mathbf{n}_i)\partial_\tau\mathbf{n}_i}_{=\mathbf{x}_i}\\ =&\mathrm{i}\frac{s}{4}\int \mathrm{d}\tau\int \mathrm{d}x\ \mathrm{d}y\ \partial_y\ \left[\partial_x\left(\mathbf{A}(\mathbf{n})\partial_\tau\mathbf{n}\right)\right]\\ =&-\mathrm{i}\frac{s}{8}\int \mathrm{d}y\ \partial_y\underbrace{\int \mathrm{d}\tau\int \mathrm{d}x\ \epsilon_{\mu,\nu}\mathbf{n}(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n})}_{=8\pi Q(y)}\\ =&-\mathrm{i}s\pi\int \mathrm{d}y\ \underbrace{\partial_y Q(y)}_{=0}\\ =&\ 0\end{aligned}$$



Identifying the Lagrangian

Final Action

$$\mathcal{S} = -\frac{\mathrm{i}}{a}\int \mathrm{d}^2x\mathrm{d}\tau \mathbf{L}\cdot(\mathbf{n}\times\partial_\tau\mathbf{n}) + \\ + \frac{Js^2}{2}\int \mathrm{d}^2x\mathrm{d}\tau(\nabla\mathbf{n})^2 + 4J\int \mathrm{d}^2x\mathrm{d}\tau\mathbf{L}^2$$

- we know

$$\mathcal{S} = \int \mathrm{d}^2x \int \mathrm{d}\tau \mathcal{L}(\mathbf{x},\tau)$$

The Lagrangian

$$\mathcal{L} = -\frac{\mathrm{i}}{a}\mathbf{L}\cdot(\mathbf{n}\times\partial_\tau\mathbf{n}) + \frac{Js^2}{2}(\nabla\mathbf{n})^2 + 4J\mathbf{L}^2$$

The Lagrangian

- the partition function is given by

$$Z = \int \mathcal{D}\mathbf{n}\mathcal{D}\mathbf{L}\mathrm{e}^{-\mathcal{S}}$$

- solve the gaussian integral

$$\int \mathcal{D}\mathbf{L}\mathrm{e}^{-\alpha\mathbf{L}^2+\beta\mathbf{L}+\gamma} \propto \mathrm{e}^{\frac{\beta^2}{4\alpha}+\gamma}$$

- and with the identity $(\mathbf{n}\times\partial_\tau\mathbf{n})^2 = (\partial_\tau\mathbf{n})^2$ we find

$$Z \propto \int \mathcal{D}\mathbf{n}\mathrm{e}^{-\int \mathrm{d}^2x\mathrm{d}\tau\mathcal{L}}$$

- with

$$\mathcal{L} = \frac{1}{16a^2J}(\partial_\tau\mathbf{n})^2 + s^2\frac{J}{2}(\nabla\mathbf{n})^2$$

Comparision of the Heisenberg antiferromagnet in d=1 and d=2

- compare Lagrangians

$$\mathcal{L}^{(d=1)} = \frac{1}{16a^2 J}(\partial_\tau \mathbf{n})^2 + s^2 \frac{J}{2}(\partial_x \mathbf{n})^2 - \mathrm{i}\frac{s}{4}\epsilon_{\mu,\nu}\mathbf{n}(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})$$
$$\mathcal{L}^{(d=2)} = \frac{1}{16a^2 J}(\partial_\tau \mathbf{n})^2 + s^2 \frac{J}{2}(\nabla \mathbf{n})^2$$

- the partition function

$$Z^{(d=1)} \propto \int \mathcal{D}\mathbf{n} e^{-\int \mathrm{d}x \mathrm{d}\tau \mathcal{L}^{(d=1)}} \propto \int \mathcal{D}\mathbf{n} e^{\mathrm{i}2\pi s Q}$$
$$Z^{(d=2)} \propto \int \mathcal{D}\mathbf{n} e^{-\int \mathrm{d}x \mathrm{d}\tau \mathcal{L}^{(d=2)}}$$

The Lagrangian and the nonlinear sigma model

Final Lagrangian in d=2

$$\mathcal{L} = \frac{1}{16a^2J}(\partial_\tau \mathbf{n})^2 + s^2\frac{J}{2}(\nabla \mathbf{n})^2$$

- bring on new form

$$\mathcal{L} = \frac{1}{2g}\left[\frac{1}{c}(\partial_\tau \mathbf{n})^2 + c(\nabla \mathbf{n})^2\right]$$

with an

$$g = \frac{a}{s}2\sqrt{2}, \qquad c = 2\sqrt{2}asJ$$

- nonlinear sigma model in $d = 2 + 1$

Nonlinear sigma model d=2+1

- change coordinate

$$(x, y, c\tau) \to (x_1, x_2, x_3)$$

- new Lagrangian

$$\mathcal{L}' = \frac{c}{2g} \sum_{\mu=1}^{3} (\partial_\mu \mathbf{n})^2$$

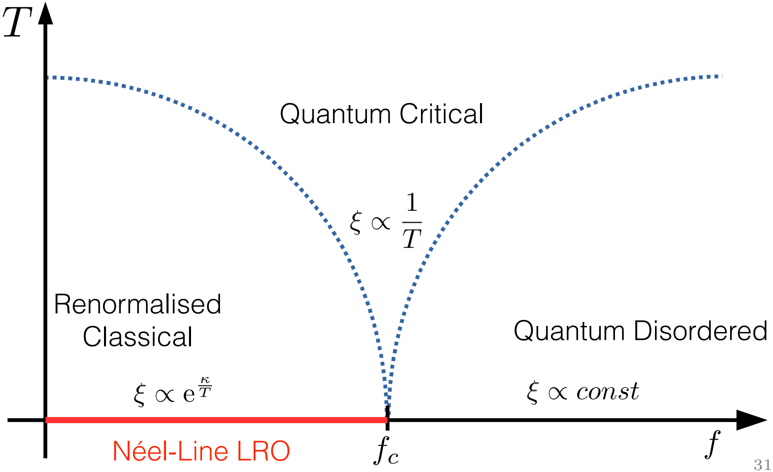
- new partition function

$$Z = \int \mathcal{D}\mathbf{n} \;\exp\left[-\frac{1}{2f} \int \mathrm{d}^3x \;\sum_{\mu=1}^{3} (\partial_\mu \mathbf{n})^2\right]$$

  with $f = g/c$
- analyzing an antiferromagnet in $d = 2$ is equivalent to analyzing a NLσM in $d = 2 + 1$

Results of NLσM

- order parameter $f(J)$
- correlation length ξ



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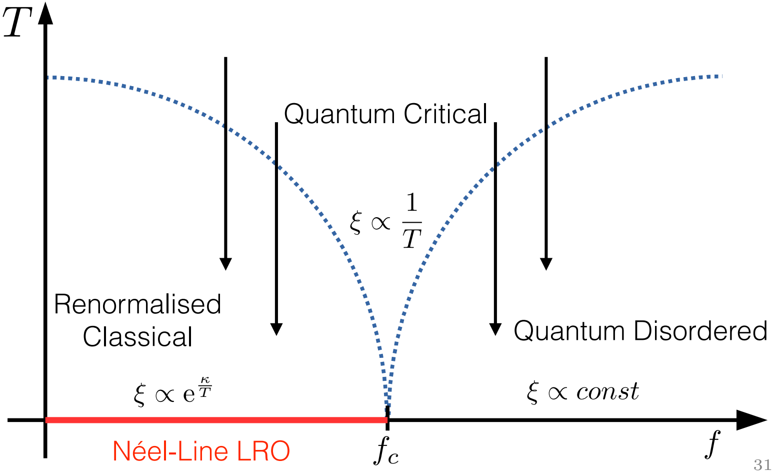

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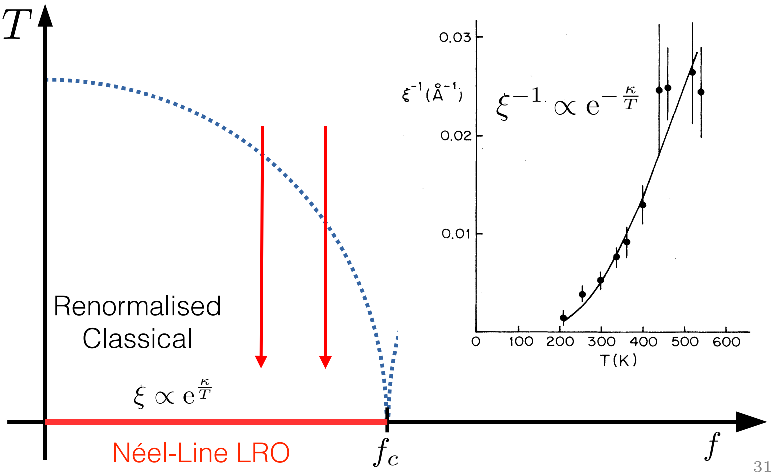

Conclusion

- we started with action

$$\mathcal{S} = \mathrm{i} \sum_{<i,j>} \int \mathrm{d}^2 x \mathrm{d}\tau \mathbf{A}_i(\mathbf{x},\tau) \cdot \partial_\tau \mathbf{s}_i(\mathbf{x},\tau) + J \sum_{<i,j>} \mathbf{s}_i \mathbf{s}_j$$

- introduced Haldane's mapping
    - split short and long range fluctuations
- Berry-phase vanishes, no topological term
- Lagrangian

$$\mathcal{L} = \frac{1}{2g} \left[\frac{1}{c} (\partial_\tau \mathbf{n})^2 + c (\nabla \mathbf{n})^2 \right]$$

- analyzing an antiferromagnet in $d = 2$ is equivalent to analyzing a NLσM in $d = 2 + 1$

Thank you

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