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Master Thesis

Properties of a supersolid in one dimension: Study on the algebraic decay of correlation functions and the stability analysis of the supersolid phase

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Declaration

I declare that I have developed and written the enclosed Master Thesis completely by myself, and have not used sources or means without declaration in the text. Any thoughts from others or literal quotations are clearly marked. The Master Thesis was not used in the same or in a similar version to achieve an academic grading or is being published elsewhere. Lastly, I assure that the electronic copy of this Master Thesis is identical in content with the enclosed version.

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Zusammenfassung in deutscher Sprache

Zu Beginn dieser Arbeit versuche ich einen möglichst leicht verständlichen Überblick über die in dieser Arbeit behandelten Themen zu geben. Diese deutsche Zusammenfassung hat dabei den Anspruch auch für nicht in der Physik ausgebildete Personen verständlich zu sein.
Ein guter Anfang hierfür ist der Titel der Arbeit: "Properties of a supersolid in one dimension: Study on the algebraic decay of correlation functions and the stability analysis of the supersolid phase".
Auf Deutsch übersetzen könnte man diesen Titel mit "Eigenschaften eines Supersolids in einer Dimension: Eine Studie über den algebraischen Zerfall der Korrelationsfunktionen und eine Stabilitätsanalyse der supersoliden Phase". Der erste spezielle Begriff der hier sofort ins Auge sticht ist der des Supersolids. Ein Supersolid ist ein spezieller quantenmechanischer Materiezustand. Um besser erklären zu können was genau ein Supersolid ausmacht, besprechen wir zuerst zwei andere Materiezustände.
Der erste Zustand ist ein Kristall. Kristalle sind Festkörper, deren elementaren Bausteine periodisch in einem Gitter angeordnet sind. Beispielhaft dafür ist Kochsalz: Dort bilden Cl^- und Na^+ Ionen in einem periodischen Gitter den Salzkristall.
Der zweite Zustand den wir betrachten wollen ist wieder exotischer, da auch er quantenmechanischer Natur ist. Wir betrachten hier ein sogenanntes Superfluid. Das übliche Beispiel für die Superfluidität ist Helium bei Temperaturen nahe dem absoluten Nullpunkt. Die definierende Eigenschaft ist hierbei, dass Superfluide keine innere Reibung besitzen, was einen verlustfreien Fluss im Superfluid möglich macht.
Einem Supersolid liegt nun der folgende Gedanke zugrunde: Ist es möglich, die Eigenschaften eines Superfluids mit denen eines Kristalls zu kombinieren, also einen Materiezustand zu erzeugen, der sich sowohl durch Periodizität als auch einen verlustfreien Fluss auszeichnet? Auf den ersten Blick scheint es unmöglich, dass Materie gleichzeitig fest und flüssig sein kann. Es wird plausibler wenn man sich genau überlegt was fest und flüssig hier bedeuten. Ein Supersolid ist fest in dem Sinne, dass eine periodische Struktur vorliegt. Würde man an einer Seite eines Supersolids schieben, so würde sich die gesamte periodische Struktur verschieben wie man es auch von einem Kristall erwarten würde. Gleichzeitig ist ein Supersolid superflüssig in dem Sinne, dass ein verlustfreier Fluss in ihm existieren kann. Ein Atom aus dem Supersolid könnte also von der einen zur anderen Seite des Supersolids fließen. Theoretisch vorhergesagt ist dieser Materiezustand schon seit 1969, experimentelle Beweise existieren jedoch erst seit 2017.

In dieser Arbeit wird ein effektives Modell für ein Supersolid untersucht. Zunächst wird die Wahl des Modellsystems anhand anderer Beschreibungen von Supersoliden motiviert. Daraufhin wird das System charakterisiert, indem Korrelationsfunktionen berechnet werden. Korrelationsfunktionen beschreiben einen Zusammenhang zwischen mehreren Größen. Ein einfaches Beispiel für eine Korrelationsfunktion wäre: Wenn ich ein Atom an Position x_A habe, mit welcher Wahrscheinlichkeit finde ich dann auch eins an Position x_B? Diese Korrelationsfunktion ausgewertet für eine Flüssigkeit würde einen konstanten Wert für jedes paar Positionen ergeben. Für einen Kristall ergibt sich ein anderes Bild. Falls die Positionen x_A und x_B der periodischen Struktur entsprechen findet man bei x_B auf jeden Fall auch ein Atom, falls nicht, findet man keins. Anhand dieses Beispiels wird klar, dass Korrelationsfunktionen Information über das System enthalten. Korrelationsfunktionen von besonderem Interesse sind diejenigen die zum Beispiel die Superfluidität charakterisieren. Für diese erwarten wir dass sie sich wie $|x_\mathrm{A} - x_\mathrm{B}|^{-\alpha}$ verhalten, wobei α eine feste aber beliebige positive Zahl ist. Solches Verhalten nennt sich algebraischer Zerfall.

Der große Vorteil das Supersolid durch ein Modellsystem zu beschreiben ist, dass es so viel einfacher ist den Effekt von Störungen am idealen Modell zu untersuchen. Dies lässt sich für die Stabilitätsanalyse ausnutzen. Dafür werden bestimmte Störungen eingeführt und untersucht ob diese das System beeinflussen. Mit der richtigen Wahl der Störung lassen sich dadurch Fragen wie "Lässt sich das System verschieben?" oder "Existiert ein verlustfreier Fluss?" beantworten. Damit lässt sich feststellen, für welche Parameter das Modellsystem tatsächlich ein Supersolid beschreibt, und wo im Phasendiagramm sich der Übergang zu benachbarten Phasen befindet.

Abstract

The concept of a supersolid is quite perplexing, as a system that has a periodic structure and is superfluid at the same time defies our classical expectation. Even so, this state of matter was predicted by theorists more than fifty years ago and, thanks to recent efforts, supersolids have now also been observed in the lab.

In this thesis we are interested in further exploring the theoretical description of a supersolid. We want to find out for what parameters in our models we actually obtain a supersolid. In order to do this we move away from typical descriptions based on the Gross-Pitaevskii equation and instead consider an effective Lagrangian, motivated by a paper by Josserand et al. With this effective Lagrangian we can then employ the path integral formalism to readily obtain correlation functions.
Further insights will prompt us to then switch to the Hamiltonian where we subsequently are able to transform our model into two decoupled Luttinger liquids.

With the system now simplified, we turn to the question of when our model system actually describes a supersolid. To answer this questions we use perturbations. We study the effects of perturbations using renormalization group theory, where it is enough for our purposes to consider only the perturbations only in lowest order. When choosing the proper perturbations their relevance or irrelevance will tell us if we have the properties required of a supersolid or not. Using this method we find several phases in the range of possible parameters: As expected there is the supersolid, but there also exist a superfluid, solid and fluid phase in our model system.

Contents

1 Introduction

Condensed Matter Physics concerns itself with the different phases of matter. From our everyday experience we intuitively understand this concept, as there is an apparent difference between the usual phases of solid, fluid and gas. The concept however is a much more general one and there is a plethora of phases known to physicists. For instance one might study the phase diagram of water some more and realize that gaseous and fluid water are not that different as well as for high pressures notice many different types of ice. Looking at solids some more, one for instance finds phase transitions from metals to superconductors or the transition between ferromagnetic and paramagnetic phases in magnetic materials. From these examples alone one can see that the concept is quite general. Physicists apply the label phase to something with distinct and (usually) interesting properties. A more rigorous mathematical description of these phases is introduced by Lev Landau in the form of Spontaneous Symmetry Breaking (SSB). Using this concept we characterize phases by their symmetries. For a transition to occur then, one or more of them must be spontaneously broken.
The typical example is again water. Looking at the liquid or gaseous phase we see that it has translational invariance. This is just a fancy way of saying that at whichever position one looks one sees the same thing: Water molecules floating around and bumping into each other. In the solid phase (Ice) this is different. If one looks closely one observes a crystal structure. Thus there is no translational invariance anymore, the system only looks the same if one moves it by an amount perfectly corresponding to the crystal. If one wants to move from the liquid to the solid phase, this translational invariance must be spontaneously broken

Physics in general can be either experimentally or theoretically driven. In the former there is some new phenomenon or experimental data that leaves people baffled and scrambling to explain it. In the latter, the development of a new theory or other mathematical tool allows new predictions which people then try to verify in the lab. In the case of supersolids it was the latter. The story begins with the discovery of superfluidity in liquid helium. The core property of a superfluid is the ability to have a flow without dissipation, meaning the flow never stops. This is demonstrated in experiments defying our very much classical intuition. After the discovery and subsequent theoretical description, physicists began to theorize whether a crystal like material could also have superfluid properties, thus bringing the idea of a supersolid alive.

The structure of this thesis is roughly as follows: First we will give an overview of supersolids and give a usual theoretical description of a dipolar supersolid, as that is the system we can actually observe in the lab. Then we will have a close look at the supersolid model we will base our later calculations on. Subsequently we shall solve the simplified model and begin to lay out perturbations to probe the stability of the superfluid. Finally we will calculate the effect of the perturbations and determine the regions of stability in the supersolid model.

2 Supersolids

In this section we take a closer look at supersolids. We first look at properties of supersolids. Although there have been new developments in the field of dipolar gasses ([1][2]), for a beginning overview, [3] is still a good reference. This next part is based on their review.

2.1 Symmetries and Order

As stated before a supersolid is supposed to have both the properties of a (crystalline) solid and a superfluid. In more concrete mathematical language this means that a supersolid must spontaneously break the characteristic symmetries of those two systems.

Solid In a (crystalline) solid, the spontaneously broken symmetry is translational invariance. One can easily see this when comparing such a solid to a liquid: The liquid can be translated by an arbitrary amount, it will still appear identical to an observer either way. For a solid however, there exists structure. As such a translation which does "nothing" must be one by exactly a multiple lattice vector. To determine whether one is in a liquid or solid phase one can simply look at the order parameter density $\rho(r)$. With the average density n one can observe the quantity

$$\rho(r) - n \tag{2.1}$$

which equates to zero for a uniform liquid phase but periodically oscillates in a solid. This condition should hold for long range and thus one speaks of (density) long-range order (LRO) or diagonal long-range order.

Superfluity The case of the superfluid is a bit more abstract. A usual description of a superfluid is a interacting Bose-Einstein Condensate (BEC). There is a distinction between the two but it is subtle and does not matter to us here. In a BEC all the constituent particles condense into the same quantum mechanical ground state of zero momentum. This means the spontaneous breaking of the $U(1)$-Gauge symmetry. In other words: One does not have the freedom to add a position dependent phase to the state as the ground state is globally phase coherent. Having the zero momentum mode macroscopically occupied

$$n(k) = \left\langle \psi^\dagger(k)\psi(k) \right\rangle \propto n_0 \delta(k) \tag{2.2}$$

means that a macroscopic part of the system is delocalized. It is then less of a surprise that

$$\left\langle \psi^\dagger(r)\psi(r') \right\rangle \to n_0 \quad \text{for} \quad |r - r'| \to \infty\,, \tag{2.3}$$

which implies that a particle can be destroyed and then created again an arbitrary distance away. This sort of order is called offdiagonal long-range order (ODLRO).

2.1.1 A Note for 1D

The comments of the previous section were made for a general superfluid. When going to low dimensions there is the Mermin-Wagner theorem to consider. The general statement is that continuous symmetries (such as $U(1)$) can not be spontaneously broken in dimensions $d \leq 2$ at finite temperature.

What we here are looking for instead is "Quasi Long Range Order", which still can be found in one dimension for $T = 0$. This means our correlation functions won't be constant but decay algebraically with some exponent instead.

2.2 Typical Theoretical Description of a Dipolar Supersolid

Here, we briefly introduce a usual theoretical description of a dipolar supersolid in a harmonic trap, which makes it effectively one dimensional. This is not a focus of this thesis. The reason for introducing this description is to clarify the connection between the effective one dimensional system we will treat later and the actual physical realization of a supersolid in the lab ([2],[1]). Such systems are described by the effective theory which we will outline in this section.

This review here is a summary of parts of [4]. We begin with the description of a BEC. Without interactions, theoretical description is simple as it works on the single particle level. The ground state simply is the state where all the bosons condense in the lowest energy state of the confining harmonic potential. When introducing interactions things become more tricky, but this is a problem that has been extensively treated [5]. When written down in terms of field operators, the problem is expressed as

$$i\hbar\partial_t\Psi = [\Psi,H] = \left[-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}} + \int dr'\Psi^\dagger V(r-r')\Psi\right]\Psi. \tag{2.4}$$

Further treatment requires approximations. The crucial thing about a BEC is the macroscopic occupation of the single particle ground state. Expressed in equations this means the ground state occupation fulfills $N_0 \gg 1$ and N_0/N finite with the total particle number N. Applying the ground state creation or annihilation operator on this kind of state effectively leaves it unaffected as $N_0 \pm 1 \approx n_0$. The operators can then just be approximated by their prefactor $\sqrt{N_0 \pm 1} \approx \sqrt{N_0}$. The field operator is the sum of the single particle states with corresponding creation/annihilation operators. The idea is now to separate the macroscopically occupied mode from the field operator

$$\Psi = \psi + \Psi' \tag{2.5}$$

where $\psi = \langle\Psi\rangle$. This also fixes the density to $n_0 = |\psi|^2$. Ψ' is a small contribution here and is set to zero. A general potential V is still hard to treat. However, as an approximate case, we consider only contact interaction in first Born approximation as well as dipolar interactions. This leads to an approximate potential of the form

$$V(r-r') = g\delta(r-r') + V_{dd}(r-r') = g\delta(r-r') + \frac{1-3\cos^2\theta}{|r-r'|^3}, \tag{2.6}$$

where θ is the angle to the polarization direction. It turns out that for a useful description one needs to include beyond mean field corrections to the BEC. Further reading can be found in [4]. This finally gives an effective description based on a extended Gross-Pitaevskii Equation (eGPE):

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}} + g\,|\psi|^2 + \Phi_{dd} + g_{qf}\,|\psi|^3\right]\psi\,, \tag{2.7}$$

where

$$\Phi_{dd} = \int dr'\psi^\dagger V_{dd}(r-r')\psi\,. \tag{2.8}$$

It is also important to note conditions for validity. With such a description we are in the weakly interacting realm, where kinetic energy dominates over potential energy. This concludes the brief outline we give of the description of a dipolar supersolid. The future goal, which we will unfortunately not yet get to in this thesis, is then to connect the parameters of this kind of description to the parameters of the model system which we will see later.

3 The Supersolid Model System

The goal of this thesis is to perform a stability analysis of this supersolid phase. To that end the description introduced above is not as useful as an effective quadratic Lagrangian field theory. There are several papers that introduce a Lagrangian for a supersolid. We will use an effective Lagrangian from a paper by Josserand et al [6] to motivate our further simplified model Lagrangian. The description by Josserand et al explicitly explores periodically modulated ground states. This is what we want as the GPE based description above also produces modulated ground states in the right parameter range.

We will begin by summarizing the derivation of Josserands effective Lagrangian. The goal here is to gain a more intuitive understanding of the physical meaning of the parameters in our model system and to be able to define sensible observables.

3.1 Derivation

As stated before we now summarize calculations by Josserand. This gives us a better understanding how our later model system is connected to the more first principle calculations which were outlined in the previous section. The system that is modeled here is a supersolid in three dimensions without confinement. We begin with the Gross-Pitaevskii equation for a weakly interacting BEC and a general interaction potential U

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\Delta + \frac{1}{2}\int U\left(|\mathbf{r}-\mathbf{r}'|\right)\left|\psi\left(\mathbf{r}'\right)\right|^2 d\mathbf{r}'\right]\psi\,. \tag{3.1}$$

This equation can be viewed as the Euler-Lagrange condition that makes the action $\mathcal{S}$ extremal.

$$\mathcal{S} = \iint dt d\mathbf{r} \underbrace{\left[\frac{i\hbar}{2}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right) - \frac{\hbar^2}{2m}|\nabla\psi|^2 - \int d\mathbf{r}' U\left(|\mathbf{r}-\mathbf{r}'|\right)\left|\psi\left(\mathbf{r}'\right)\right|^2\left|\psi\left(\mathbf{r}\right)\right|^2\right]}_{\mathcal{L}}. \tag{3.2}$$

We now take the Ansatz

$$\psi = \sqrt{\rho}e^{i\phi}\,, \tag{3.3}$$

where ρ is positive and ρ as well as ϕ are dependent on time and position, and use it to rewrite the Lagrangian as

$$\mathcal{L} = \int dr\left[-\left[\hbar\rho\partial_t\phi + \frac{\hbar^2}{2m}\left(\rho\left(\nabla\phi\right)^2 + \frac{1}{4\rho}\left(\nabla\rho\right)^2\right)\right] - \frac{1}{2}\int dr' U(|r-r'|)\rho(r)\rho(r')\right]. \tag{3.4}$$

For this new Lagrangian the Euler Lagrange Equations are

$$\partial_t\rho + \frac{\hbar}{m}\nabla\left(\rho\nabla\phi\right) = 0, \tag{3.5}$$

$$\hbar\partial_t\phi + \frac{\hbar^2}{2m}\left(\nabla\phi\right)^2 + \frac{\hbar^2}{4m}\left(\frac{\left(\nabla\rho\right)^2}{2\rho^2} - \frac{\nabla^2\rho}{\rho}\right) + \int dr' U(|r-r'|)\rho(r') = 0. \tag{3.6}$$

Josserand et al point out that studies of superfluids based on equation 3.1 show periodicity after a critical density. Motivated by this we now search for a ground state in the form $\psi_0 \mathrm{e}^{-iE_0t/\hbar}$, where ψ_0 is lattice periodic for some lattice vectors. Ground state means here that given an average density the combination of lattice parameters and ψ_0 has the smallest possible energy. We should stress again that there is no external potential forcing the periodicity. It is an intrinsic property instead.

An important distinction to classical crystals arises here. There we have an integer (or simple fraction) number of atoms per unit cell. This is not necessarily the case here, where the lattice parameters and the density are independent. There is now in total three sets of parameters for the ground state: The average density n, the position of the lattice and the global phase. When one considers low-frequency perturbations of this ground state these become the three slowly varying fields $n(r,t)$, $\Phi(r,t)$ and the displacement field $u(r,t)$ of the lattice. In figure 1 we can see an example how a possible ground state density might be constructed from these fields.

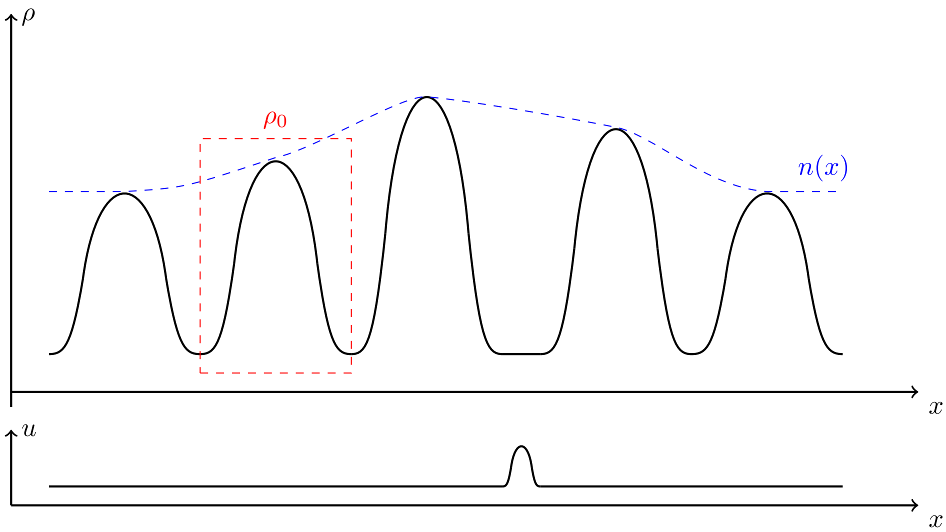

Figure 1: An explanatory sketch of how the density ρ could be build up from the fields δn and u with the help of ρ_0. In principle the ground state could look totally different, the only restriction here is that ρ_0 must be a periodic function.

Effective Lagrangian The goal is now an effective Lagrangian for those three fields. The technique used to do this is called Homogenization. It has its origin in fluid mechanics and is a useful tool to separate the long range behavior from short range effects due to in our case the lattice. This is particularly useful as our fields are in the approximation only supposed to be slowly varying anyways.

The Ansatz for the density and phase is as follows

$$\rho(r,t) \approx \rho_0(r - u(r,t)|n(r,t)) + \tilde{\rho}(r - u,n,t) \quad , \tag{3.7}$$

$$\phi(r,t) \approx \Phi(r,t) + \tilde{\phi}(r - u). \tag{3.8}$$

The $\tilde{\phi}(r - u)$ and $\tilde{\rho}(r - u,n,t)$ are small and vary quickly compared to the three fields. Thus begins the arduous task of inserting this Ansatz into the Lagrangian, collecting and simplifying therms. We skip this part in this review, the interested reader is again referred to [6] and the references within.

We obtain

$$\mathcal{L}_{\text{eff}} = -\hbar n \partial_t \Phi - \frac{\hbar^2}{2m}\left[n\left(\nabla\Phi\right)^2 - \rho_{ik}\left(\nabla\Phi - \frac{m}{\hbar}\frac{Du}{Dt}\right)_i\left(\nabla\Phi - \frac{m}{\hbar}\frac{Du}{Dt}\right)_k\right] - \mathcal{E}(n) - \frac{1}{2}\lambda_{ijkl}\epsilon_{ik}\epsilon_{jl} - \mu n \epsilon_{ll} \tag{3.9}$$

with

$$\frac{Du}{Dt} = \partial_t u + \frac{\hbar}{m}\nabla\Phi \cdot \nabla u. \tag{3.10}$$

The exact definition of the parameters is not relevant to us as we are only interested in their physical meaning. If needed they can be found in [6]. The two definitions we want for later replacements are

$$\epsilon_{ik} = -\frac{1}{2}(\partial_i u_k + \partial_k u_i) + \frac{1}{2}\partial_l u_i \partial_l u_k, \tag{3.11}$$

$$\mathcal{E}(n) = \mu n - \frac{1}{2V}\int dr \rho_0(r) \int dr' U(|r - r'|)\rho_0(r'). \tag{3.12}$$

$\mathcal{E}(n)$ corresponds to an internal energy, solely dependent on n. $\mu n \epsilon_{ll}$ is related to the compression, μ the chemical potential and the Lagrange multiplier to the density. ρ_{ij} can be interpreted as the "lattice part" of the supersolid since it has the property that in the considered cases of $\rho_{ij} = \rho(n)\delta_{i,j}$ for $\rho(n) \to n$ the system behaves like a ordinary solid. λ_{ijkl} as it is written in the Lagrangian has the interpretation of an elastic density energy.

3.2 Reduction to a 1D Quadratic Theory

We now have obtained Josserands effective Lagrangian. The goal is now to use this as motivation for our one dimensional quadratic model Lagrangian. We want to be able to compare to the GPE based description from section 2.2 and to be able to apply the results to dipolar supersolids in harmonic confinement. Making the Lagrangian quadratic lets us solve it analytically.

We begin by reducing the effective Lagrangian to one dimension

$$\mathcal{L}_{\text{eff}} = -\hbar n \partial_t \Phi - \frac{\hbar^2}{2m}\left[n\left(\partial_x \Phi\right)^2 - \rho_{xx}\left(\partial_x \Phi - \frac{m}{\hbar}\partial_t u - \partial_x \Phi \partial_x u\right)^2\right] - \mathcal{E}(n) - \frac{1}{2}\lambda \epsilon_{xx}\epsilon_{xx} - \mu n \epsilon_{xx}. \tag{3.13}$$

We consider only small perturbations in the three fields, up to quadratic order. For that purpose we make the substitution $n(r,t) \to n + \delta n(r,t)$, where n is the average density and δn a local variation with spatial average zero. From this point onward n will always refer to the average density. There are several substitutions and simplifications to do:

- $\mathcal{E}(n)$: The first term μn drops out completely. After the substitution, there is one constant term and one where the integral over it gives zero, both of which can be dropped. The second term is of greater interest. It is quadratic in ρ_0 and in case of U constant simplifies to $n(x,t)^2$. Again the terms we would keep would be $(\delta n)^2$.
  The replacement we choose is $\mathcal{E}(n) \to \kappa(\delta n)^2$, which corresponds to an attractive (for κ positive) homogeneous interaction in the unit cell.
- ρ_{xx}: This quantity assumes the role of a parameter. As discussed previously, for $\rho_{xx} \to n$ the system behaves like a ordinary solid. As such the replacement is $\rho_{xx} \to n_L$ where n_L is a parameter that indicates the solid character of the system. Its range is zero to n.
- ϵ_{xx}: We only consider terms at most quadratic in the fields. As ϵ_{xx} never appears without another field, we can simplify with the replacement $\epsilon_{xx} \to -\partial_x u$.
- $\mu n \epsilon_{xx} \to \xi \delta n \partial_x u$: This replacement is only $\mu \to -\xi$ and applying the ones before. Terms with the average density are dropped or incorporated into λ.

Applying these simplifications we arrive at the model Lagrangian density

$$\mathcal{L} = -\hbar \delta n \partial_t \Phi - \frac{\hbar^2}{2m}\left[n\left(\partial_x \Phi\right)^2 - \frac{n_L}{\hbar^2}\left(\hbar \partial_x \Phi - m \partial_t u\right)^2\right] - \frac{\kappa}{2}\left(\delta n\right)^2 - \frac{\lambda}{2}\left(\partial_x u\right)^2 - \xi \delta n \partial_x u. \tag{3.14}$$

3.3 Connection to Physical Observables

Generally we are interested in calculating physically relevant expectation values in our system. One reason for this is to possibly calculate the order parameters introduced in section 2.1. To do this we need to be able to express the physically relevant fields through the fields contained in our Lagrangian. The two important fields here are the phase field Φ and the density ρ. The phase field is simple, as it is still part of the Lagrangian. The density is a bit more challenging to consider. Keeping only the long range part from equation 3.7 we obtain

$$\rho \approx \rho_0(x - u(x,t)|n(x,t)), \tag{3.15}$$

where $\rho_0(x|n)$ is a periodic function in x with period a and the corresponding reciprocal lattice vector $k_0 = \pi/a$. It can therefore be expanded in a Fourier series

$$\rho = \sum_m c_m(n) \mathrm{e}^{imk_0(x-u)} \tag{3.16}$$

with

$$c_m(n(x,t)) = \int dy \rho_0(y|n(x,t)) \mathrm{e}^{imk_0 y}. \tag{3.17}$$

Considering only the longest range oscillations this reduces to

$$\rho \approx n(x,t) + c_1 \cos\left(k_0(x-u)\right) = n + \delta n(x,t) + 2c_1 \cos\left(k_0(x-u)\right) \tag{3.18}$$

where we neglect the dependence of c_1 on $n(x,t)$ and replace it by only the average density n. $n(x,t)$ is an average or smeared density here. To get back a more discrete view of the density operator we follow a procedure by Haldane [7]. We introduce a new field θ as the integrated density

$$\partial_x \theta = \delta n, \qquad \int dx \delta n = \theta \implies \theta + nx = \int dx' n(x',t)\,. \tag{3.19}$$

We reintroduce discretness by locating the particle at the points where the integrated density has integer values.

$$n(x,t) \to n(x,t) \sum_j \delta(\theta + nx - j). \tag{3.20}$$

We can simplify further:

$$\sum_j \delta(\theta + nx - j) = \sum_j \int \frac{dk}{2\pi} \mathrm{e}^{ik(\theta+nx-j)}. \tag{3.21}$$

This term can be evaluated further using Poisson summation, yielding

$$\sum_j \delta(\theta + nx - j) = \sum_{l\in\mathbb{Z}} \mathrm{e}^{2\pi i l(\theta+nx)}\,. \tag{3.22}$$

Thus in total the replacement reads

$$n(x,t) \to n(x,t) \sum_{l\in\mathbb{Z}} \mathrm{e}^{2\pi i l(\theta+nx)} = n(x,t) \left[1 + \sum_{l=1} 2\cos\left(2\pi l\left(\theta + nx\right)\right)\right], \tag{3.23}$$

which gives us the final form of the density

$$\rho \approx \left[n + \delta n(x,t)\right] \left[1 + \sum_{l=1} 2\cos\left(2\pi l\left(\theta + nx\right)\right)\right] + 2c_1 \cos\left(k_0(x-u)\right). \tag{3.24}$$

4 Studying the Model Lagrangian

At this point in the thesis we have finished the fundamentals: We have the model Lagrangian and understand its connection to more fundamental descriptions of a supersolid. We also know how to write down physically meaningful correlation functions. The next step is to study the model Lagrangian and to actually calculate correlation functions using the path integral formalism.

4.1 Correlation Functions in the Path Integral Formalism

The great advantage of having an effective quadratic Lagrangian theory is that ground state correlation functions are much simpler to calculate. We use the path integral formalism to calculate correlation functions. This section provides only a brief overview about the most important concepts, detailed introductions can be found in lecture notes or books such as [8],[9]. In the formalism we define a generating functional

$$Z[\{J_i\}] = \left[\prod_k \int \mathcal{D}\phi_k\right] \mathrm{e}^{\frac{i}{\hbar}S[\{\phi_j\}]+\int dX J_i\phi_i}, \tag{4.1}$$

where the J_i are functions called sources and $X = (t,x)$. The generating functional is closely related to the partition function Z via $Z = Z[\{J_i = 0\}]$. Correlation functions are defined via functional integrals as

$$\left\langle \prod_i \phi_i(X_i) \right\rangle = \frac{1}{Z}\left[\prod_k \int \mathcal{D}\phi_k\right] \prod_i \phi_i(X_i)\, \mathrm{e}^{\frac{i}{\hbar}S[\{\phi_j\}]}. \tag{4.2}$$

The generating functional can be used to find a different expression of the correlation functions

$$\left\langle \prod_i \phi_i(X_i) \right\rangle = \frac{1}{Z}\left[\prod_i \frac{\delta}{\delta J_i(X_i)}\right] Z[\{J_i\}]\Bigg|_{J_i=0}. \tag{4.3}$$

This is in principle hard to evaluate, but for a quadratic theory there are major simplifications. Let us view the special case of a quadratic Lagrangian in the case of two fields and two dimensions, so $i \in \{1,2\}$ and $Y = (t,x)$. Then the action is of the form

$$\begin{aligned} S[\phi_1,\phi_2] &= \frac{1}{2}\iint dY dY' \begin{pmatrix} \phi_1(Y), & \phi_2(Y) \end{pmatrix} G^{-1}(Y,Y') \begin{pmatrix} \phi_1(Y') \\ \phi_2(Y') \end{pmatrix} && (4.4)\\ &= \frac{1}{2}\iint dY dY'\ \mathbf{v}^T(Y) G^{-1}(Y,Y')\mathbf{v}(Y') && (4.5) \end{aligned}$$

Under the assumption that G^{-1} is invertible (meaning $G^{-1}G = \delta(Y - Y')$) and symmetric, the transformation $\tilde{\mathbf{v}} = \mathbf{v} - i\hbar G\mathbf{J}$ gives

$$\frac{i}{\hbar}S[\phi_1,\phi_2]+\int dY \mathbf{J}\mathbf{v} = \frac{1}{2}\frac{i}{\hbar}\iint dY dY'\ \tilde{\mathbf{v}}^T(Y)G^{-1}(Y,Y')\tilde{\mathbf{v}}(Y')+\frac{1}{2}i\hbar\iint dY dY'\ \mathbf{J}^T(Y)G(Y,Y')\mathbf{J}(Y') \tag{4.6}$$

This transformation gives us a much simpler expression for the generating functional

$$\begin{aligned} Z[\{J_i\}] &= \left[\left[\prod_k \int \mathcal{D}\tilde{\phi}_k\right] \mathrm{e}^{\frac{i}{\hbar}\left(S[\{\tilde{\phi}_j\}]\right)}\right] \mathrm{e}^{\frac{i\hbar}{2}\iint dY dY'\ \mathbf{J}^T(Y)G(Y,Y')\mathbf{J}(Y')} \\ &= Z\mathrm{e}^{\frac{1}{2}\iint dY dY'\ \mathbf{J}^T(Y)[i\hbar G(Y,Y')]\mathbf{J}(Y')} && (4.7) \end{aligned}$$

This allows us to calculate correlation functions easily, for instance the two point correlater

$$\langle \phi_i(X)\phi_j(X')\rangle = i\hbar G_{ij}(X,X'). \tag{4.8}$$

In the same way one can calculate the Fourier transformed correlation

$$Z[\{\overline{J}_i\}] = \left[\prod_k \int \mathcal{D}\overline{\phi}_k\right] \mathrm{e}^{\frac{i}{\hbar}S[\{\overline{\phi}_j\}]+\int \frac{dK}{(2\pi)^2}\overline{J}_i^\dagger\overline{\phi}_i} = Z\mathrm{e}^{\frac{1}{2}\int \frac{dK}{4\pi^2}\ \overline{\mathbf{J}}^\dagger(K)[i\hbar G(K)]\overline{\mathbf{J}}(K)}, \tag{4.9}$$

$$\left\langle \prod_i \overline{\phi}_i(K_i) \right\rangle = \frac{1}{Z} \left[\prod_i \left(4\pi^2\right) \frac{\delta}{\delta \overline{J}_i(X_i)} \right] Z[\{\overline{J}_i\}] \bigg|_{\overline{J}_i=0}, \tag{4.10}$$

$$\left\langle \overline{\phi}_i(K) \overline{\phi}_j(K') \right\rangle = i\hbar G_{ij}(K) 4\pi^2 \delta(K+K'). \tag{4.11}$$

In similar fashion to X we define $K = (\omega,k)$. When calculating the expectation value of fields in the exponent there is a great simplification one can make for a quadratic theory, which is usually mentioned as a consequence of the Wick-Theorem:

$$\left\langle \exp\left(\sum_i m_i \phi_i(X_i) \right) \right\rangle = \exp\left(\frac{1}{2} \left\langle \left[\sum_i m_i \phi_i(X_i) \right]^2 \right\rangle \right). \tag{4.12}$$

The proof of this is very simple in the path integral formalism. Simply observe that

$$\frac{1}{Z} Z[\{J_i\}] \bigg|_{J_i = -i\hbar m_i \delta(X-X_i)} = \left\langle \exp\left(\sum_i m_i \phi_i(X_i) \right) \right\rangle. \tag{4.13}$$

But also

$$\frac{1}{Z} Z[\{J_i\}] \bigg|_{J_i = -i\hbar m_i \delta(X-X_i)} = \exp\left(\frac{1}{2} \sum_{i,j} m_i m_j \left[i\hbar G(X_i,X_j) \right] \right) = \exp\left(\frac{1}{2} \left\langle \left[\sum_i m_i \phi_i(X_i) \right]^2 \right\rangle \right). \tag{4.14}$$

Generally, not setting the J_i to zero is a good way to calculate correlation functions which include fields in the exponent.

4.2 Calculating Momentum Space Correlations

After the brief introduction to the path integral formalism we now proceed to calculate the correlation functions of the quadratic Lagrangian

$$\mathcal{L} = -\hbar \delta n \partial_t \Phi - \frac{\hbar^2}{2m} \left[n \left(\partial_x \Phi\right)^2 - \frac{n_L}{\hbar^2} \left(\hbar \partial_x \Phi - m \partial_t u\right)^2 \right] - \frac{\kappa}{2} \left(\delta n\right)^2 - \frac{\lambda}{2} \left(\partial_x u\right)^2 - \xi \delta n \partial_x u. \tag{4.15}$$

The first thing to notice about this Lagrangian is that it consists of three fields. This is unexpected since this Lagrangian describes a supersolid, which breaks two symmetries: The $U(1)$ gauge symmetry and translational symmetry. For each broken symmetry one expects a massless Goldstone mode, so we would expect to see only two fields in the description. It turns out that only two of our three fields are independent. δn becomes dependent quite naturally when we transform to the Hamiltonian, as the Lagrangian does not contain its derivatives. To transform to the Hamiltonian density, we first need the canonical momenta

$$\Pi = \frac{\partial \mathcal{L}}{\partial \left(\partial_t \Phi\right)} = -\hbar \delta n,$$

$$p = \frac{\partial \mathcal{L}}{\partial \left(\partial_t u\right)} = -n_L \left(\hbar \partial_x \Phi - m \partial_t u\right).$$

This gives the Hamiltonian density

$$\mathcal{H} = \partial_t \Phi \frac{\partial \mathcal{L}}{\partial \partial_t \Phi} + \partial_t u \frac{\partial \mathcal{L}}{\partial \partial_t u} - \mathcal{L} \tag{4.16}$$

$$= \frac{\kappa}{2\hbar^2} \Pi^2 + \frac{\hbar^2}{2m} n \left(\partial_x \Phi\right)^2 + \frac{1}{2mn_L} p^2 + \frac{\lambda}{2} \left(\partial_x u\right)^2 - \frac{\xi}{\hbar} \Pi \partial_x u + \frac{\hbar}{m} p \partial_x \Phi, \tag{4.17}$$

where we now only have two fields and two momenta, δn depending on the momentum Π. We can obtain a Lagrangian that contains only two fields by simply transforming back. To do this, we need Hamiltons equations of motion:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial \Pi} &= -\frac{\xi}{\hbar}\partial_x u + \frac{\kappa}{\hbar^2}\Pi = \partial_t \Phi, \\
\frac{\delta \mathcal{H}}{\delta \Phi} &= -\frac{\hbar^2}{m} n \partial_x^2 \Phi - \frac{\hbar}{m}\partial_x p = -\partial_t \Pi, \\
\frac{\partial \mathcal{H}}{\partial p} &= \frac{1}{m n_L} p + \frac{\hbar}{m}\partial_x \Phi = \partial_t u, \\
\frac{\delta \mathcal{H}}{\delta u} &= -\lambda \partial_x^2 u + \frac{\xi}{\hbar}\partial_x \Pi = -\partial_t p.
\end{aligned}$$

This allows us to calculate

$$\begin{aligned}
\mathcal{L} &= \Pi \frac{\partial \mathcal{H}}{\partial \Pi} + p\frac{\partial \mathcal{H}}{\partial p} - \mathcal{H} && (4.18)\\
&= \frac{\hbar^2}{2\kappa}\left(\partial_t \Phi\right)^2 - \frac{\hbar^2}{2m}\left(n - n_L\right)\left(\partial_x \Phi\right)^2 + \frac{m n_L}{2}\left(\partial_t u\right)^2 - \frac{1}{2\kappa}\left(\lambda\kappa - \xi^2\right)\left(\partial_x u\right)^2 && (4.19)\\
&\quad + \frac{\hbar \xi}{\kappa}\left(\partial_t \Phi\right)\left(\partial_x u\right) - \hbar n_L \left(\partial_t u\right)\left(\partial_x \Phi\right). && (4.20)
\end{aligned}$$

The fact that we now only need to deal with two fields instead of three makes the calculation of correlation functions simpler.

Dimensionless Units To further ease calculation we introduce dimensionless units.

$$\phi_1 := \Phi, \quad \phi_2 := \frac{\sqrt{nm\kappa}}{\hbar} u, \tag{4.21}$$

$$\hat{t} := \frac{\kappa n}{\hbar} t, \quad \hat{x} = \frac{\sqrt{nm\kappa}}{\hbar} x, \tag{4.22}$$

$$E := \kappa n, \quad \gamma := \frac{n_L}{n}, \quad \epsilon_1 := \frac{\xi}{E}, \quad \epsilon_2 := \frac{\lambda}{En}, \tag{4.23}$$

$$s_0 = \sqrt{\frac{\hbar^2 n}{\kappa m}} = \frac{\hbar}{E} n \sqrt{\frac{E}{m}}. \tag{4.24}$$

This leads to our Lagrangian taking form

$$\mathcal{L}(\hat{x},\hat{t}) = \frac{En}{2}\left[\begin{array}{l} \left(\partial_{\hat{t}}\phi_1\right)^2 - (1-\gamma)\left(\partial_{\hat{x}}\phi_1\right)^2 + \gamma\left(\partial_{\hat{t}}\phi_2\right)^2 - \left(\epsilon_2 - \epsilon_1^2\right)\left(\partial_{\hat{x}}\phi_2\right)^2 \\ +2\epsilon_1\left(\partial_{\hat{x}}\phi_2\right)\left(\partial_{\hat{t}}\phi_1\right) - 2\gamma\left(\partial_{\hat{t}}\phi_2\right)\left(\partial_{\hat{x}}\phi_1\right) \end{array}\right]. \tag{4.25}$$

We introduce the Fourier transform as

$$\overline{\phi}_i = \mathcal{F}\left[\phi_i\right](\omega,k) = \iint dx dt \phi_i e^{-i(kx+\omega t)}, \tag{4.26}$$

$$\phi_i = \mathcal{F}^{-1}\left[\overline{\phi}_i\right](x,t) = \iint \frac{dk}{2\pi}\frac{d\omega}{2\pi}\overline{\phi}_i e^{i(kx+\omega t)}. \tag{4.27}$$

This allows one to write the action as

$$\begin{aligned}
S/\hbar &= \frac{1}{\hbar}\iint dx dt \mathcal{L} = s_0 \iint d\hat{x} d\hat{t} \mathcal{L}(\hat{x},\hat{t}) && (4.28)\\
&= \frac{s_0}{2}\iint \frac{d\hat{k}}{2\pi}\frac{d\hat{\omega}}{2\pi}\ \overline{\phi}_1\overline{\phi}_1^*\left[\hat{\omega}^2 - (1-\gamma)\hat{k}^2\right] + \overline{\phi}_2\overline{\phi}_2^*\left[\gamma\hat{\omega}^2 - (\epsilon_2 - \epsilon_1^2)\hat{k}^2\right] && (4.29)\\
&\qquad + \overline{\phi}_2\overline{\phi}_1^*\left[\epsilon_1 - \gamma\right]\hat{\omega}\hat{k} + \overline{\phi}_1\overline{\phi}_2^*\left[\epsilon_1 - \gamma\right]\hat{\omega}\hat{k}.
\end{aligned}$$

This expression lends itself to a matrix form

$$S/\hbar = \frac{s_0}{2}\iint \frac{d\hat{k}}{2\pi}\frac{d\hat{\omega}}{2\pi}\begin{pmatrix}\overline{\phi}_1^* & \overline{\phi}_2^*\end{pmatrix}\begin{pmatrix}\hat{\omega}^2-(1-\gamma)\hat{k}^2 & [\epsilon_1-\gamma]\,\hat{\omega}\hat{k}\\ [\epsilon_1-\gamma]\,\hat{\omega}\hat{k} & \gamma\hat{\omega}^2-(\epsilon_2-\epsilon_1^2)\hat{k}^2\end{pmatrix}\begin{pmatrix}\overline{\phi}_1\\ \overline{\phi}_2\end{pmatrix} \tag{4.30}$$

$$= \frac{s_0}{2}\iint \frac{d\hat{k}}{2\pi}\frac{d\hat{\omega}}{2\pi}\begin{pmatrix}\overline{\phi}_1^* & \overline{\phi}_2^*\end{pmatrix} G^{-1}(\hat{\omega},\hat{k})\begin{pmatrix}\overline{\phi}_1\\ \overline{\phi}_2\end{pmatrix}. \tag{4.31}$$

As discussed in the previous section, to obtain the correlation functions one now only needs to find G. This is relatively simple to write down as

$$G(\hat{\omega},\hat{k}) = \frac{1}{\det(G^{-1}(\hat{\omega},\hat{k}))}\begin{pmatrix}\gamma\hat{\omega}^2-(\epsilon_2-\epsilon_1^2)\hat{k}^2 & -[\epsilon_1-\gamma]\,\hat{\omega}\hat{k}\\ -[\epsilon_1-\gamma]\,\hat{\omega}\hat{k} & \hat{\omega}^2-(1-\gamma)\hat{k}^2\end{pmatrix}. \tag{4.32}$$

One needs to be careful as not to forget to invert s_0 as well. The correct generating functional reads

$$Z[\overline{J}_1,\overline{J}_2] = Z\exp\left(-\frac{i}{2s_0}\int\frac{d\hat{K}}{4\pi^2}\overline{\mathbf{J}}^\dagger(\hat{K})G(\hat{K})\overline{\mathbf{J}}(\hat{K})\right). \tag{4.33}$$

The determinant still needs to be calculated to have the full expression for G:

$$\det(G^{-1}(\hat{\omega},\hat{k})) = \gamma\hat{\omega}^4+(1-\gamma)(\epsilon_2-\epsilon_1^2)\hat{k}^4-\left[\gamma(1-\gamma)+(\epsilon_2-\epsilon_1^2)+(\epsilon_1-\gamma)^2\right]\hat{\omega}^2\hat{k}^2 \tag{4.34}$$

$$= \gamma\left(\hat{\omega}^2-\hat{v}_+^2\hat{k}^2\right)\left(\hat{\omega}^2-\hat{v}_-^2\hat{k}^2\right), \tag{4.35}$$

where

$$\hat{v}_\pm^2 = \frac{1}{2\gamma}\left[\epsilon_2+\gamma-2\epsilon_1\gamma\pm\sqrt{(\epsilon_2+\gamma-2\epsilon_1\gamma)^2-4\gamma\,(1-\gamma)\,(\epsilon_2-\epsilon_1^2)}\right]. \tag{4.36}$$

These are the same velocities that Josserand [6] also observes in his calculations (after accounting for the substitutions we have made), which is a nice sanity check. Also we can see that one obtains two real velocities ($\hat{v}_\pm^2 > 0$) iff $(\epsilon_2-\epsilon_1^2) > 0$ (see App. A.1). In figure 2 we show the two velocities for a fixed γ. We can now explicitly write down the two point correlation functions

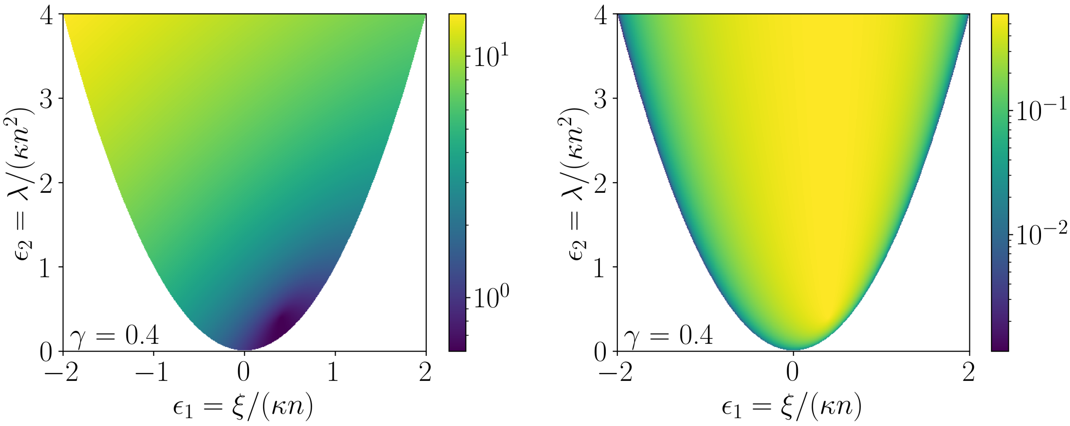

Figure 2: $\hat{v}_\pm^2$ shown for fixed $\gamma = 0.4$ and variable ϵ_1, ϵ_2. The plot on the left shows $\hat{v}_+^2$ while on the right $\hat{v}_-^2$ is shown. Values are only given for $\epsilon_2-\epsilon_1^2 > 0$ since only then both velocities are real. One can see that $\hat{v}_-^2$ goes to zero at that border.

$$\left\langle \overline{\phi}_1(\hat{\omega},\hat{k})\overline{\phi}_1(\hat{\omega}',\hat{k}')\right\rangle = \frac{4\pi^2 i}{s_0}\frac{\gamma\hat{\omega}^2 - (\epsilon_2 - \epsilon_1^2)\hat{k}^2}{\gamma\left(\hat{\omega}^2 - \hat{v}_+^2\hat{k}^2\right)\left(\hat{\omega}^2 - \hat{v}_-^2\hat{k}^2\right)}\delta(\hat{k}+\hat{k}')\delta(\hat{\omega}+\hat{\omega}'), \tag{4.37}$$

$$\left\langle \overline{\phi}_2(\hat{\omega},\hat{k})\overline{\phi}_2(\hat{\omega}',\hat{k}')\right\rangle = \frac{4\pi^2 i}{s_0}\frac{\hat{\omega}^2 - (1-\gamma)\hat{k}^2}{\gamma\left(\hat{\omega}^2 - \hat{v}_+^2\hat{k}^2\right)\left(\hat{\omega}^2 - \hat{v}_-^2\hat{k}^2\right)}\delta(\hat{k}+\hat{k}')\delta(\hat{\omega}+\hat{\omega}'), \tag{4.38}$$

$$\left\langle \overline{\phi}_1(\hat{\omega},\hat{k})\overline{\phi}_2(\hat{\omega}',\hat{k}')\right\rangle = \frac{4\pi^2 i}{s_0}\frac{-\left[\epsilon_1 - \gamma\right]\hat{\omega}\hat{k}}{\gamma\left(\hat{\omega}^2 - \hat{v}_+^2\hat{k}^2\right)\left(\hat{\omega}^2 - \hat{v}_-^2\hat{k}^2\right)}\delta(\hat{k}+\hat{k}')\delta(\hat{\omega}+\hat{\omega}'). \tag{4.39}$$

This can be written in a more compact form (see appendix A.2)

$$\left\langle \overline{\phi}_1(\hat{\omega},\hat{k})\overline{\phi}_1(\hat{\omega}',\hat{k}')\right\rangle = \frac{4\pi^2 i}{s_0}\left[\sum_{\pm}\frac{\alpha_{11}^{\pm}\hat{v}_{\pm}}{\hat{\omega}^2 - \hat{v}_{\pm}^2\hat{k}^2}\right]\delta(\hat{k}+\hat{k}')\delta(\hat{\omega}+\hat{\omega}'), \tag{4.40}$$

$$\left\langle \overline{\phi}_2(\hat{\omega},\hat{k})\overline{\phi}_2(\hat{\omega}',\hat{k}')\right\rangle = \frac{4\pi^2 i}{s_0}\left[\sum_{\pm}\frac{\alpha_{22}^{\pm}\hat{v}_{\pm}}{\hat{\omega}^2 - \hat{v}_{\pm}^2\hat{k}^2}\right]\delta(\hat{k}+\hat{k}')\delta(\hat{\omega}+\hat{\omega}'), \tag{4.41}$$

$$\left\langle \overline{\phi}_1(\hat{\omega},\hat{k})\overline{\phi}_2(\hat{\omega}',\hat{k}')\right\rangle = \frac{4\pi^2 i}{s_0}\left[\sum_{\pm}\frac{\alpha_{12}^{\pm}\hat{\omega}/\hat{k}}{\hat{\omega}^2 - \hat{v}_{\pm}^2\hat{k}^2}\right]\delta(\hat{k}+\hat{k}')\delta(\hat{\omega}+\hat{\omega}'). \tag{4.42}$$

with

$$\alpha_{11}^{\pm} = \mp\frac{\frac{1}{\gamma}(\epsilon_2 - \epsilon_1^2) - \hat{v}_{\pm}^2}{\hat{v}_{\pm}\left(\hat{v}_+^2 - \hat{v}_-^2\right)}, \tag{4.43}$$

$$\alpha_{22}^{\pm} = \mp\frac{1}{\gamma}\frac{(1-\gamma) - \hat{v}_{\pm}^2}{\hat{v}_{\pm}\left(\hat{v}_+^2 - \hat{v}_-^2\right)}, \tag{4.44}$$

$$\alpha_{12}^{\pm} = \mp\frac{1}{\gamma}\frac{\left[\epsilon_1 - \gamma\right]}{\hat{v}_+^2 - \hat{v}_+^2}. \tag{4.45}$$

There are two things to notice with these correlation functions: Firstly, they have divergences at the characteristic velocities. That in itself is not a problem, but Fourier transforming then becomes a tedious task. Here it is useful to use a Wick-Rotation and move to imaginary time. This avoids the divergences and makes integration easier. The second thing to notice is the peculiar form of the correlation functions. When comparing their form to textbooks like [8], they appear to be the sum of two simpler correlation functions for free bosonic fields. This indicates the existence of a transformation, which allows us to separate our Lagrangian into two simple independent ones. Interesting is also that one of the fields appears like the canonically conjugate field of the other. We shall first discuss the Wick-Rotation and imaginary time and then in the next section look for the transformation.

4.3 Imaginary Time

Imaginary time might seem very odd at first, especially when trying to attach physical meaning to the concept. However, as a mathematical tool it is incredibly useful, as it simplifies calculations enormously. Take for instance the integrals with integrands $\sim 1/(\hat{\omega}^2 - \hat{v}_{\pm}^2\hat{k}^2)$. A switch to imaginary time would in this case give integrands $\sim 1/(\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2)$, which gets rid of most of the divergences. A sketch of this is presented in figure 3. We can see there that for each fixed $\hat{k}$ the integration with real and imaginary time gives the same result (after taking care of the divergences correctly) due to contour integration. The only thing to be careful about is the divergence at the origin: As long as it exists we can not rotate the original plane into the imaginary time plane without crossing a divergence. Also we would still have a divergence in the

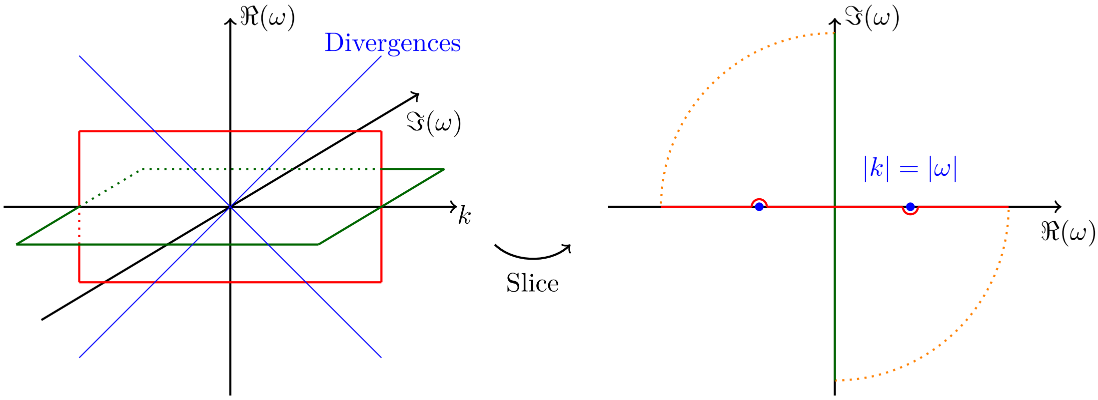

Figure 3: Integration domains before and after imaginary time is introduced. On the left one can see a slice of the right sketch for one fixed k. The integration paths for real and imaginary time can be seen to give the same result due to complex contour integration.

new integration domain. As it turns out, correlation functions with a divergence at the origin are not physical and do diverge. This gives some confidence that we can switch to imaginary time for the calculation of correlation functions. The argument presented here is not rigorous and one should still be careful when analyzing the results.
We now introduce imaginary time as

$$\tau = it, \quad t = -i\tau, \tag{4.46}$$

which then means for the derivatives

$$\partial_\tau = -i\partial_t, \quad \partial_t = i\partial_\tau. \tag{4.47}$$

These definitions also make clear that imaginary time is made dimensionless exactly the same way normal time is. This gives the new Lagrangian

$$\mathcal{L}(\hat{x},\hat{\tau}) = \frac{En}{2}\left[\begin{array}{l} -\left(\partial_{\hat{\tau}}\phi_1\right)^2 - (1-\gamma)\left(\partial_{\hat{x}}\phi_1\right)^2 - \gamma\left(\partial_{\hat{\tau}}\phi_2\right)^2 - \left(\epsilon_2 - \epsilon_1^2\right)\left(\partial_{\hat{x}}\phi_2\right)^2 \\ +2i\epsilon_1\left(\partial_{\hat{x}}\phi_2\right)\left(\partial_{\hat{\tau}}\phi_1\right) - 2i\gamma\left(\partial_{\hat{\tau}}\phi_2\right)\left(\partial_{\hat{x}}\phi_1\right) \end{array}\right]. \tag{4.48}$$

We can again write down the Fourier transformed fields

$$\overline{\phi}_i = \mathcal{F}\left[\phi_i\right](q,k) = \iint dx d\tau \phi_i e^{-i(kx+q\tau)}, \tag{4.49}$$

$$\phi_i = \mathcal{F}^{-1}\left[\overline{\phi}_i\right](x,\tau) = \iint \frac{dk}{2\pi}\frac{dq}{2\pi}\overline{\phi}_i e^{i(kx+q\tau)}, \tag{4.50}$$

and then directly see the new action from this

$$\begin{aligned} \mathcal{S} = iS/\hbar &= \frac{s_0}{2}\iint \frac{d\hat{k}}{2\pi}\frac{d\hat{q}}{2\pi}\begin{pmatrix} \overline{\phi}_1^* & \overline{\phi}_2^* \end{pmatrix}\begin{pmatrix} -\hat{q}^2 - (1-\gamma)\hat{k}^2 & i\left[\epsilon_1 - \gamma\right]\hat{q}\hat{k} \\ i\left[\epsilon_1 - \gamma\right]\hat{q}\hat{k} & -\gamma\hat{q}^2 - (\epsilon_2 - \epsilon_1^2)\hat{k}^2 \end{pmatrix}\begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \end{pmatrix} \\ &= -i\frac{s_0}{2}\iint \frac{d\hat{k}}{2\pi}\frac{d\hat{q}}{2\pi}\begin{pmatrix} \overline{\phi}_1^* & \overline{\phi}_2^* \end{pmatrix} G^{-1}(\hat{q},\hat{k})\begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \end{pmatrix}. \end{aligned} \tag{4.51}$$

With the determinant

$$\det\left(G^{-1}(\hat{q},\hat{k})\right) = \gamma\left(\hat{q}^2 + \hat{v}_+^2\hat{k}^2\right)\left(\hat{q}^2 + \hat{v}_-^2\hat{k}^2\right), \tag{4.52}$$

we then also trivially know the inverse

$$G_{11}(\hat{q},\hat{k}) = -\sum_{\pm} \frac{\alpha_{11}^{\pm}\hat{v}_{\pm}}{\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2}, \tag{4.53}$$

$$G_{22}(\hat{q},\hat{k}) = -\sum_{\pm} \frac{\alpha_{22}^{\pm}\hat{v}_{\pm}}{\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2}, \tag{4.54}$$

$$G_{12}(\hat{q},\hat{k}) = -i\sum_{\pm} \frac{\alpha_{12}^{\pm}\hat{q}/\hat{k}}{\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2}. \tag{4.55}$$

Finally this gives the two point correlators

$$\left\langle \overline{\phi}_1(\hat{q},\hat{k})\overline{\phi}_1(\hat{\omega}',\hat{k}')\right\rangle = \frac{4\pi^2}{s_0}\left[\sum_{\pm} \frac{\alpha_{11}^{\pm}\hat{v}_{\pm}}{\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2}\right]\delta(\hat{k}+\hat{k}')\delta(\hat{q}+\hat{q}'), \tag{4.56}$$

$$\left\langle \overline{\phi}_2(\hat{q},\hat{k})\overline{\phi}_2(\hat{\omega}',\hat{k}')\right\rangle = \frac{4\pi^2}{s_0}\left[\sum_{\pm} \frac{\alpha_{22}^{\pm}\hat{v}_{\pm}}{\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2}\right]\delta(\hat{k}+\hat{k}')\delta(\hat{q}+\hat{q}'), \tag{4.57}$$

$$\left\langle \overline{\phi}_1(\hat{q},\hat{k})\overline{\phi}_2(\hat{\omega}',\hat{k}')\right\rangle = \frac{4\pi^2}{s_0}i\left[\sum_{\pm} \frac{\alpha_{12}^{\pm}\hat{q}/\hat{k}}{\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2}.\right]\delta(\hat{k}+\hat{k}')\delta(\hat{q}+\hat{q}'). \tag{4.58}$$

For further simplicity when calculating future correlation functions we write down the generating functional

$$Z[\{\overline{J}_i\}] = \left[\prod_k \int D\overline{\phi}_k\right] \mathrm{e}^{\mathcal{S}+\int \frac{dQ}{4\pi^2}\overline{\phi}_i\overline{J}_i} = Z\exp\left(-\frac{1}{2s_0}\int \frac{d\hat{Q}}{4\pi^2}\overline{\boldsymbol{J}}(\hat{Q})G(\hat{Q})\overline{\boldsymbol{J}}(\hat{Q})\right), \tag{4.59}$$

with $\hat{Q} = (\hat{q},\hat{k})$ as well as the definition for observables

$$\left\langle \prod_k \overline{\phi}_k(\hat{Q}_k)\right\rangle = \left[\prod_k 4\pi^2 \frac{\delta}{\delta\overline{J}_k(\hat{Q}_k)}\right] Z[\{\overline{J}_i\}]\Bigg|_{\overline{J}_k=0}. \tag{4.60}$$

4.4 Real-Space Correlations

With this it is now also possible to calculate the real-space (but still imaginary time) correlation functions by Fourier transforming again. One can see that the simple quadratic correlations diverge:

$$\left\langle \phi_i(\hat{X})\phi_i(\hat{X}')\right\rangle = \int \frac{d\hat{Q}}{4\pi^2}\frac{1}{s_0}\left[\sum_{\pm} \frac{\alpha_{11}^{\pm}\hat{v}_{\pm}}{\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2}\right]\mathrm{e}^{i\hat{Q}(\hat{X}-\hat{X}')} = \infty. \tag{4.61}$$

This is not to much of a concern, as these do not constitute observables. More interesting are terms of the form

$$\frac{1}{2}\left\langle (\phi_i(\hat{X}) - \phi_i(\hat{X}'))^2\right\rangle = \frac{1}{s_0}\int \frac{d\hat{Q}}{4\pi^2}\left[1-\cos(\hat{Q}(\hat{X}-\hat{X}'))\right]\left[\sum_{\pm} \frac{\alpha_{11}^{\pm}\hat{v}_{\pm}}{\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2}\right]. \tag{4.62}$$

These still diverge, but there is a distinction now. The first integrand had a divergence at zero but also did not tend to zero quickly enough in the limit of $|\hat{Q}| \to \infty$. The second integrand fixes the first but still has the later issue. However,this is where we can begin arguing from a physical point of view that there is need for a short range cutoff, since our effective theory is not expected to describe the system precisely at short range. We can then introduce a short range cutoff, which in turn leads to a UV cutoff α in momentum space. For ease of calculation we choose this cutoff as

$$\mathrm{e}^{-\hat{\alpha}|\hat{k}|}. \tag{4.63}$$

With the integral now convergent we can calculate this correlation function. The calculation can be found in appendix A.3. The result obtained is

$$\frac{1}{2}\left\langle(\phi_i(\hat{X})-\phi_i(\hat{X}'))^2\right\rangle = \frac{1}{2\pi s_0}\sum_{\pm}\alpha_{ii}^{\pm}\ln\left(\sqrt{(1+\hat{v}_\pm\left|\Delta\hat{\tau}\right|/\hat{\alpha})^2+(\Delta\hat{x}/\hat{\alpha})^2}\right). \tag{4.64}$$

For $((\Delta\hat{x})^2+(\Delta\hat{\tau})^2)/\hat{\alpha}^2 \gg 1$, this is approximated by

$$\frac{1}{2}\left\langle(\phi_i(\hat{X})-\phi_i(\hat{X}'))^2\right\rangle \approx \frac{1}{2\pi s_0}\sum_{\pm}\alpha_{ii}^{\pm}\ln\left(\sqrt{\hat{v}_\pm^2\left|\Delta\hat{\tau}\right|^2/\hat{\alpha}^2+\left(\Delta\hat{x}\right)^2/\hat{\alpha}^2}\right). \tag{4.65}$$

This kind of correlation function also has physical meaning. In section 2.1 we introduced off-diagonal long range order, the correlation function this order is tied to is

$$\left\langle\psi(X)\psi^\dagger(X')\right\rangle = \left\langle\sqrt{\rho(X)\rho(X')}\mathrm{e}^{i(\Phi(X)-\Phi(X'))}\right\rangle. \tag{4.66}$$

When we consider only the longest range contribution to this expectation value this simplifies to

$$\left\langle\psi(X)\psi^\dagger(X')\right\rangle \approx n\left\langle\mathrm{e}^{i(\Phi(X)-\Phi(X'))}\right\rangle = n\mathrm{e}^{-\frac{1}{2}\left\langle(\Phi(X)-\Phi(X'))^2\right\rangle}. \tag{4.67}$$

This means that with the result from above we obtain in the limit $\Delta x \to \infty$

$$\left\langle\psi(x)\psi^\dagger(x')\right\rangle \approx n\left|\frac{\Delta\hat{x}}{\hat{\alpha}}\right|^{-\sum_\pm \frac{1}{4\pi s_0}\alpha_{11}^{\pm}}. \tag{4.68}$$

We see that we do have quasi off-diagonal long range order.

5 Hamiltonian Treatment

As mentioned in section 4.2 before, the correlation functions we have calculated before look suspiciously similar to correlation functions for 1D free bosonic fields. To be more precise, ϕ_1 and ϕ_2 seem like conjugate fields in these correlation functions. With conjugate we mean something along the lines of $[\partial_x\phi_1,\phi_2'] \sim \delta(X-X')$ (see again [8]). This intuition turns out to be correct, in the following we describe the transformation.

The goal of this section is to find a way to simplify the problem. We have technically solved the problem on the Lagrangian side, however certain correlation functions containing terms such as δn are unpleasant to evaluate. The goal is still to analyze perturbations, simplifying the problem now will make this much easier later.

5.1 Transformation

With this motivation let us begin by first restating the Hamiltonian we have thus far

$$\mathcal{H} = \frac{\kappa}{2\hbar^2}\Pi^2 + \frac{\hbar^2}{2m}n\left(\partial_x\Phi\right)^2 + \frac{1}{2mn_L}p^2 + \frac{\lambda}{2}\left(\partial_x u\right)^2 - \frac{\xi}{\hbar}\Pi\partial_x u + \frac{\hbar}{m}p\partial_x\Phi\,. \tag{5.1}$$

In line with our motivation, we now define a ”new” field and its canonical conjugate

$$\partial_x\theta = -\frac{1}{\hbar}\Pi, \quad \partial_x\Phi = -\frac{1}{\hbar}\Pi_\theta. \tag{5.2}$$

As the observant reader may have noticed, this field is not exactly new as we have introduced it in section 3.3 already, then motivated by Haldane [7]. The fact that this field has a physical meaning already leads us to favor this definition instead of trying a similar approach with the second field u.

To more precisely state the motivation: When considering the correlation functions from section 4.2 and comparing them to the usual correlation functions for Luttinger Liquids, the two fields seem to be different. One is conjugate to the other. With this definition we switch one of the fields for its canonical conjugate. We can hope that this allows us to treat the fields more easily in the Hamiltonian framework, as they now "serve the same purpose".
The second motivation is that this replacement separates the momenta and coordinates in the Hamiltonian.
Before that we still need to check the validity of our slightly unusual replacement. We begin with the known commutator

$$[\Pi,\Phi'] = \delta(X - X'), \tag{5.3}$$

where it is obvious that interchanging X and X' has no effect. Here Φ' is short for $\Phi(X')$. Now consider

$$\partial_{x'}\left[\Pi,\Phi'\right] = -\frac{1}{\hbar}\left[\Pi,\Pi_\theta\right] = \partial_x\left[\theta,\Pi_\theta\right]. \tag{5.4}$$

The left hand side is known and especially also

$$\partial_{x'}\delta(x - x') = -\partial_x\delta(x - x') \tag{5.5}$$

In total we can now conclude that θ and Π_θ are canonically conjugate, as (theoretically only up to a constant)

$$[\Pi_\theta,\theta] = \delta(x - x'). \tag{5.6}$$

As we now know that the transformation is in a sense canonical, we can proceed by writing down this new Hamiltonian

$$\mathcal{H} = \frac{\kappa}{2}\left(\partial_x\theta\right)^2 + \frac{n}{2m}\left(\Pi_\theta\right)^2 + \frac{1}{2mn_L}p^2 + \frac{\lambda}{2}\left(\partial_x u\right)^2 + \xi\partial_x\theta\partial_x u - \frac{1}{m}p\Pi_\theta. \tag{5.7}$$

We can now conveniently write this Hamiltonian in matrix form and switch to dimensionless units

$$\mathcal{H} = \frac{n}{2m}\begin{pmatrix} \Pi_\theta & p/n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1/\gamma \end{pmatrix}\begin{pmatrix} \Pi_\theta \\ p/n \end{pmatrix} + \frac{\kappa}{2}\begin{pmatrix} \partial_x\theta & \partial_x(nu) \end{pmatrix}\begin{pmatrix} 1 & \epsilon_1 \\ \epsilon_1 & \epsilon_2 \end{pmatrix}\begin{pmatrix} \partial_x\theta \\ \partial_x(nu) \end{pmatrix} \tag{5.8}$$

$$= \frac{1}{2}\frac{\hbar^2 n^3}{m}\begin{pmatrix} \hat{\Pi}_\theta & \hat{p} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1/\gamma \end{pmatrix}\begin{pmatrix} \hat{\Pi}_\theta \\ \hat{p} \end{pmatrix} + \frac{1}{2}\kappa n^2\frac{m\kappa}{n\hbar^2}\begin{pmatrix} \partial_{\hat{x}}\hat{\theta} & \partial_{\hat{x}}\hat{u} \end{pmatrix}\begin{pmatrix} 1 & \epsilon_1 \\ \epsilon_1 & \epsilon_2 \end{pmatrix}\begin{pmatrix} \partial_{\hat{x}}\hat{\theta} \\ \partial_{\hat{x}}\hat{u} \end{pmatrix} \tag{5.9}$$

$$= \frac{1}{2}\kappa n^2\left[\frac{n\hbar^2}{m\kappa}v_1^T M_1 v_1 + \frac{m\kappa}{n\hbar^2}v_2^T M_2 v_2\right]. \tag{5.10}$$

In this context $\hat{u} = nu$, $\Pi_\theta = \hbar n\hat{\Pi}_\theta$ and $p/n = \hbar n\hat{p}$ (also $\hat{\theta} = \theta$ for aesthetic consistency).

Canonical Transformation Our aim now is to find another canonical transformation that simultaneously diagonalizes both M_1 and M_2. If both matrices would be diagonalized by the same orthogonal transformation the problem would be simple. However, we know that two matrices have a common Eigenbasis (e.g. are simultaneously unitarily diagonalizable) only if they commute. This is in general not the case for our two matrices. The idea that works here is to use scaling transformations

$$\hat{\Pi}'_\theta = \frac{1}{s_1}\hat{\Pi}_\theta \qquad \hat{\theta}' = s_1\hat{\theta}$$
$$\hat{p}' = \frac{1}{s_2}\hat{p} \qquad \hat{u}\prime = s_2\hat{u}.$$

In matrix form this reads

$$S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1/s_1 & 0 \\ 0 & 1/s_2 \end{pmatrix}. \tag{5.11}$$

We use the scaling transformation to make the matrices commute. This allows us to simultaneously diagonalize them with an orthogonal transformation W as the matrices are symmetric. It turns out that we need another scaling transformation Q afterward to get rid of the imaginary units we have picked up during the transformation. As this calculation is quite lengthy it can be found in appendix B. The first scaling is applied in appendix B.1, the diagonalization is done in appendix B.2 and the final scaling is applied in appendix B.3.

The full transformation is then given by

$$W_\phi = QW^TS, \quad W_\Pi = Q^{-1}W^TS^{-1}, \tag{5.12}$$

where W_ϕ transforms the fields to the new ones while W_Π does the same for the momenta. Both can be found in appendix B.4. Generally for a canonical transformation, we want to conserve the commutator. Say for instance we have $\Pi'_i = U_{ij}\Pi_j$ and $\phi'_k = P_{kl}\phi_l$. Then the commutator would read

$$[\Pi'_i,\phi'_k] = U_{ij}P_{kl}\,[\Pi_j,\phi_l] = U_{ij}(P^T)_{jk} \stackrel{!}{=} \delta_{ik}. \tag{5.13}$$

This condition in matrix form is $UP^T = I$. In our case we calculate

$$W_\phi W_\Pi^T = QW^TS(S^{-1})^TW(Q^{-1})^T = I \tag{5.14}$$

and see that this kind of transformation is canonical.

Summarizing the calculations in the appendix, we find a new diagonal Hamiltonian of the form

$$\mathcal{H} = \frac{1}{2}\kappa n^2 \sum_\pm \left[\lambda_{1\pm}\Pi_\pm^2 + \lambda_{2\pm}\left(\partial_{\hat{x}}\phi_\pm\right)^2\right], \tag{5.15}$$

where the eigenvalues (with $\delta = \pm$) are

$$\lambda_{1\delta} = \frac{Q_\delta^2 \mathrm{sign}\left(\gamma - \epsilon_1\right)}{2\sqrt{\gamma\left|\left(\epsilon_2 - \epsilon_1\right)\left(\gamma - \epsilon_1\right)\right|}}\left[\left(\gamma + \epsilon_2 - 2\epsilon_1\right) + \delta\sqrt{\Delta}\right], \tag{5.16}$$

$$\lambda_{2\delta} = \frac{Q_\delta^2 \mathrm{sign}\left(\epsilon_2 - \epsilon_1\right)}{2\sqrt{\gamma\left|\left(\epsilon_2 - \epsilon_1\right)\left(\gamma - \epsilon_1\right)\right|}}\left[\left(2\epsilon_2\gamma - \epsilon_1\left(\epsilon_2 + \gamma\right)\right) - \delta\epsilon_1\sqrt{\Delta}\right]. \tag{5.17}$$

and

$$\Delta = \left(\gamma - \epsilon_2\right)^2 + 4\gamma\left(\epsilon_2 - \epsilon_1\right)\left(\gamma - \epsilon_1\right). \tag{5.18}$$

Additionally

$$Q_\delta^2 = \begin{cases} -1 & \text{if } \mathrm{sign}\left(\gamma - \epsilon_1\right) = -\delta \wedge \mathrm{sign}\left(\epsilon_2 - \epsilon_1\right) = \delta \\ 1 & \text{else} \end{cases} = \begin{cases} \delta\mathrm{sign}\left(\gamma - \epsilon_1\right) & \text{if } Q^2 = -1 \\ 1 & \text{else} \end{cases}. \tag{5.19}$$

The definition of the new fields can be found in section 5.3.

5.2 Parameter Analysis

Now that the the Hamiltonian has been separated, we shall take a look at the parameters and bring them in to a more commonly used form, which is convenient for further study and comparison. To begin with we analyze the sign of the eigenvalues, starting with $\lambda_{1\pm}$. To that end we evaluate the sign of the term in brackets in equation 5.16. Luckily, after some calculation we see that

$$\left(\gamma + \epsilon_2 - 2\epsilon_1 + \sqrt{\Delta}\right)\left(\gamma + \epsilon_2 - 2\epsilon_1 - \sqrt{\Delta}\right) = 4\left(1-\gamma\right)\left(\epsilon_2 - \epsilon_1\right)\left(\gamma - \epsilon_1\right) .$$

This tells us that if the rhs. of this equation is negative, the sign of $\lambda_{1\delta}$ will be δ. Otherwise the sign is determined by $\gamma + \epsilon_2 - 2\epsilon_1$. Outside of the case that either of $\lambda_{1\delta} = 0$ we have that

$$\operatorname{sign}\left(\gamma + \epsilon_2 - 2\epsilon_1 + \delta\sqrt{\Delta}\right) = \begin{cases} \delta & \text{if } Q^2 = -1 \\ \operatorname{sign}\left(\gamma + \epsilon_2 - 2\epsilon_1\right) & \text{if } Q^2 = 1 \end{cases} . \tag{5.20}$$

We now want to simplify the case $Q^2 = 1$. To do that let us have a look at figure 4. It shows the

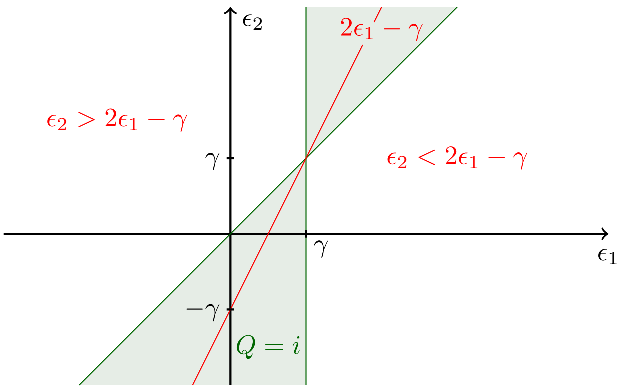

Figure 4: Sketch of the parameter space to illustrate replacements

space of parameters ϵ_1 and ϵ_2. The green shaded region is when $Q^2 = -1$. On the non shaded region on the left the sign expression in 5.20 equates to $+1$. On the right it becomes -1. We can now immediately see that there are a lot of lines we can draw through the shaded region which all give this result. In particular sign$(\epsilon_1 - \gamma)$ gives the same sign as sign$\left(\gamma + \epsilon_2 - 2\epsilon_1\right)$ in the non shaded region. With that and equation 5.19 we can rewrite equation 5.20 as

$$\operatorname{sign}\left(\gamma + \epsilon_2 - 2\epsilon_1 + \delta\sqrt{\Delta}\right) = \begin{cases} \delta & \text{if } Q^2 = -1 \\ \operatorname{sign}\left(\gamma - \epsilon_1\right) & \text{if } Q^2 = 1 \end{cases} = \operatorname{sign}\left(\gamma - \epsilon_1\right) Q_\delta^2. \tag{5.21}$$

This implies that

$$\operatorname{sign}(\lambda_{1\delta}) = 1.$$

We turn our attention to $\lambda_{2\delta}$. To that end we calculate

$$\lambda_{1\delta}\lambda_{2\delta} = \frac{1}{2\gamma}\left[\gamma + \epsilon_2 - 2\gamma\epsilon_1 + \delta\sqrt{\Delta}\right], \tag{5.22}$$

and also

$$\frac{1}{4\gamma^2}\left[\gamma + \epsilon_2 - 2\gamma\epsilon_1 + \sqrt{\Delta}\right]\left[\gamma + \epsilon_2 - 2\gamma\epsilon_1 - \sqrt{\Delta}\right] = \frac{1}{\gamma}\left(1-\gamma\right)\left(\epsilon_2 - \epsilon_1^2\right).$$

Since $\lambda_{1\delta}$ is always positive, we can immediately conclude from these calculations that

$$\text{sign}(\lambda_{2\delta}) = \begin{cases} \text{sign}(\gamma + \epsilon_2 - 2\gamma\epsilon_1) & \text{if } (\epsilon_2 - \epsilon_1^2) > 0 \\ \delta & \text{if } (\epsilon_2 - \epsilon_1^2) < 0 \end{cases} .$$

Since $\gamma + \epsilon_2 - 2\gamma\epsilon_1 = (\epsilon_2 - \epsilon_1^2) + \gamma(1-\gamma) + (\epsilon_1 - \gamma)^2$ we see that this simplifies to

$$\text{sign}(\lambda_{2\delta}) = \begin{cases} 1 & \text{if } (\epsilon_2 - \epsilon_1^2) > 0 \\ \delta & \text{if } (\epsilon_2 - \epsilon_1^2) < 0 \end{cases} .$$

This negative sign that appears for $\delta = -1$ is a result of the fact that one of the velocities becomes imaginary for $(\epsilon_2 - \epsilon_1^2) < 0$. This excludes this region from a space of "sensible" parameters. This means that for "sensible" parameters $(\epsilon_2 - \epsilon_1^2 > 0)$ we have that

$$\text{sign}(\lambda_{2\delta}) = 1. \tag{5.23}$$

In this case we can switch to a more common representation of the parameters by defining

$$\hat{v}_\pm^2 = \lambda_{1\pm}\lambda_{2\pm}, \quad \hat{K}_\pm^2 = \frac{\lambda_{1\pm}}{\lambda_{2\pm}}, \tag{5.24}$$

such that

$$\mathcal{H} = \frac{1}{2}\kappa n^2 \sum_\pm \hat{v}_\pm \left[\hat{K}_\pm \Pi_\pm^2 + \frac{1}{\hat{K}_\pm} \left(\partial_{\hat{x}}\phi_\pm\right)^2 \right]. \tag{5.25}$$

It is quite nice to see that the same velocities as in the Lagrangian also naturally appear here. The new parameters $\hat{K}_\pm$ are shown in figure 5. This finishes the transformation. We can see that the final result is a Hamiltonian, which is the sum of two decoupled Luttinger Liquids. The fact that we have decoupled the Hamiltonian will simplify all future calculations, since we can now define $\partial_{\hat{x}}\Theta_\pm = \Pi_\pm$. This gives the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\kappa n^2 \sum_\pm \hat{v}_\pm \left[\hat{K}_\pm \left(\partial_{\hat{x}}\Theta_\pm\right)^2 + \frac{1}{\hat{K}_\pm} \left(\partial_{\hat{x}}\phi_\pm\right)^2 \right]. \tag{5.26}$$

We will see in section 5.4 that we can calculate correlation functions for both $\phi_\pm$ and $\theta_\pm$. This makes it possible to easily calculate correlation functions which were troublesome to evaluate with only the Lagrangian before.

5.3 Connecting to the Original Parameters

Having a simplified theory we now want to express our original fields in the new ones. From the transformation we know that

$$W_\phi^{-1} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \begin{pmatrix} \hat{\theta} \\ \hat{u} \end{pmatrix}, \quad W_\Pi^{-1} \begin{pmatrix} \Pi_+ \\ \Pi_- \end{pmatrix} = \begin{pmatrix} \hat{\Pi}_\theta \\ \hat{p} \end{pmatrix}. \tag{5.27}$$

As it is a canonical transformation we also have $W_\phi^{-1} = W_\Pi$ and $W_\Pi^{-1} = W_\phi$. This is shown in appendix B.4, where we also see that

$$\hat{\Pi}_\theta = \frac{1}{s_0} \sum_\pm (-a_\pm)\Pi_\pm, \quad \hat{u} = s_0 \sum_\pm b_\pm \phi_\pm, \quad \theta = s_0 \sum_\pm c_\pm \phi_\pm. \tag{5.28}$$

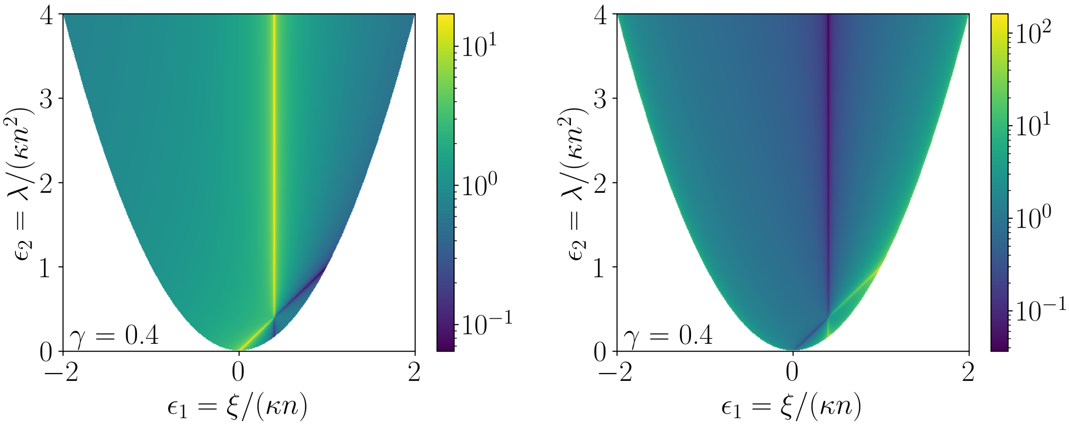

Figure 5: Plots for $\hat{K}_\pm$ with a fixed $\gamma = 0.4$ and variable ϵ_1, ϵ_2. The visible lines in this plot are a remnant of the transformation. When combined with the parameters of the fields to calculate physical expectation values later, these lines vanish.

where

$$a_\pm = \pm \text{sign}(\epsilon_2 - \epsilon_1) Q^2_{\pm 1} \sqrt[4]{\frac{1}{\gamma} \left| \frac{\gamma - \epsilon_1}{\epsilon_2 - \epsilon_1} \right|} \sqrt{\frac{\left| \sqrt{\Delta} \pm \beta_0 \right|}{2\sqrt{\Delta}}}, \tag{5.29}$$

$$b_\pm = Q^2_{\mp 1} \sqrt[4]{\frac{1}{\gamma} \left| \frac{\gamma - \epsilon_1}{\epsilon_2 - \epsilon_1} \right|} \sqrt{\frac{\left| \sqrt{\Delta} \mp \beta_0 \right|}{2\sqrt{\Delta}}}, \tag{5.30}$$

$$c_\pm = \mp \text{sign}\,(\epsilon_2 - \epsilon_1) \sqrt[4]{\gamma \left| \frac{\epsilon_2 - \epsilon_1}{\gamma - \epsilon_1} \right|} \sqrt{\frac{\left| \sqrt{\Delta} \pm \beta_0 \right|}{2\sqrt{\Delta}}}. \tag{5.31}$$

We now fully connect this to the original fields. For this first see that

$$\partial_x \Phi = -\frac{1}{\hbar} \Pi_\theta = -n \hat{\Pi}_\theta = n \sqrt{\frac{\kappa m}{\hbar^2 n}} \sum_\pm a_\pm \Pi_\pm = n \sqrt{\frac{\kappa m}{\hbar^2 n}} \sum_\pm a_\pm \partial_{\hat{x}} \Theta_\pm.$$

With this we can now easily write down the three fields

$$\Phi = \sum_\pm a_\pm \Theta_\pm, \quad u = \hat{u}/n = \frac{1}{n} \sqrt{\frac{\hbar^2 n}{\kappa m}} \sum_\pm b_\pm \phi_\pm = \frac{1}{n} s_0 \sum_\pm b_\pm \phi_\pm, \tag{5.32}$$

$$\delta n = \partial_x \theta = \sqrt{\frac{\hbar^2 n}{\kappa m}} \sqrt{\frac{n \kappa m}{\hbar^2}} \sum_\pm c_\pm \partial_{\hat{x}} \phi_\pm = n \sum_\pm c_\pm \partial_{\hat{x}} \phi_\pm. \tag{5.33}$$

5.4 Expectation Values

In this section we now use the newly diagonal Hamiltonian to calculate correlation functions. We will see that it reproduces the results from the original Lagrangian but the calculation is simpler and more flexible.

Expectation value calculations now become exceedingly simple as one can simply look at textbooks for the solutions (see [8] for instance). We go trough the steps here anyways. The goal we have is to calculate not only expectation values for the field $\phi_\pm$ but also for the conjugate fields, since the phase field Φ is connected to them. For this purpose we express the Hamiltonian as follows

$$\mathcal{H} = \frac{1}{2}\kappa n^2 \sum_{\pm} \hat{v}_\pm \left[\hat{K}_\pm (\partial_{\hat{x}}\Theta_\pm)^2 + \frac{1}{\hat{K}_\pm} (\partial_{\hat{x}}\phi_\pm)^2 \right]. \tag{5.34}$$

We switch now to the Lagrangian. However, we do not fully do the replacement but keep the four fields

$$\mathcal{L} = \frac{1}{2}\kappa n^2 \sum_{\pm} \left[2\partial_{\hat{t}}\phi_\pm \partial_{\hat{x}}\Theta - \hat{v}_\pm \hat{K}_\pm (\partial_{\hat{x}}\Theta)^2 - \frac{\hat{v}_\pm}{\hat{K}_\pm} (\partial_{\hat{x}}\phi_\pm)^2 \right]. \tag{5.35}$$

In imaginary time the action becomes

$$\mathcal{S} = iS/\hbar = \frac{s_0}{2} \iint \frac{d\hat{Q}}{4\pi^2} \begin{pmatrix} \overline{\Theta}_\pm & \overline{\phi}_\pm \end{pmatrix} \underbrace{\begin{pmatrix} -\hat{v}_\pm \hat{K}_\pm \hat{k}^2 & i\hat{q}\hat{k} \\ i\hat{q}\hat{k} & -\frac{\hat{v}_\pm}{\hat{K}_\pm}\hat{k}^2 \end{pmatrix}}_{:=M_\pm^{-1}} \begin{pmatrix} \overline{\Theta}_\pm \\ \overline{\phi}_\pm \end{pmatrix}. \tag{5.36}$$

As usual, we introduce a generating functional

$$Z[\overline{J}_{\phi\pm}, \overline{J}_{\Theta\pm}] = \left[\iint D\overline{\phi}_\pm \iint D\overline{\Theta}_\pm \right] \, \mathrm{e}^{\mathcal{S} + \int \frac{d\hat{Q}}{4\pi^2} \sum_\pm \left[\overline{\phi}_\pm \overline{J}^\dagger_{\phi\pm} + \overline{\Theta}_\pm \overline{J}^\dagger_{\Theta\pm} \right]} \tag{5.37}$$

$$= Z \exp\left(\frac{1}{2} \int \frac{d\hat{Q}}{4\pi^2} \sum_\pm \overline{\boldsymbol{J}}_\pm(-\hat{Q}) \left[-\frac{1}{s_0} M_\pm(\hat{Q}) \right] \overline{\boldsymbol{J}}_\pm(\hat{Q}) \right), \tag{5.38}$$

where the inverse of $M_\pm^{-1}$ is given by

$$M_\pm = \frac{1}{\hat{k}^2(\hat{v}_\pm^2 \hat{k}^2 + \hat{q}^2)} \begin{pmatrix} -\frac{\hat{v}_\pm}{\hat{K}_\pm}\hat{k}^2 & -i\hat{q}\hat{k} \\ -i\hat{q}\hat{k} & -\hat{v}_\pm \hat{K}_\pm \hat{k}^2 \end{pmatrix}. \tag{5.39}$$

This then leads to the correlation functions

$$\left\langle \overline{\phi}_\pm(\hat{Q}) \overline{\phi}_\pm(\hat{Q}') \right\rangle = -\frac{1}{s_0} (M_\pm)_{22}(\hat{Q}) 4\pi^2 \delta(\hat{Q} + \hat{Q}') = \frac{1}{s_0} \hat{K}_\pm \frac{\hat{v}_\pm}{\hat{v}_\pm^2 \hat{k}^2 + \hat{q}^2} 4\pi^2 \delta(\hat{Q} + \hat{Q}'), \tag{5.40}$$

$$\left\langle \overline{\Theta}_\pm(\hat{Q}) \overline{\Theta}_\pm(\hat{Q}') \right\rangle = -\frac{1}{s_0} (M_\pm)_{11}(\hat{Q}) 4\pi^2 \delta(\hat{Q} + \hat{Q}') = \frac{1}{s_0} \frac{1}{\hat{K}_\pm} \frac{\hat{v}_\pm}{\hat{v}_\pm^2 \hat{k}^2 + \hat{q}^2} 4\pi^2 \delta(\hat{Q} + \hat{Q}'), \tag{5.41}$$

$$\left\langle \overline{\Theta}_\pm(\hat{Q}) \overline{\phi}_\pm(\hat{Q}') \right\rangle = -\frac{1}{s_0} (M_\pm)_{12}(\hat{Q}) 4\pi^2 \delta(\hat{Q} + \hat{Q}') = \frac{1}{s_0} i \frac{\hat{v}_\pm \hat{q}/\hat{k}}{\hat{v}_\pm^2 \hat{k}^2 + \hat{q}^2} 4\pi^2 \delta(\hat{Q} + \hat{Q}'). \tag{5.42}$$

We can now check our calculations for consistency. We already have correlation functions calculated via the Lagrangian in section 4.3. From section 5.3 we know how to express the original fields in the new ones. With this we can write down the correlation functions for the original fields

$$\left\langle \overline{u}(\hat{Q}) \overline{u}(\hat{Q}') \right\rangle = \frac{1}{s_0} \frac{\hbar^2}{\kappa m n} \sum_\pm b_\pm^2 \hat{K}_\pm \frac{\hat{v}_\pm}{\hat{v}_\pm^2 \hat{k}^2 + \hat{q}^2} 4\pi^2 \delta(\hat{Q} + \hat{Q}'), \tag{5.43}$$

$$\left\langle \overline{\Phi}(\hat{Q}) \overline{\Phi}(\hat{Q}') \right\rangle = \frac{1}{s_0} \sum_\pm \frac{a_\pm^2}{\hat{K}_\pm} \frac{\hat{v}_\pm}{\hat{v}_\pm^2 \hat{k}^2 + \hat{q}^2} 4\pi^2 \delta(\hat{Q} + \hat{Q}'), \tag{5.44}$$

$$\left\langle \overline{\Phi}(\hat{Q}) \overline{u}(\hat{Q}') \right\rangle = \frac{1}{s_0} i \sqrt{\frac{\hbar^2}{\kappa m n}} \sum_\pm (a_\pm b_\pm) \frac{\hat{v}_\pm \hat{q}/\hat{k}}{\hat{v}_\pm^2 \hat{k}^2 + \hat{q}^2} 4\pi^2 \delta(\hat{Q} + \hat{Q}'). \tag{5.45}$$

When we now compare this to section 4.3 we see that we need to check whether it is true that

$$\frac{a_\pm^2}{\hat{K}_\pm} = \alpha_{11}^\pm, \quad b_\pm^2 \hat{K}_\pm = \alpha_{22}^\pm, \quad a_\pm b_\pm = \pm\frac{\gamma - \epsilon_1}{2\sqrt{\Delta}} = \alpha_{12}^\pm. \tag{5.46}$$

After some calculation (see appendix B.6) we see that we indeed have

$$b_\pm^2 \hat{K}_\pm = \pm\frac{v_\pm^2 - (1-\gamma)}{v_\pm\sqrt{\Delta}}, \qquad \frac{a_\pm^2}{\hat{K}_\pm} = \pm\frac{\gamma v_\pm^2 - (\epsilon_2 - \epsilon_1^2)}{v_\pm\sqrt{\Delta}}, \tag{5.47}$$

as well as

$$a_\pm b_\pm = \pm\frac{\gamma - \epsilon_1}{2\sqrt{\Delta}}. \tag{5.48}$$

As such we see that after the transformation we can calculate expectation values more easily and they give the same results in the end.

Density Correlation There is one important observable which we have not touched thus far. The representation of the density field

$$\rho \approx [n + \delta n(x,t)] \left[1 + \sum_{l=1} 2\cos\left(2\pi l\left(\theta + nx\right)\right)\right] + 2c_1 \cos\left(k_0(x-u)\right). \tag{5.49}$$

The object of interest are density-density correlations $\langle \rho\rho' \rangle$. There are several types of correlation functions which appear in the calculation of this term. We are interested in the long range behavior and intend to only keep the longest range terms.
The actual calculations are somewhat lengthy and can be found in appendix B.5. In total we see that there are only three correlation functions contributing

$$\left\langle \delta n \delta n'/n^2 \right\rangle = s_0 \sum_\pm \frac{\hat{K}_\pm c_\pm^2}{2\pi} \frac{\hat{v}_\pm^2\left(|\Delta\hat{\tau}| + \alpha\right)^2 - |\Delta\hat{x}|^2}{\left(\hat{v}_\pm^2(|\Delta\hat{\tau}| + \alpha)^2 + |\Delta\hat{x}|^2\right)^2} \tag{5.50}$$

$$\left\langle \cos\left(2\pi l n x + 2\pi l s_0 \boldsymbol{c}\boldsymbol{\phi}\right)^2 \right\rangle = \cos\left(2\pi l n \Delta x\right) \prod_\pm \left[\sqrt{\frac{(\Delta\hat{x})^2 + (\hat{v_\pm^2}\,|\Delta\hat{\tau}| + \alpha)^2}{\alpha^2}}\right]^{-2\pi s_0 l^2 c_\pm^2 \hat{K}_\pm} \tag{5.51}$$

$$\left\langle \cos\left(k_0 x - s_0\frac{k_0}{n}\boldsymbol{b}\boldsymbol{\phi}\right)^2 \right\rangle = \cos\left(k_0 \Delta x\right) \prod_\pm \left[\sqrt{\frac{(\Delta\hat{x})^2 + (\hat{v_\pm^2}\,|\Delta\hat{\tau}| + \alpha)^2}{\alpha^2}}\right]^{-2\pi s_0 (\frac{k_0}{2\pi n})^2 b_\pm^2 \hat{K}_\pm} \tag{5.52}$$

The square in the expectation values here means the term multiplied by itself but with $\hat{X}'$ instead of $\hat{X}$. If we consider the limit of $\Delta\hat{x} \to \infty$, the correlation functions simplify

$$\left\langle \delta n \delta n'/n^2 \right\rangle = -s_0 \sum_\pm \frac{\hat{K}_\pm c_\pm^2}{2\pi} (\Delta\hat{x})^{-2} \tag{5.53}$$

$$\left\langle \cos\left(2\pi l n x + 2\pi l s_0 \boldsymbol{c}\boldsymbol{\phi}\right)^2 \right\rangle = \cos\left(2\pi l n \Delta x\right) \left|\frac{\Delta\hat{x}}{\alpha}\right|^{-2\pi s_0 l^2 \sum_\pm c_\pm^2 \hat{K}_\pm} \tag{5.54}$$

$$\left\langle \cos\left(k_0 x - s_0\frac{k_0}{n}\boldsymbol{b}\boldsymbol{\phi}\right)^2 \right\rangle = \cos\left(k_0 \Delta x\right) \left|\frac{\Delta\hat{x}}{\alpha}\right|^{-2\pi s_0 (\frac{k_0}{2\pi n})^2 \sum_\pm b_\pm^2 \hat{K}_\pm} \tag{5.55}$$

We want to include only the longest range contribution and set $l = 1$. The question then becomes which of the three values

$$\left\{\ 1, \quad \pi s_0 l^2 \textstyle\sum_\pm c_\pm^2 \hat{K}_\pm, \quad \pi s_0 (\frac{k_0}{2\pi n})^2 \sum_\pm b_\pm^2 \hat{K}_\pm\ \right\}, \tag{5.56}$$

is the smallest. We skip the analysis of these exponents here as we will do it in section 6.3.2.

6 Stability Analysis

After analyzing the system and introducing transformations to simplify it we are now at the point where we can begin to characterize the phases in the system and analyze where in parameter space they exist. From this information we can infer about the stability of for instance the supersolid phase, for which parameters it exists and when the system transitions in and out of it.

To do this we first need to be able to decide when we have which phase. There are several ways to do this. A usual one are order parameters, which would constitute evaluating the right correlation functions. The method we use here is to probe the system with perturbations. The idea behind this is very basic. Imagine simply poking an unknown system. If the whole system moves one might say it is a solid. If, however, instead the system deforms and does not move then it is not.

We still need a way to actually treat the perturbation, since they will break the nice quadratic form of the Hamiltonian. The way we treat this here is perturbation theory and renormalization group.

6.1 Renormalization Group

The renormalization group (RG) is a label that is applied to many similar but slightly different methods. On one hand it is used to describe the procedure of dealing with infinities in Quantum Field Theory by renormalizing bare parameters. On the other hand there is the problem of critical phenomena in statistical mechanics. There we have the renormalization group as a iterative procedure which integrates out degrees of freedom to obtain an effective low energy theory. For the reader interested in leaning more there are Lectures[10], thesis' [11] or books [8],[9] which cover this topic.
Our focus here is on a method called Wilson RG the outline of which we discuss briefly. The usual setting is that we have an action

$$S[\{\phi_i\},\{g_i\},\Lambda] \tag{6.1}$$

depending on some fields ϕ_i and parameters g_i as well as a a cutoff Λ in momentum or real space. The idea is now that we try to lower that cutoff $\Lambda \to \Lambda'$. This can usually be done by splitting of part of the action we call ΔS here

$$S[\{\phi_i\},\{g_i\},\Lambda] = S[\{\phi_i\},\{g_i\},\Lambda'] + \Delta S[\{\phi_i\},\{g_i\},\Lambda,\Lambda']. \tag{6.2}$$

For instance when Λ is a momentum cutoff, this difference is a integral over the momentum shell. The idea is now to integrate out this difference and incorporate it into the action in such a way that

$$S[\{\phi_i\},\{g_i\},\Lambda] = S[\{\phi_i'\},\{g_i\}',\Lambda'], \tag{6.3}$$

meaning that the general form of the action is kept invariant but only the couplings or fields change. This step is usually actually done for the partition function

$$Z = \int D\phi_i \ \mathrm{e}^{S[\{\phi_i\},\{g_i\},\Lambda]}, \tag{6.4}$$

as there these calculations become expectation values. The last step is then to rescale the cutoff back to its original value. With that we have completed one renormalization cycle. Integrating out some degrees of freedom has brought us back to an action of the same form with different coupling constants and fields

$$S[\{\phi_i\},\{g_i\},\Lambda] \to S[\{\phi_i'\},\{g_i\}',\Lambda]. \tag{6.5}$$

It is important to note that there is no guarantee that this kind of procedure works for a particular system. The procedure might seem quite general here but there is no guarantee that is possible to incorporate the shrinking of the cutoff into the parameters.

It is simple to see that this procedure can be repeated indefinitely and thus leads to a flow of the couplings and fields. Along this flow we find different models with the same low energy behavior, which may be more easily solvable. An overview of the procedure in flowchart form is found in figure 6.

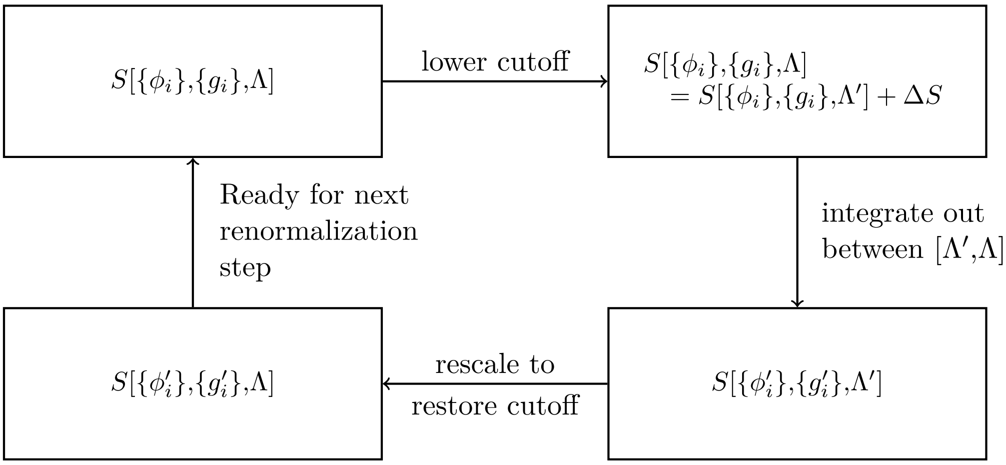

Figure 6: A flow chart for the RG procedure. Starting in the top left the cycle begins and can be continued indefinitely. The idea for such a sketch is from [12].

6.2 Coupled Sine-Gordon

All the perturbations we will later consider lead to a similar problem: Two free bosonic fields coupled to each other via a cosine term. This is called coupled sine-Gordon, alluding to the sine-Gordon problem which is only one free field with a perturbation of cosine form. For comparison

$$\mathcal{L} = \frac{1}{K}\left[v(\partial_x\phi)^2 + \frac{1}{v}(\partial_\tau\phi)^2\right] + g\cos(m\phi), \tag{6.6}$$

would be the Lagrangian of a sine-Gordon theory, while the problem we face here is of the form

$$\mathcal{L} = \frac{1}{2}\sum_{\pm}\frac{1}{\hat{K}_\pm}\left[\hat{v}_\pm(\partial_{\hat{x}}\phi_\pm)^2 + \frac{1}{\hat{v}_\pm}(\partial_{\hat{\tau}}\phi_\pm)^2\right] + g\cos\left(\sum_{\pm} m_\pm\phi_\pm\right). \tag{6.7}$$

In the following we first treat this model problem and then use the results for the different perturbations. The action has two components

$$\mathcal{S}_0 = s_0\iint d\hat{x}d\hat{\tau}\frac{1}{2}\sum_{\pm}\frac{1}{\hat{K}_\pm}\left[\hat{v}_\pm(\partial_{\hat{x}}\phi_\pm)^2 + \frac{1}{\hat{v}_\pm}(\partial_{\hat{\tau}}\phi_\pm)^2\right], \tag{6.8}$$

and

$$\mathcal{S}_g = g\iint d\hat{x}d\hat{\tau}\cos(\boldsymbol{m\phi}). \tag{6.9}$$

We treat the perturbations with a Wilson renormalization scheme in momentum space. To begin with we introduce a high energy cutoff in momentum space. We have needed this for the calculation of correlation functions before (see section 4.4), here we use a hard cutoff instead of the exponential decay which was more useful before. It is not entirely clear which velocity $\hat{v}_0$ should be used to relate frequency and momentum as there are two velocities in the system. We leave this open for now as it turns out not to matter. Define for now

$$\Omega: \; 0 < \hat{q}^2 + \hat{v}_0^2 \hat{k}^2 < \Lambda^2. \tag{6.10}$$

This means also

$$\phi_\pm = \iint_\Omega \frac{d\hat{q}}{2\pi} \frac{d\hat{k}}{2\pi} \overline{\phi}_\pm \mathrm{e}^{i(\hat{q}\hat{\tau} + \hat{k}\hat{x})}, \tag{6.11}$$

which leads to the transformed action

$$\mathcal{S}_0 = s_0 \iint_\Omega \frac{d\hat{q}}{2\pi} \frac{d\hat{k}}{2\pi} \frac{1}{2} \sum_\pm \frac{1}{\hat{K}_\pm} \left[\hat{v}_\pm \hat{k}^2 + \frac{1}{\hat{v}_\pm} \hat{q}^2 \right] \overline{\phi}_\pm \overline{\phi}_\pm^\dagger. \tag{6.12}$$

The RG scheme works by setting a lower cutoff and integrating out the high frequency component. We therefore introduce

$$\Omega^l: \quad 0 < \hat{q}^2 + \hat{v}_0^2 \hat{k}^2 < \Lambda'^2, \tag{6.13}$$

$$\Omega^h: \; \Lambda'^2 < \hat{q}^2 + \hat{v}_0^2 \hat{k}^2 < \Lambda^2. \tag{6.14}$$

Naturally we can then introduce the functions

$$\phi_\pm^l = \iint_{\Omega^l} \frac{d\hat{q}}{2\pi} \frac{d\hat{k}}{2\pi} \overline{\phi}_\pm \mathrm{e}^{i(\hat{q}\hat{\tau} + \hat{k}\hat{x})}, \tag{6.15}$$

$$\phi_\pm^h = \iint_{\Omega^h} \frac{d\hat{q}}{2\pi} \frac{d\hat{k}}{2\pi} \overline{\phi}_\pm \mathrm{e}^{i(\hat{q}\hat{\tau} + \hat{k}\hat{x})}, \tag{6.16}$$

to immediately see that

$$\mathcal{S}_0[\phi_\pm] = \mathcal{S}_0[\phi_\pm^l] + \mathcal{S}_0[\phi_\pm^h] = \mathcal{S}_0^l + \mathcal{S}_0^h, \tag{6.17}$$

as the difference is only one of integration domain when in Fourier space. For the same reason it is clear that

$$\phi_\pm = \phi_\pm^l + \phi_\pm^h. \tag{6.18}$$

We now consider the quantity

$$\frac{1}{Z_0} Z_g = \frac{1}{Z_0} \iint D\phi_\pm \; \mathrm{e}^{\mathcal{S}_0 + \mathcal{S}_g} = \left\langle \mathrm{e}^{\mathcal{S}_g} \right\rangle. \tag{6.19}$$

The goal is now to split of the high energy behavior of this quantity and integrate it out. We choose this particular term since the calculations simplify to expectation values very quickly. Since we are only interested in the critical behavior and not in a specific expectation value, this is enough.

We split the fields in low and high frequency parts and integrate out the high frequency contribution.

$$\frac{1}{Z_0} Z_g = \frac{1}{Z_0^l} \iint D\phi_\pm^l \mathrm{e}^{\mathcal{S}_0^l} \frac{1}{Z_0^h} \iint D\phi_\pm^h \; \mathrm{e}^{\mathcal{S}_0[\phi_\pm^h] + \mathcal{S}_g[\phi_\pm]} = \frac{1}{Z_0^l} \iint D\phi_\pm^l \mathrm{e}^{\mathcal{S}_0^l} \left\langle \mathrm{e}^{\mathcal{S}_g} \right\rangle^h, \tag{6.20}$$

where $\langle A \rangle^h = \iint D\phi_\pm^h A \exp(\mathcal{S}_0^h)$ is the expectation value but only over the high frequency fields. For small perturbations ($g \ll 1$) we treat this expectation in only first order as this is enough for our purposes. In appendix C we begin to outline how to extend this to second order. For now we have

$$\left\langle \mathrm{e}^{\mathcal{S}_g} \right\rangle^h \approx 1 + g \iint d\hat{x} d\hat{\tau} \left\langle \cos\left(\boldsymbol{m} \left(\boldsymbol{\phi}^h + \boldsymbol{\phi}^l \right) \right) \right\rangle^h. \tag{6.21}$$

The expectation value can be simplified using equation 4.12

$$\left\langle \cos\left(\boldsymbol{m}\left(\boldsymbol{\phi}^h + \boldsymbol{\phi}^l\right)\right)\right\rangle^h = \cos\left(\boldsymbol{m}\boldsymbol{\phi}^l\right) \exp\left(-\frac{1}{2}\sum_{\pm} m_{\pm}^2 \left\langle \left[\phi_{\pm}^h\right]^2\right\rangle^h\right). \tag{6.22}$$

We now need to take a closer look at the high frequency expectation values,

$$\left\langle (\phi_{\pm}^h)^2\right\rangle^h = \frac{\hat{K}_{\pm}}{s_0} \iint_{\Omega^h} \frac{d\hat{q}}{2\pi}\frac{d\hat{k}}{2\pi}\frac{\hat{v}_{\pm}}{\hat{q}^2 + \hat{v}_{\pm}^2\hat{k}^2} = \frac{\hat{K}_{\pm}}{4\pi^2 s_0}\int_{\Lambda'}^{\Lambda} dr\frac{1}{r}\int_0^{2\pi} d\varphi \frac{\delta_{\pm}}{\cos^2(\varphi) + \delta_{\pm}^2\sin^2(\varphi)} \tag{6.23}$$

$$= \frac{\hat{K}_{\pm}}{4\pi^2 s_0}\ln(\Lambda/\Lambda')2\delta_{\pm}\int_{-\infty}^{\infty} da \frac{1}{1+\delta_{\pm}^2 a^2} = 2\pi\frac{\hat{K}_{\pm}}{4\pi^2 s_0}\ln(\Lambda/\Lambda'), \tag{6.24}$$

where $\delta_{\pm} = \hat{v}_{\pm}/\hat{v}_0$. In total we see that

$$\left\langle \mathrm{e}^{\mathcal{S}_g}\right\rangle^h \approx 1 + \left[g\mathrm{e}^{-(\pi/s_0)\left(\sum_{\pm}(\frac{m_{\pm}}{2\pi})^2\hat{K}_{\pm}\right)\ln(\Lambda/\Lambda')}\right]\iint d\hat{x}d\hat{\tau}\cos\left(\boldsymbol{m}\boldsymbol{\phi}^l\right). \tag{6.25}$$

We have now dealt with the high frequency expectation values. To be able to identify the new action with the old one we still need to rescale. This means we want a map

$$\phi_{\pm}^l(\hat{x},\hat{\tau}) \to \phi_{\pm}'(\hat{x}',\hat{\tau}'). \tag{6.26}$$

This behavior is needed to keep the cosine term invariant so we can identify it later. When we look at the definition in equation 6.15 we see that

$$\phi_{\pm}^l(\hat{x},\hat{\tau}) = \iint_{\Omega^l}\frac{d\hat{q}}{2\pi}\frac{d\hat{k}}{2\pi}\overline{\phi}_{\pm}\mathrm{e}^{i(\hat{q}\hat{\tau}+\hat{k}\hat{x})} \tag{6.27}$$

$$= \iint_{\Omega}\frac{d\hat{q}}{2\pi}\frac{d\hat{k}}{2\pi}\left[\frac{\Lambda'}{\Lambda}\right]^2\overline{\phi}_{\pm}\left(\frac{\Lambda'}{\Lambda}\hat{Q}\right)\mathrm{e}^{i(\hat{q}\hat{\tau}+\hat{k}\hat{x})(\Lambda'/\Lambda)} = \phi_{\pm}'\left(\frac{\Lambda'}{\Lambda}\hat{x},\frac{\Lambda'}{\Lambda}\hat{\tau}\right), \tag{6.28}$$

where

$$\overline{\phi}_{\pm}'(\hat{Q}) = \left[\frac{\Lambda'}{\Lambda}\right]^2\overline{\phi}_{\pm}\left(\frac{\Lambda'}{\Lambda}\hat{Q}\right). \tag{6.29}$$

Introducing the scaling to equation 6.25 we obtain

$$\left\langle \mathrm{e}^{\mathcal{S}_g}\right\rangle^h \approx 1 + \left[g\mathrm{e}^{\left[2-(\pi/s_0)\left(\sum_{\pm}(\frac{m_{\pm}}{2\pi})^2\hat{K}_{\pm}\right)\right]\ln(\Lambda/\Lambda')}\right]\iint d\hat{x}d\hat{\tau}\cos\left(\boldsymbol{m}\boldsymbol{\phi}'\right) \approx \mathrm{e}^{\mathcal{S}_{\tilde{g}}}, \tag{6.30}$$

with the new constant

$$\tilde{g} = g\mathrm{e}^{\left[2-(\pi/s_0)\left(\sum_{\pm}(\frac{m_{\pm}}{2\pi})^2\hat{K}_{\pm}\right)\right]\ln(\Lambda/\Lambda')}. \tag{6.31}$$

For $\mathcal{S}_0$ the scaling works nicely with $\mathcal{S}_0[\phi_{\pm}^l] = \mathcal{S}_0[\phi_{\pm}']$, since the scaling of the second derivatives cancels with the integrals. After resealing we obtain the same partition function as in the beginning, only with a different parameter

$$\frac{1}{Z_0}Z_g \to \frac{1}{Z_0}Z_{\tilde{g}}. \tag{6.32}$$

This completes the renormalization step. By parameterizing the new cutoff with $\Lambda/\Lambda' = e^l$ we find the flow equation for the coupling

$$\partial_l\tilde{g} = \left[2 - (\pi/s_0)\left(\sum_{\pm}\left(\frac{m_{\pm}}{2\pi}\right)^2\hat{K}_{\pm}\right)\right]\tilde{g}, \quad \tilde{g}(l=0) = g. \tag{6.33}$$

Depending on the sign of this prefactor the coupling either flows to zero or infinity. If it flows to zero, the perturbation is considered irrelevant. It does not affect the low energy behavior of the system and can be neglected. In the other case the perturbation is relevant and has an effect on the system. In a first approximation one can then say that the fields will take on a value that minimizes the cosine meaning the field inside of the cosine is pinned to a fixed value.

6.3 Perturbations

In this section we finally use perturbations as a probe for the stability of our supersolid. The reasoning behind this we have already introduced in the beginning of this section. In the following we give three perturbations.

Impurity The impurity (I) is the first perturbation we consider. The idea here is that if the system does not sense the impurity (it is irrelevant) then we can move it around without moving the system. Akin to the analogy from before this means the system does not have solid character. Mathematically the impurity is implemented as a very localized potential V. This gives an additional term in the Hamiltonian which is the overlap of this potential and the density

$$\iint dxdt\ V(x)\rho(x,t). \tag{6.34}$$

To simplify slightly, we take the impurity as delta localized at zero meaning we set $V(x) = g\delta(x)$ and insert the definition for ρ from equation 3.24

$$S' = g\int dt\,[n+\delta n]\,[1+2\cos(2\pi\theta)] + 2c_1\cos(k_0 u)\,. \tag{6.35}$$

Josephson Junction The Josephson Junction (JJ) is the perturbation we use to check for superfluidity. The idea is to take two separate yet identical copies of the system, put them next to each other and then try to couple the phase of one to the other. When these systems are in a superfluid state we expect the phases to align. If they do not then we know that the superfluidity is absent in the systems. Mathematically this coupling is archived by introducing

$$S' = g\int dt\ \cos(\Phi_1 - \Phi_2)\,. \tag{6.36}$$

Periodic Modulation As a third perturbation we consider an external periodic modulation and probe its effect on the system. For this purpose one can think of perturbing the system by introducing a weak periodic modulation (an optical lattice for instance) of the form

$$S' = g\iint dxdt\ \cos(k_{\text{ext}}x)\,\rho(x,t), \tag{6.37}$$

and then observing whether this weak perturbation is enough to bring the system into a periodic state.

6.3.1 Periodic Modulation

We consider this perturbation only briefly and focus on the physical interpretation as the effect of an external optical lattice is intuitive: If the perturbation is relevant then the system conforms to the lattice, if it is irrelevant then the lattice can be neglected. The question now becomes when we obtain relevant perturbations. We have defined the density as

$$\rho \approx [n + \delta n(x,t)] \left[1 + \sum_{l=1} 2\cos\left(2\pi l \left(\theta + nx\right)\right)\right] + 2c_1 \cos\left(k_0(x-u)\right). \tag{6.38}$$

Being brief and handwavy here, the relevant contribution from the density comes either from the term $\cos\left(k_0(x-u)\right)$ or $\cos\left(2\pi\left(\theta + nx\right)\right)$. In case the relevance of the perturbation arises form the former then the greatest overlap is for $k_{\text{ext}} = k_0$ and the system conforms to the modulation already intrinsic to our supersolid. If the relevance is due to the latter term then the greatest overlap is given for $k_{\text{ext}} = 2\pi n$. This is the periodicity where every atom sits in its own potential well and we call this configuration a Mott Insulator.
We will not discuss this perturbation in further detail here, as the other two perturbations let us better characterize the phases.

6.3.2 Impurity

We now consider the impurity. The perturbation to our action reads

$$S' = g \int d\hat{\tau}\, [n + \delta n]\left[1 + 2\cos\left(2\pi\theta\right)\right] + 2c_1 \cos\left(k_0 u\right). \tag{6.39}$$

We can see already that this action looks very similar to the one we have already treated in section 6.2. The difference is the extra $n + \delta n$ term as well as the fact that the dimension has been reduced to 1D as the integral is evaluated at $x = 0$. This perturbation consists of several fields that can become relevant. For this perturbation, we consider each term on its own. In the end however we want to only consider the most relevant term in the perturbation, meaning the term with the greatest prefactor in the flow equation. This is sensible as this term will grow the fastest in the renormalization flow and subsequently dominate over other possibly also relevant terms in the action.

Consider first the rightmost term $\cos(k_0 u)$. If we again reference section 6.2 we see that almost nothing changes when we use this one dimensional perturbation instead of the two dimensional one considered there. The only thing which changes is that in the end when rescaling we here obtain a factor of 1 in the flow equation for g while with the two dimensional perturbation there was a factor of 2. The flow equation for this term then reads

$$\cos(k_0 u): \quad \partial_l \tilde{g} = \left[1 - \pi s_0 / \left(\lambda_0 n\right)^2 \left(\sum_{\pm} b_{\pm}^2 \hat{K}_{\pm}\right)\right] \tilde{g}. \tag{6.40}$$

For the same reason we can write down the flow equation

$$\cos(2\pi\theta): \quad \partial_l \tilde{g} = \left[1 - \pi s_0 \left(\sum_{\pm} c_{\pm}^2 \hat{K}_{\pm}\right)\right] \tilde{g}, \tag{6.41}$$

for the term $2n\cos(2\pi\theta)$ in the perturbation. The term n in the action we can neglect since it is only a shift in the total energy, irrelevant to the dynamics. The two terms which are left are not as well covered by the calculation in section 6.2. When following the same procedure as before we need to calculate different high frequency correlation functions instead of the correlation function in equation 6.22. For the term δn we evaluate

$$\langle \delta n \rangle^h = \delta n^l + \left\langle \partial_{\hat{x}} \boldsymbol{c} \boldsymbol{\phi}^h \right\rangle^h = \delta n^l. \tag{6.42}$$

The calculation is trivial as expectation values of only one field always give zero in a quadratic theory. This leads to the flow equation

$$\delta n: \quad \partial_l \tilde{g} = 0, \tag{6.43}$$

which means the operator is marginal. We will neglect this operator for now, a more precise treatment would require calculating the RG flow to higher orders. The last term in the perturbation is $2\delta n \cos(2\pi\theta)$. Here we evaluate the high frequency expectation value

$$\langle \delta n \cos(2\pi\theta) \rangle^h = \delta n^l \langle \cos(2\pi\theta) \rangle^h + \left\langle \delta n^h \cos(2\pi\theta) \right\rangle^h = \delta n^l \langle \cos(2\pi\theta) \rangle^h . \tag{6.44}$$

The expectation value is not quite as trivial. After Fourier transformation we see that the expectation value becomes

$$\left\langle \delta n^h \cos(2\pi\theta) \right\rangle^h \sim \int_{\Omega^h} \frac{d\hat{Q}}{4\pi^2} \hat{k} \left[\sum_{\pm} c_\pm \left\langle \overline{\phi}_\pm(\hat{Q}) \overline{\phi}_\pm(-\hat{Q}) \right\rangle \right] \left\langle \mathrm{e}^{2\pi i \theta} \right\rangle^h , \tag{6.45}$$

and is therefore zero due to a parity argument. This leads to the flow equation

$$\delta n \cos(2\pi\theta): \quad \partial_l \tilde{g} = -\pi s_0 \left(\sum_{\pm} c_\pm^2 \hat{K}_\pm \right) \tilde{g}. \tag{6.46}$$

We can clearly see that when comparing this to equation 6.41 that $\cos(2\pi\theta)$ is always a more relevant perturbation than $\delta n \cos(2\pi\theta)$. To summarize, the expressions we still need to analyze further are

$$\begin{aligned} \cos(2\pi\theta): \quad \partial_l \tilde{g} &= \left[1 - \pi s_0 \left(\textstyle\sum_\pm c_\pm^2 \hat{K}_\pm \right) \right] \tilde{g} := [1 - C_0]\, \tilde{g} \\ \cos(k_0 u): \quad \partial_l \tilde{g} &= \left[1 - \pi s_0 / (\lambda_0 n)^2 \left(\textstyle\sum_\pm b_\pm^2 \hat{K}_\pm \right) \right] \tilde{g} := [1 - B_0]\, \tilde{g} \end{aligned} \tag{6.47}$$

These equations give the conditions for relevance and irrelevance of the two operators. $\cos(2\pi\theta)$ is relevant for $C_0 < 1$ and irrelevant for $C_0 > 1$. For $\cos(k_0 u)$ the condition is $B_0 < 1$ for relevance and $B_0 > 1$ for irrelevance. Before we proceed with further analysis it is important to take a moment and think about what the different cases of relevance and irrelevance mean here.

- Both $\cos(k_0 u)$ and $\cos(2\pi\theta)$ irrelevant: In this case the system is not affected by the perturbation. We cannot move the system by moving the impurity, so we argue there is no solid character.
- $\cos(k_0 u)$ relevant, $\cos(2\pi\theta)$ irrelevant: The system is affected by the perturbation. The field which is pinned is the displacement field u. For our model this means that the inbuilt periodicity is kept perfectly, there are no fluctuations of displacement possible anymore. This gives the system a solid character in this case
- $\cos(k_0 u)$ irrelevant, $\cos(2\pi\theta)$ relevant: Again the system is affected by the perturbation. However, the field pinned is θ, the integrated particle number. As motivated in section 6.3.1 this also means a solid character but with a different lattice.
- Both $\cos(k_0 u)$ and $\cos(2\pi\theta)$ relevant: As we have argued before, this case equals one of the two previous cases, depending on which one of the two operators is the most relevant.

Analysis We want to now first consider the last case where both $\cos(k_0 u)$ and $\cos(2\pi\theta)$ are relevant. To understand this case we need to know which of the two quantities C_0 and B_0 is smaller as this determines which is most relevant. In order to do this we first define

$$B := \sum_{\pm} b_{\pm}^2 \hat{K}_{\pm} = \frac{1}{(\hat{v}_+ + \hat{v}_-)\sqrt{\gamma}} \left[\frac{1}{\sqrt{\gamma}} + \sqrt{\frac{1-\gamma}{\epsilon_2 - \epsilon_1^2}} \right], \tag{6.48}$$

$$C := \sum_{\pm} c_{\pm}^2 \hat{K}_{\pm} = \frac{1}{(\hat{v}_+ + \hat{v}_-)\sqrt{\gamma}} \left[\sqrt{\gamma} + \epsilon_2 \sqrt{\frac{1-\gamma}{\epsilon_2 - \epsilon_1^2}} \right]. \tag{6.49}$$

Calculations for this can be found in appendix B.6. The quantity to consider is now the fraction

$$\frac{C_0}{B_0} = \frac{\pi s_0 C}{\pi s_0 \left(\frac{k_0}{2\pi n}\right)^2 B} = \left(\frac{2\pi n}{k_0}\right)^2 \frac{C}{B} = (\lambda_0 n)^2 \frac{C}{B}, \tag{6.50}$$

where λ_0 is the characteristic wavelength in our supersolid. This means it is also the width of one droplet, which means that $\lambda_0 n = \#$Particles per droplet. In the experiment this is a large number (see [2]). We are interested in finding out when $C_0/B_0 = 1$ as this gives us the line that separates the regime where $C_0 > B_0$ from the one where $C_0 < B_0$. To do this we now analyze

$$\frac{C}{B} = \frac{\gamma\sqrt{\epsilon_2 - \epsilon_1^2} + \epsilon_2\sqrt{\gamma(1-\gamma)}}{\sqrt{\epsilon_2 - \epsilon_1^2} + \sqrt{\gamma(1-\gamma)}} := \Gamma. \tag{6.51}$$

This analysis is done by finding lines of fixed Γ for constant γ. Restructuring equation 6.51 gives

$$(\gamma - \Gamma)\sqrt{\epsilon_2 - \epsilon_1^2} = (\Gamma - \epsilon_2)\sqrt{\gamma(1-\gamma)}. \tag{6.52}$$

There is only a solution if $\text{sign}(\gamma - \Gamma) = \text{sign}(\Gamma - \epsilon_2)$. For that case we can square the equation to obtain

$$(\gamma - \Gamma)^2 \left(\epsilon_2 - \epsilon_1^2\right) = (\Gamma - \epsilon_2)^2 \gamma(1-\gamma). \tag{6.53}$$

Writing this in a smart way shows that the equation actually has the form of a shifted Ellipse

$$\epsilon_1^2(\gamma - \Gamma)^2 + \gamma(1-\gamma)\left[\epsilon_2 - \frac{1}{2}\frac{\Gamma^2 + \gamma^2(1-2\Gamma)}{\gamma(1-\gamma)}\right]^2 = \frac{1}{4}\frac{(\Gamma^2 + \gamma^2 - 2\Gamma\gamma^2)^2}{\gamma(1-\gamma)} - \Gamma^2\gamma(1-\gamma). \tag{6.54}$$

We see here that near the origin Γ has to be small while it becomes large away from it. To obtain $C_0/B_0 = 1$ we need to choose $\Gamma = 1/(\lambda_0 n)^2$, which is a small number. For this reason we consider now the case $\Gamma < \gamma$. We know from equation 6.52 that we must then obey $\epsilon_2 < \Gamma$. This means that the equipotential line for $\Gamma = 1/(\lambda_0 n)^2$ lies in small region around the origin bounded by $\epsilon_2 < 1/(\lambda_0 n)^2$ and $|\epsilon_1| < 1/(\lambda_0 n)$. This behavior can be observed in figure 7, where we plot equipotential lines for different Γ with a fixed $\gamma = 0.4$. Below the line with $\Gamma = 1/(\lambda_0 n)^2$ we have $C_0 < B_0$ while for other values of ϵ_1 and ϵ_2 we have $C_0 > B_0$.

We still need to consider the other case of $\Gamma > \gamma$. It is still the case that below the equipotential line $\Gamma = 1/(\lambda_0 n)^2$ we have $C_0 < B_0$ and above it $C_0 > B_0$. The difference is that the equipotential lines are not bound to the origin anymore. We now that γ is small, since $\gamma < \Gamma = 1/(\lambda_0 n)^2$. It is therefore instructive to consider the case $\gamma = 0$. Then equation 6.51 shows that $C_0/B_0 = 0$ since we have $\epsilon_2 - \epsilon_1^2 > 0$. To be more precise we have $C_0 < B_0 = \infty$.

In summary we have two cases. The first case is for $\gamma \approx 0$ ($\gamma < 1/(\lambda_0 n)^2$) where we have that $C_0 < B_0$ for most parameters. The behavior of the impurity is then governed by the term

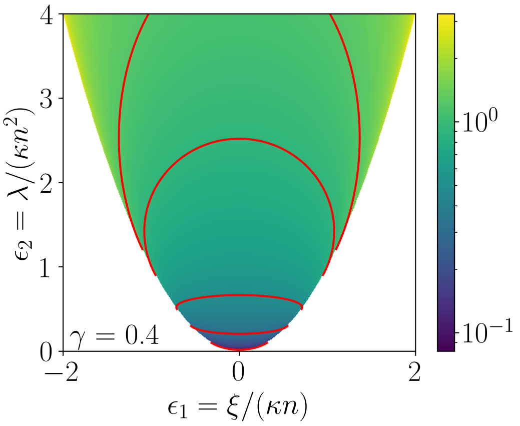

Figure 7: Shown here is the value of $\Gamma = C/B$ for a fixed $\gamma = 0.4$ and variable ϵ_1, ϵ_2. The red lines are equipotential lines for $\Gamma \in \{0.1,0.3,0.5,0.9,1.2\}$. The line closest to the origin corresponds to $\Gamma = 0.1$. The further away from the origin the equipotential line is the larger the corresponding Γ. One can observe that as Γ becomes small the corresponding equipotential lines are closely confined to the origin.

$\cos(2\pi\theta)$. The second case is for $\gamma > 1/(\lambda_0 n)^2$. There we have $B_0 < C_0$ for most parameters, except for very close to the origin. In this case the impurity is then governed by the term $\cos(k_0 u)$.

So far we have only investigated which of the two is larger. We have not yet answered the question when $\cos(k_0 u)$ or $\cos(2\pi\theta)$ are actually relevant. To answer it we have to find out when we have $B_0 = 1$ and $C_0 = 1$ respectively. As before this is best achieved by calculating equipotential lines for B and C while γ is also kept fixed. Unfortunately the calculations here are harder than for C/B before and can be found in appendix B.6. The results of the calculations are parameterized equipotential lines:

- For B with $\varphi \in (-\pi/2,\pi/2)$

$$\epsilon_1 = \gamma + \frac{1}{\sqrt{\gamma}B}\sin(\varphi) + \sqrt{\gamma(1-\gamma)}\tan(\varphi) \tag{6.55}$$

$$\epsilon_2 = \frac{1}{\gamma B^2}\cos^2(\varphi) + \epsilon_1^2(\varphi)\,. \tag{6.56}$$

- For C with $\varphi \in (-\arccos(\sqrt{1-\gamma}/C), \arccos(\sqrt{1-\gamma}/C))$

$$T = \frac{\sqrt{\gamma}}{2}\frac{(1-2\gamma)\cos(2\varphi) - \left[1 + 2\sqrt{\gamma(1-\gamma)}\sin(2\varphi)\right]}{\sqrt{1-\gamma} - C\cos(\varphi)} \tag{6.57}$$

$$\epsilon_1 = \gamma + \left[\sqrt{\gamma(1-\gamma)} + T\right]\tan(\varphi) \tag{6.58}$$

$$\epsilon_2 = T^2 + \epsilon_1^2(\varphi)\,. \tag{6.59}$$

We see here that there are no parameters such that C is smaller than $\sqrt{1-\gamma}$. This gives the condition $C > \sqrt{1-\gamma}$. As an example B and C are plotted for $\gamma = 0.4$ in figure 8 with equipotential lines $C = 1.1$ and $B = 1.1$ respectively. The general behavior we can observe in this plot is that both B and C are smallest for large ϵ_1 and ϵ_2 away from the border where $\epsilon_2 = \epsilon_1^2$. Close to this edge we usually find large values of B and C. We now know the behavior of C and

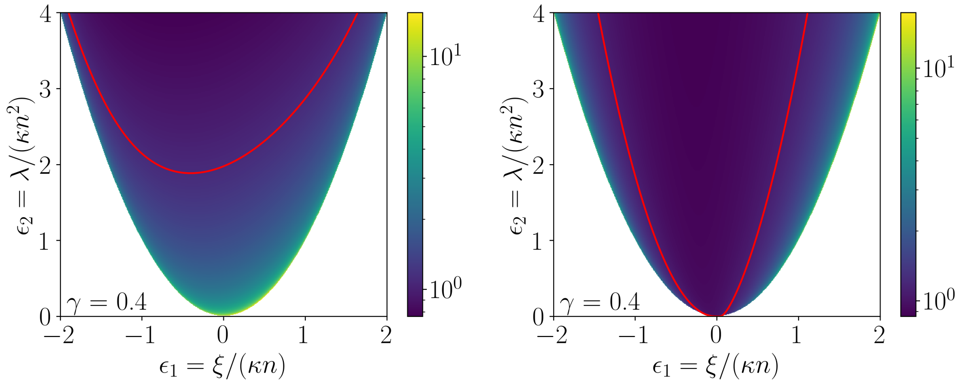

Figure 8: Plots with the values of B (left) and C (right) for fixed $\gamma = 0.4$ and variable ϵ_1, ϵ_2. The red lines are example equipotential lines for a value of $B = 1.1$ and $C = 1.1$.

B but for relevance of the perturbation we need to consider

$$C_0 = \pi s_0 C, \quad B_0 = \pi s_0/(\lambda_0 n)^2 B. \tag{6.60}$$

To further analyze this we need to gain intuition for the size of $s_0 = \sqrt{\frac{\hbar^2 n}{\kappa m}}$. This parameter is effectively the ration of kinetic energy $\hbar^2 n^2/m$ to the potential energy (of the pure density interaction) κn. Thus we call $s_0 \ll 1$ strongly interacting while $s_0 \gg 1$ is called weakly interacting. We will discuss both cases here, however due to how we derived our model we can only expect valid results in the weakly interacting regime. We now take a look at the relevance of the impurity in the two cases we have found,

- $\boldsymbol{\gamma < 1/(\lambda_0 n)^2}$: In this case the impurity is governed by $\cos(2\pi\theta)$. We also know that $C > \sqrt{1-\gamma}$ which, since we are in the case of $\gamma \approx 0$ gives us $C > 1$. This means that $C_0 > \pi s_0$. When we are weakly interacting this inequality means that the impurity is then irrelevant. In the strongly interacting case the impurity can be relevant.
- $\boldsymbol{\gamma > 1/(\lambda_0 n)^2}$: In this case the impurity is governed by $\cos(k_0 u)$. We see that in the strongly interacting case the impurity is then almost always relevant since the factor $s_0/(\lambda_0 n)^2$ is extremely small. In the weakly interacting case the exact size of $s_0/(\lambda_0 n)^2$ determines when the impurity is relevant or not. Generally we can say that it will become relevant for sufficiently large ϵ_1 and ϵ_2 when $\epsilon_2 \neq \epsilon_1^2$.

6.3.3 Josephson Junction

In the case of the Josephson Junction we have two systems with actions $\mathcal{S}_1$ and $\mathcal{S}_2$ which we couple at $\hat{x} = 0$ with the term

$$S' = g \int d\hat{\tau} \ \cos\left(\Phi_1 - \Phi_2\right). \tag{6.61}$$

The two actions for the left and right side respectively are

$$\mathcal{S}_{\hat{x}<0}[\phi_{1\pm}] = s_0 \int d\hat{\tau} \int_{-\infty}^{0} d\hat{x} \frac{1}{2} \sum_{\pm} \frac{1}{\hat{K}_\pm} \left[\hat{v}_\pm (\partial_{\hat{x}} \phi_{1\pm})^2 + \frac{1}{\hat{v}_\pm} (\partial_{\hat{\tau}} \phi_{1\pm})^2\right], \tag{6.62}$$

$$\mathcal{S}_{\hat{x}>0}[\phi_{2\pm}] = s_0 \int d\hat{\tau} \int_{0}^{\infty} d\hat{x} \frac{1}{2} \sum_{\pm} \frac{1}{\hat{K}_\pm} \left[\hat{v}_\pm (\partial_{\hat{x}} \phi_{2\pm})^2 + \frac{1}{\hat{v}_\pm} (\partial_{\hat{\tau}} \phi_{2\pm})^2\right]. \tag{6.63}$$

In this definition $\phi_{1\pm}$ is defined for $\hat{x} < 0$ and $\phi_{2\pm}$ for $\hat{x} > 0$. It is sensible to express the action here not in the $\phi_\pm$ but instead in the $\Theta_\pm$ fields

$$\mathcal{S}_{\hat{x}<0}[\Theta_{1\pm}] = s_0 \int d\hat{\tau} \int_{-\infty}^{0} d\hat{x} \frac{1}{2} \sum_\pm \hat{K}_\pm \left[\hat{v}_\pm (\partial_{\hat{x}} \Theta_{1\pm})^2 + \frac{1}{\hat{v}_\pm} (\partial_{\hat{\tau}} \Theta_{1\pm})^2\right], \tag{6.64}$$

$$\mathcal{S}_{\hat{x}>0}[\Theta_{2\pm}] = s_0 \int d\hat{\tau} \int_{0}^{\infty} d\hat{x} \frac{1}{2} \sum_\pm \hat{K}_\pm \left[\hat{v}_\pm (\partial_{\hat{x}} \Theta_{2\pm})^2 + \frac{1}{\hat{v}_\pm} (\partial_{\hat{\tau}} \Theta_{2\pm})^2\right], \tag{6.65}$$

since the perturbation can also be expressed in them

$$S' = g \int d\hat{\tau} \ \cos\left(\sum_\pm c_\pm \left(\Theta_{1\pm} - \Theta_{2\pm}\right)\right). \tag{6.66}$$

We now define new fields

$$\Theta_{s\pm} = \frac{1}{2} \left[\Theta_{1\pm}(-\hat{x}) + \Theta_{1\pm}(x)\right], \quad \Theta_{a\pm} = \Theta_{1\pm}(-\hat{x}) - \Theta_{1\pm}(x). \tag{6.67}$$

These new fields are defined for $\hat{x} > 0$. They are not symmetrical or antisymmetrical although the definitions are inspired by the idea. In these new fields the perturbation reads

$$S' = g \int d\hat{\tau} \ \cos\left(\sum_\pm c_\pm \Theta_{a\pm}\right) \tag{6.68}$$

and the original action becomes

$$\mathcal{S}_{\hat{x}<0}[\Theta_{1\pm}] + \mathcal{S}_{\hat{x}>0}[\Theta_{2\pm}] = 2\mathcal{S}_{\hat{x}>0}[\Theta_{s\pm}] + 2\mathcal{S}_{\hat{x}>0}[\Theta_{a\pm}/2]. \tag{6.69}$$

These terms look very similar to calculations we do in appendix C from which we could obtain the flow equations. We can however also make the connection to section 6.2 again. For that we see that for the evaluation of the critical point we only care about the action containing $\Theta_{a\pm}$ since the perturbation also only contains these fields. When we now define

$$\Theta'_\pm(\hat{x}) = \Theta_{a\pm}(\hat{x}) \text{ for } \hat{x} > 0, \quad \Theta'_\pm(\hat{x}) = \Theta_{a\pm}(-\hat{x}) \text{ for } \hat{x} < 0, \tag{6.70}$$

we can write this relevant part of the action as

$$\mathcal{S} = s_0 \iint d\hat{\tau} d\hat{x} \frac{1}{2} \sum_\pm \frac{\hat{K}_\pm}{4} \left[\hat{v}_\pm (\partial_{\hat{x}} \Theta'_\pm)^2 + \frac{1}{\hat{v}_\pm} (\partial_{\hat{\tau}} \Theta'_\pm)^2\right] + g \int d\hat{\tau} \ \cos\left(\sum_\pm c_\pm \Theta'_\pm\right). \tag{6.71}$$

This is exactly the form of action we consider in section 6.2 and as such we obtain the flow equations

$$\partial_l \tilde{g} = \left[1 - 1/(\pi s_0) \left(\sum_\pm \frac{a_\pm^2}{\hat{K}_\pm}\right)\right] \tilde{g}, \quad \tilde{g}(l = 0) = g. \tag{6.72}$$

Before we go further with the analysis, let us consider the physical meaning of this perturbation. If the perturbation is irrelevant we simply keep the two separate systems. In case the perturbation is relevant the field $\Phi_1 - \Phi_2$ is pinned in a minimum of the cosine. This means we now have one global phase instead of two phases for the two systems. We interpret this as superfluid character.

Analysis For further analysis we define

$$A = \sum \frac{a_\pm^2}{\hat{K}_\pm} = \frac{1}{(v_+ + v_-)\sqrt{\gamma}} \left[\sqrt{\gamma} + \sqrt{\frac{\epsilon_2 - \epsilon_1^2}{1-\gamma}} \right], \tag{6.73}$$

such that the condition for relevance becomes $1 = A_0 = \frac{1}{\pi s_0} A$. We will then try to find equipotential lines for A with a fixed γ. This is done in appendix B.6. We obtain the equipotential lines

$$\epsilon_2 = \epsilon_1^2 + (1-\gamma) \left[\frac{|\epsilon_1 - \gamma|\, A}{\sqrt{1-(1-\gamma)A^2}} - \sqrt{\gamma} \right]^2, \tag{6.74}$$

with the conditions

$$|\epsilon_1 - \gamma|\, A > \sqrt{\gamma}\sqrt{1-(1-\gamma)A^2}, \quad A < 1/\sqrt{1-\gamma}. \tag{6.75}$$

To gain intuition for the behavior of A we plot it for $\gamma = 0.4$ with the equipotential line $A = 1.1$ in figure 9. In the plot we can see the effect of the condition $|\epsilon_1 - \gamma| > \sqrt{\gamma}\sqrt{1/A - (1-\gamma)}$. Small

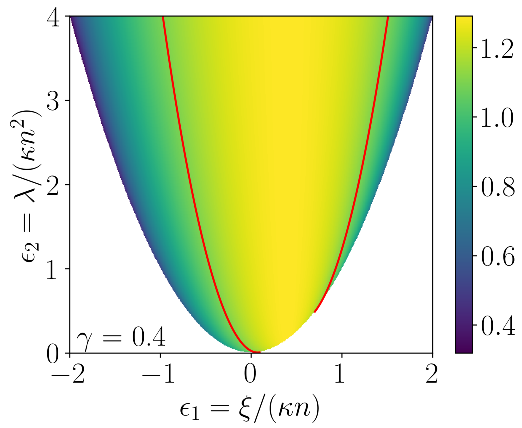

Figure 9: The plot shows the value of A for fixed $\gamma = 0.4$ and variable ϵ_1, ϵ_2. The red line is an example equipotential line for a value of $A = 1.1$.

values of A can only be found near the edge where $|\epsilon_1 - \gamma|$ is large. The largest values of A are found near $\epsilon_1 = \gamma$ but their value is still bound by the second condition $A < 1/\sqrt{1-\gamma}$. It is also instructive to consider the edge cases here

$$\gamma = 0: \ \ A = \sqrt{\frac{\epsilon_2 - \epsilon_1^2}{\epsilon_2}}; \qquad \gamma = 1: \ \ A \sim 1/\sqrt{1-\gamma} = \infty. \tag{6.76}$$

We see that there is one special case where $\gamma = 1$. There the perturbation is always irrelevant since $A_0 = \infty > 1$. Away from this extreme case we need to again take the value of the prefactor s_0 into account. In the weakly interacting case ($s_0 \gg 1$) far enough away from $\gamma = 1$ we realize that since $A < 1/\sqrt{1-\gamma}$ the perturbation will always be relevant. In the strongly interacting case the perturbation is usually irrelevant except for near the edge of the plot where $\epsilon_1 = \gamma$.

6.4 Phase Diagram

We now combine the insights from both perturbations to obtain the phase diagram. We consider the three cases we have observed so far, beginning with

Case 1: $\gamma < 1/(\lambda_0 n)^2$ We have established already that in this case γ is very small and that the relevant contribution arises from the θ field, which counts the number of particles. The critical value for the parameters is given by

$$1 = C_0 = \frac{\pi s_0}{(\hat{v}_+ + \hat{v}_-)\sqrt{\gamma}} \left[\sqrt{\gamma} + \epsilon_2 \sqrt{\frac{1-\gamma}{\epsilon_2 - \epsilon_1^2}} \right] \approx \pi s_0 \left[\sqrt{\frac{\epsilon_2}{\epsilon_2 - \epsilon_1^2}} + \sqrt{\gamma/\epsilon_2} \right]. \tag{6.77}$$

In this same regime the critical parameters for the Josephson Junction are

$$1 = A_0 = \frac{1}{\pi s_0} \frac{1}{(v_+ + v_-)\sqrt{\gamma}} \left[\sqrt{\gamma} + \sqrt{\frac{\epsilon_2 - \epsilon_1^2}{1-\gamma}} \right] \approx \frac{1}{\pi s_0} \left[\sqrt{\frac{\epsilon_2 - \epsilon_1^2}{\epsilon_2}} + \sqrt{\gamma/\epsilon_2} \right]. \tag{6.78}$$

Except for a small correction we see that we have the relation

$$C_0 \approx \frac{1}{A_0}. \tag{6.79}$$

This is particularly nice as we see here the usual duality between impurity and Josephson Junction. The argument is that a strong impurity effectively splits the system in half with only a weak coupling between the two. For the parameters this means that the Josephson Junction becomes a relevant perturbation exactly when the impurity stops being relevant. We see that this is given for $\gamma = 0$ in our system as if $C_0 > 1$ then $A_0 = 1/C_0 < 1$ and vice versa.
On one side we have a regime where the Josephson Junction is relevant and the impurity is irrelevant. This is the superfluid (SF), as we have no solid but only superfluid character. The other regime has those roles reversed. We call this regime Mott Insulator (MI), motivated by the brief discussion in section 6.3.1. There we had seen that the relevance of $\cos(2\pi\theta)$ leads to a Mott Insulator state when considering a weak optical lattice as perturbation.
For γ not exactly zero, there is a small region between (SF) and (MI) where neither of the perturbations are relevant. The first guess is that this is then a normal fluid (F).

Case 2: $1/(\lambda_0 n)^2 < \gamma < 1$ In this regime the two parameters to observe are

$$A_0 = 1/(\pi s_0)A, \quad B_0 = \frac{\pi s_0}{(\lambda_0 n)^2} B. \tag{6.80}$$

We begin by observing the edge cases of the parameter $s_0 = \sqrt{\frac{\hbar^2 n}{\kappa m}}$. This parameter is effectively the ration of kinetic energy $\hbar^2 n^2/m$ to the potential energy (of the pure density interaction) κn. Thus we call $s_0 \ll 1$ strongly interacting while $s_0 \gg 1$ is called weakly interacting.

- **Weakly interacting ($s_0 \gg 1$)**:
  - In this case $\frac{\pi s_0}{(\lambda_0 n)^2}$ is not necessarily small. Therefore B needs to be sufficiently small to ensure the condition for relevance $B_0 < 1$. This requires the other lattice parameters ϵ_1 and ϵ_2 to be sufficiently large.
  - Since we already know that $A < 1/\sqrt{1-\gamma}$, we obtain that A_0 is always relevant as long as $s_0 > 1/(\pi\sqrt{1-\gamma})$, which means that in this case, since $\gamma \neq 1$, A_0 is always relevant.
- **Strongly interacting ($s_0 \ll 1$)**:
  - Contrary to before $\frac{\pi s_0}{(\lambda_0 n)^2}$ is now definitely very small. This means $B_0 < 1$ and thus the impurity is relevant for practically all ϵ_1 and ϵ_2

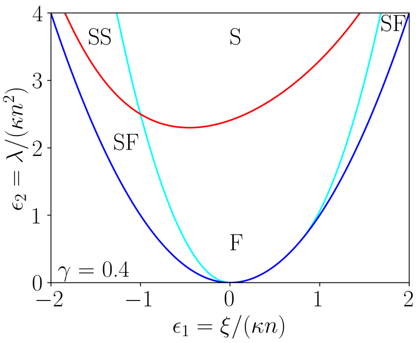

Figure 10: Diagram showing equipotential lines for $A = 1$ (cyan) and $B = 1$ (red) for a fixed $\gamma = 0.4$ and variables ϵ_1 and ϵ_2. The phase labels are Solid (S), Fluid (F), Superfluid (SF) and Supersolid (SS).

– For the Josephson Junction to be relevant here, A needs to be very small. This can only be the case on the edge of the ϵ_2-ϵ_1 diagram, where $\epsilon_1 - \gamma$ is large.

To understand the intermediate part we can take a look at figure 10. This figure shows the equipotential lines for $A = 1$ and $B = 1$. This corresponds to the critical lines in the special case of $\pi s_0 = 1$ and $\lambda_0 n = 1$. It is not a realistic choice of parameters however we can use it to understand the dependence of A and B on ϵ_1 and ϵ_2. First it needs to be noted that B is smaller in the upper part of the diagram, while it diverges toward the edge. Generally we can see that B_0 is relevant for large ϵ_1 and ϵ_2. Therefore we see in figure 10 that the phases with solid character lie in the upper region of the diagram. For A this is different: It is largest in the middle of the diagram and becomes smaller toward the edge. It is generally smallest for $\epsilon_1 - \gamma$ large. For this reason we see in figure 10 that in the middle of the diagram we have the non superfluid phases. Those four regions stay roughly the same for different γ. The value of γ plays a role in precisely determining the shape of these areas and whether they are actually large enough to matter.

Case 3: $\gamma \approx 1$ In the case of $\gamma = 1$ we have that $A = \infty$ meaning the Josephson Junction is irrelevant. This is in line with our expectation as for $n_L = n$ we expect a normal solid and therefore superfluidity should be absent here. Whether we have an actual solid or a normal fluid in this case still depends on the exact lattice parameters, but superfluidity does not exist.

To summarize, we have the following phases in the phase diagram

- **Fluid:** This phase appears when both the Josephson Junction ($A_0 > 1$) and the Impurity ($B_0 > 1$) are irrelevant in the case of $\gamma > 1/(\lambda n)^2$. The impurity can be pulled through the system but there is no superfluidity at the same time. Therefore the characterization as a normal fluid.
- **Mott Insulator:** Here $\cos(2\pi\theta)$ ($C_0 < 1$) is most relevant which is (almost) only the case for $\gamma < 1/(\lambda n)^2$. At the same time the phase requires $A_0 > 1$. The system can then be pushed via the impurity but is not superfluid. This speaks to a solid character, which is obtained due to the perturbation pertaining to modulation with period $1/n$. This leads to the label Mott Insulator.
- **Superfluid:** The superfluid is the case when only $A_0 < 1$. For $\gamma < 1/(\lambda n)^2$ it usually exists when there is no Mott Insulator. When $\gamma > 1/(\lambda n)^2$ the borders between different phases are more complex, but generally the superfluid vanishes as $\gamma \to 1$.

- **Solid/Droplet:** This is the phase characterized by relevance of the impurity ($B_0 < 1$) and irrelevance of the Josephson Junction ($A_0 > 1$). As in the Mott Insulator case we can push the system via the impurity, however the source of the interaction is from the periodic modulation with frequency k_0.

- **Supersolid:** The last case is then obviously when both perturbations are relevant at the same time, which gives us the characteristically unintuitive behavior of the supersolid.

6.4.1 Phases in the Weakly Interacting Regime

For the previous analysis we have still allowed the strongly interacting case of $s_0 \ll 1$. However, as argued before, the motivation for our effective Lagrangian is based on theory which is weakly interacting. For the sake of consistency we should also consider the possible phases when we keep to the weakly interacting regime of $s_0 \gg 1$.

$\boldsymbol{\gamma < 1/(\lambda_0 n)^2}$**:** In this region of the phase diagram only the superfluid phase (SF) exists when $s_0 \gg 1$.

$\boldsymbol{1/(\lambda_0 n)^2 < \gamma < 1}$**:** As long as we are not to close to $\gamma = 1$ the Josephson Junction is always relevant for $s_0 \gg 1$. Whether the impurity is relevant depends on the value of $\frac{\pi s_0}{(\lambda_0 n)^2}$. If it is large then the impurity is only relevant for very large ϵ_1 and ϵ_2. When it is small then the impurity is relevant for most parameters. We see that we have two phases in this case. The supersolid phase (SS) when the impurity is relevant and the superfluid phase (SF) when it is not.

$\boldsymbol{\gamma \approx 1}$**:** Toward this limit the Josephson Junction becomes irrelevant. The condition for relevance for the impurity becomes

$$1 > B_0 = \frac{\pi s_0}{(\lambda_0 n)^2} \frac{1}{\hat{v}_+}. \tag{6.81}$$

We see that the impurity is relevant except for two cases. Case one is the system being extremely weakly interacting such that $\frac{\pi s_0}{(\lambda_0 n)^2} \ll 1$. The other case is when the parameters are chosen such that $\hat{v}_+ \approx 0$. These two cases make intuitive sense as for those choices we expect a weak solid character. When the impurity is relevant we have a solid phase (S) here while we have a fluid phase (F) for a irrelevant impurity.

In total if we choose the parameters ϵ_1 and ϵ_2 such that the lattice is not too weak then we can observe a transition from Superfluid to Supersolid to Solid by varying γ from zero to one. For parameters that make a weak lattice we do not transition into a solid state but at some point only loose the superfluidity and transition to a normal fluid.

7 Summary and Outlook

In this thesis we have introduced a model Lagrangian for a supersolid. We have then proceeded to analyze it and calculated correlation functions. This allowed us to find a suitable transformation which simplified the model by transforming it into two decoupled Luttinger liquids.
In such a simplified model we have easy access to all correlation functions. This allowed us to then effectively use renormalization group theory to deal with perturbations. In this thesis we only considered them to first order, since it was enough for us to obtain only the critical points, where the perturbations become relevant. The physical meaning of the relevance of our perturbations, namely the impurity and the Josephson Junction, has been discussed but not

yet more precisely explored in higher orders. This was however still enough to determine a phase diagram for our effective model. We have found that, as long as our choice of parameters guarantees a sufficiently strong lattice, when sweeping the parameter n_L from zero to n we obtain a transition from a superfluid via a supersolid toward a normal solid state. This is in line with our expectations of n_L characterizing the "solid part" of our system. If the parameters are however chosen such that the lattice is weak, then we do not obtain a solid character, instead at some point transitioning from a superfluid to a normal fluid.

Future work on this topic will most likely focus on better connecting this toy model to actual experimental parameters. Even though our model is based on a more microscopic model, as we see in section 3, the connection to parameters which are actually useful in the lab is not immediately clear. We would like to connect our model parameters to the parameters of other models which are more closely related to experiments, such as the model referenced in section 2.2.

The closer connection to an experiment would then in turn allow us to restrict the choices of parameters to only ones, which are accessible experimentally. This would allow us to properly predict phase diagrams and check the validity of this model.

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A Auxiliary Calculations for the Model Lagrangian

A.1 Real Velocities

We check the sign of

$$\hat{v}_\pm^2 = \frac{1}{2\gamma}\left[\epsilon_2 + \gamma - 2\gamma\epsilon_1 \pm \sqrt{(\epsilon_2 + \gamma - 2\gamma\epsilon_1)^2 - 4\gamma(1-\gamma)(\epsilon_2 - \epsilon_1^2)}\right], \tag{A.1}$$

to see for which parameters one obtains real velocities. For that purpose, calculate

$$\hat{v}_+^2 \hat{v}_-^2 = \frac{1}{\gamma}(1-\gamma)(\epsilon_2 - \epsilon_1^2). \tag{A.2}$$

When this expression is negative exactly one of the $\hat{v}_\pm^2$ is negative. We want the chance for both velocities to be positive so we require $(\epsilon_2 - \epsilon_1^2) > 0$. This turns out to be enough of a condition to make both velocities positive as we observe that then

$$\epsilon_2 + \gamma - 2\gamma\epsilon_1 = (\epsilon_2 - \epsilon_1^2) + \gamma(1-\gamma) + (\epsilon_1 - \gamma)^2 > 0 \tag{A.3}$$

One might still worry that the term in the square root may become negative. Fortunately, with the same idea as before, we can see that that is not the case, since

$$\begin{aligned}(\epsilon_2 + \gamma - 2\gamma\epsilon_1)^2 - 4\gamma(1-\gamma)(\epsilon_2 - \epsilon_1^2) &= \left[(\epsilon_2 - \epsilon_1^2) + \gamma(1-\gamma) + (\epsilon_1 - \gamma)^2\right]^2 - 4\gamma(1-\gamma)(\epsilon_2 - \epsilon_1^2) \\ &= \left[(\epsilon_2 - \epsilon_1^2) - \gamma(1-\gamma) + (\epsilon_1 - \gamma)^2\right]^2 + 4\gamma(1-\gamma)(\epsilon_1 - \gamma)^2 > 0\end{aligned}$$

A.2 Partial Fraction Decomposition

Here we want to simplify the expressions

$$\left\langle \overline{\phi}_1(\hat{\omega},\hat{k})\overline{\phi}_1(\hat{\omega}',\hat{k}')\right\rangle = \frac{i}{4\pi^2}\frac{1}{s_0}\frac{\gamma\hat{\omega}^2 - (\epsilon_2 - \epsilon_1^2)\hat{k}^2}{\gamma\left(\hat{\omega}^2 - \hat{v}_+^2\hat{k}^2\right)\left(\hat{\omega}^2 - \hat{v}_-^2\hat{k}^2\right)}\delta(\hat{k}+\hat{k}')\delta(\hat{\omega}+\hat{\omega}') \tag{A.4}$$

$$\left\langle \overline{\phi}_2(\hat{\omega},\hat{k})\overline{\phi}_2(\hat{\omega}',\hat{k}')\right\rangle = \frac{i}{4\pi^2}\frac{1}{s_0}\frac{\hat{\omega}^2 - (1-\gamma)\hat{k}^2}{\gamma\left(\hat{\omega}^2 - \hat{v}_+^2\hat{k}^2\right)\left(\hat{\omega}^2 - \hat{v}_-^2\hat{k}^2\right)}\delta(\hat{k}+\hat{k}')\delta(\hat{\omega}+\hat{\omega}') \tag{A.5}$$

$$\left\langle \overline{\phi}_1(\hat{\omega},\hat{k})\overline{\phi}_2(\hat{\omega}',\hat{k}')\right\rangle = \frac{i}{4\pi^2}\frac{1}{s_0}\frac{-\left[\epsilon_1 - \gamma\right]\hat{\omega}\hat{k}}{\gamma\left(\hat{\omega}^2 - \hat{v}_+^2\hat{k}^2\right)\left(\hat{\omega}^2 - \hat{v}_-^2\hat{k}^2\right)}\delta(\hat{k}+\hat{k}')\delta(\hat{\omega}+\hat{\omega}') \tag{A.6}$$

We illustrate the calculation via equation A.4. The goal is to bring the equation to the form

$$\frac{\gamma\hat{\omega}^2 - (\epsilon_2 - \epsilon_1^2)\hat{k}^2}{\gamma\left(\hat{\omega}^2 - \hat{v}_+^2\hat{k}^2\right)\left(\hat{\omega}^2 - \hat{v}_-^2\hat{k}^2\right)} = \frac{\alpha_{11}^+\hat{v}_+}{\hat{\omega}^2 - \hat{v}_+^2\hat{k}^2} + \frac{\alpha_{11}^-\hat{v}_-}{\hat{\omega}^2 - \hat{v}_-^2\hat{k}^2}. \tag{A.7}$$

Now the challenge is to determine the newly introduced coefficients. They fulfill

$$\alpha_{11}^+\hat{v}_+ + \alpha_{11}^-\hat{v}_- = 1 \tag{A.8}$$

$$\hat{v}_-^2\hat{v}_+\alpha_{11}^+ + \hat{v}_+^2\hat{v}_-\alpha_{11}^- = \frac{1}{\gamma}(\epsilon_2 - \epsilon_1^2) \tag{A.9}$$

This can be solved to obtain

$$\alpha_{11}^\pm = \mp\frac{\frac{1}{\gamma}(\epsilon_2 - \epsilon_1^2) - \hat{v}_\pm^2}{\hat{v}_\pm\left(\hat{v}_+^2 - \hat{v}_-^2\right)}. \tag{A.10}$$

The same can be done for equation A.5, there we have

$$\alpha_{22}^{+}\hat{v}_{+} + \alpha_{22}^{-}\hat{v}_{-} = \frac{1}{\gamma} \tag{A.11}$$

$$\hat{v}_{-}^{2}\hat{v}_{+}\alpha_{22}^{+} + \hat{v}_{+}^{2}\hat{v}_{-}\alpha_{22}^{-} = \frac{1}{\gamma}(1-\gamma) \tag{A.12}$$

This yields

$$\alpha_{22}^{\pm} = \mp\frac{1}{\gamma}\frac{(1-\gamma)-\hat{v}_{\pm}^{2}}{\hat{v}_{\pm}\left(\hat{v}_{+}^{2}-\hat{v}_{-}^{2}\right)}. \tag{A.13}$$

Equation A.6 is different. There a sensible decomposition is

$$\begin{aligned}\frac{-\left[\epsilon_1-\gamma\right]\hat{\omega}\hat{k}}{\gamma\left(\hat{\omega}^2-\hat{v}_{+}^{2}\hat{k}^2\right)\left(\hat{\omega}^2-\hat{v}_{-}^{2}\hat{k}^2\right)} &= -\frac{1}{\gamma}\frac{\left[\epsilon_1-\gamma\right]}{\hat{v}_{+}^{2}-\hat{v}_{+}^{2}}\left[\frac{\hat{\omega}\hat{k}}{\hat{k}^2\left(\hat{\omega}^2-\hat{v}_{+}^{2}\hat{k}^2\right)}-\frac{\hat{\omega}\hat{k}}{\hat{k}^2\left(\hat{\omega}^2-\hat{v}_{-}^{2}\hat{k}^2\right)}\right]\\ &= \sum_{\pm}\frac{\alpha_{12}^{\pm}\hat{\omega}/\hat{k}}{\left(\hat{\omega}^2-\hat{v}_{\pm}^{2}\hat{k}^2\right)}, \quad \text{with } \alpha_{12}^{\pm} = \mp\frac{1}{\gamma}\frac{\left[\epsilon_1-\gamma\right]}{\hat{v}_{+}^{2}-\hat{v}_{+}^{2}}\end{aligned} \tag{A.14}$$

A.3 Real Space Correlations

Here we calculate

$$\frac{1}{2}\left\langle(\phi_i(\hat{X})-\phi_i(\hat{X}'))^2\right\rangle = \frac{1}{s_0}\int\frac{d\hat{Q}}{4\pi^2}\left[1-\cos(\hat{Q}(\hat{X}-\hat{X}'))\right]\left[\sum_{\pm}\frac{\alpha_{11}^{\pm}\hat{v}_{\pm}}{\hat{q}^2+\hat{v}_{\pm}^{2}\hat{k}^2}\right]\mathrm{e}^{-\hat{\alpha}|\hat{k}|} \tag{A.15}$$

Consider first only

$$\int\frac{d\hat{Q}}{4\pi^2}\hat{v}_{\pm}\frac{1-\cos(\hat{Q}(\hat{X}-\hat{X}'))}{\hat{q}^2+\hat{v}_{\pm}^{2}\hat{k}^2}\mathrm{e}^{-\hat{\alpha}|\hat{k}|} = \iint\frac{d\hat{k}'}{2\pi}\frac{d\hat{q}}{2\pi}\frac{1-\cos(\hat{q}\Delta\hat{\tau}+\hat{k}'\Delta\hat{x}/\hat{v}_{\pm})}{\hat{q}^2+\hat{k}'^2}\mathrm{e}^{-\hat{\alpha}|\hat{k}'|/\hat{v}_{\pm}} \tag{A.16}$$

$$= \iint\frac{d\hat{k}'}{2\pi}\frac{d\hat{q}}{2\pi}\frac{1-\cos(\hat{q}\Delta\hat{\tau})\cos(\hat{k}'\Delta\hat{x}/\hat{v}_{\pm})}{\hat{q}^2+\hat{k}'^2}\mathrm{e}^{-\hat{\alpha}|\hat{k}'|/\hat{v}_{\pm}} \tag{A.17}$$

The last equal sign follows by parity. Apply the transformation $a = \frac{\hat{q}}{\hat{k}'}$ gives

$$\iint\frac{d\hat{k}'}{2\pi}\frac{da}{2\pi}\frac{1}{\hat{k}'}\frac{1-\cos(\Delta\hat{\tau}\hat{k}'a)\cos(\hat{k}'\Delta\hat{x}/\hat{v}_{\pm})}{1+a^2}\mathrm{e}^{-\hat{\alpha}|\hat{k}'|/\hat{v}_{\pm}}. \tag{A.18}$$

The first term is simple, as

$$\int_{-\infty}^{\infty}\frac{da}{2\pi}\frac{1}{1+a^2} = \frac{1}{2\pi}\left[\arctan(a)\right]_{\infty}^{\infty} = \frac{1}{2}. \tag{A.19}$$

For the other term, we employ some helpful transformations. First we need

$$\frac{1}{1+a^2} = -\frac{1}{2i}\left[\frac{1}{a+i}-\frac{1}{a-i}\right] = -\frac{1}{2i}\sum_{\pm}\left[\pm\frac{1}{a\pm i}\right]. \tag{A.20}$$

Then one can also see that

$$\cos(\Delta\hat{\tau}\hat{k}'a) = \cos(\Delta\hat{\tau}\hat{k}'(a\pm i))\cos(\Delta\hat{\tau}\hat{k}'(\mp i)) - \sin(\Delta\hat{\tau}\hat{k}'(a\pm i))\sin(\Delta\hat{\tau}\hat{k}'(\mp i)) \tag{A.21}$$

$$= \cos(\Delta\hat{\tau}\hat{k}'(a\pm i))\cosh(\Delta\hat{\tau}\hat{k}') \pm i\sin(\Delta\hat{\tau}\hat{k}'(a\pm i))\sinh(\Delta\hat{\tau}\hat{k}'). \tag{A.22}$$

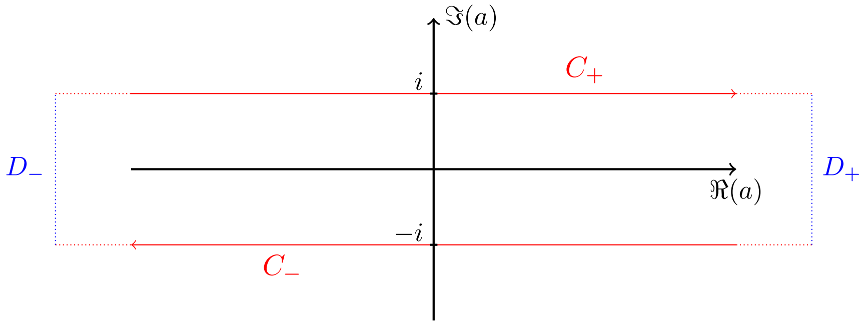

Figure 11: Sketch of the integration domain.

This leads to two types of integrals. One is of the following form

$$-\frac{1}{2i}\sum_{\pm}\int\frac{da}{2\pi}\left[\pm\frac{\cos(\Delta\hat{\tau}\hat{k}'(a\pm i))}{a\pm i}\right]=-\frac{1}{2i}\sum_{\pm}\int_{\mathcal{C}_\pm}\frac{da_\pm}{2\pi}\frac{\cos(\Delta\hat{\tau}\hat{k}'a_\pm)}{a_\pm}, \tag{A.23}$$

where $\mathcal{C}_\pm$ is the contour that runs from $\mp(\infty-i)$ to $\pm(\infty+i)$. For more clarity this is also sketched in figure 11. There we can already see the next idea: Our integrand tends towards zero as $a_\pm\to\infty$. There is also only a finite distance between the two contours, thus the integral over the connection between the two contours (dubbed $D_\pm$ in the sketch) also tends to zero. Thus the sum can be written as one contour integral

$$-\frac{1}{2i}\int_{\mathcal{C}}\frac{da}{2\pi}\frac{\cos(\Delta\hat{\tau}\hat{k}'a)}{a}=\frac{1}{2} \tag{A.24}$$

using the residue theorem. The other integral is

$$-\frac{1}{2}\sum_{\pm}\int\frac{da}{2\pi}\frac{\sin(\Delta\hat{\tau}\hat{k}'(a\pm i))}{a\pm i}=-\frac{1}{2}\sum_{\pm}\int_{-\infty\pm i}^{\infty\pm i}\frac{da_\pm}{2\pi}\frac{\sin(\Delta\hat{\tau}\hat{k}'a_\pm)}{a_\pm}=-\frac{1}{2}. \tag{A.25}$$

This was slightly more straight forward, as it is known that

$$\int_{-\infty}^{\infty}da\frac{\sin(a)}{a}=\pi \tag{A.26}$$

and one can see fairly quickly that this does not change if the integration is shifted in the imaginary direction by a finite value. With this equation A.18 becomes

$$\int_0^\infty\frac{d\hat{k}'}{2\pi}\frac{1}{\hat{k}'}\left[1-\cos(\hat{k}'\Delta\hat{x}/\hat{v}_\pm)\mathrm{e}^{-|\Delta\hat{\tau}|\hat{k}'}\right]\mathrm{e}^{-\hat{\alpha}\hat{k}'/\hat{v}_\pm}. \tag{A.27}$$

We can now finally move to solve this last integral

$$\int_0^\infty\frac{d\hat{k}'}{2\pi}\frac{1}{\hat{k}'}\left[1-\cos(\hat{k}'\Delta\hat{x}/\hat{v}_\pm)\mathrm{e}^{-|\Delta\hat{\tau}|\hat{k}'}\right]\mathrm{e}^{-\hat{\alpha}\hat{k}'/\hat{v}_\pm}=\int_{\hat{\alpha}}^\infty d\hat{\alpha}'\frac{1}{\hat{v}_\pm}\int_0^\infty\frac{d\hat{k}'}{2\pi}\left[1-\cos(\hat{k}'\Delta\hat{x}/\hat{v}_\pm)\mathrm{e}^{-|\Delta\hat{\tau}|\hat{k}'}\right]\mathrm{e}^{-\hat{\alpha}'\hat{k}'/\hat{v}_\pm} \tag{A.28}$$

$$=\int_{\hat{\alpha}}^\infty d\hat{\alpha}'\frac{1}{2\pi\hat{v}_\pm}\left[\frac{\hat{v}_\pm}{\hat{\alpha}'}-\frac{1}{2}\left(\frac{\hat{v}_\pm}{\hat{\alpha}'+\hat{v}_\pm|\Delta\hat{\tau}|+i\Delta\hat{x}}+\frac{\hat{v}_\pm}{\hat{\alpha}'+\hat{v}_\pm|\Delta\hat{\tau}|-i\Delta\hat{x}}\right)\right] \tag{A.29}$$

$$=\frac{1}{2\pi}\int_{\hat{\alpha}}^\infty d\hat{\alpha}'\left[\frac{1}{\hat{\alpha}'}-\frac{\hat{\alpha}'+\hat{v}_\pm|\Delta\hat{\tau}|}{(\hat{\alpha}'+\hat{v}_\pm|\Delta\hat{\tau}|)^2+(\Delta\hat{x})^2}\right] \tag{A.30}$$

$$=\frac{1}{2\pi}\left[\ln\left(\sqrt{[\hat{\alpha}+\hat{v}_\pm|\Delta\hat{\tau}|]^2+\Delta\hat{x}^2}\right)-\ln(\alpha)\right]\approx\frac{1}{2\pi}\ln\left(\sqrt{\hat{v}_\pm^2(|\Delta\hat{\tau}|)^2+(\Delta\hat{x})^2}\right) \tag{A.31}$$

This gives us the full correlation function as

$$\frac{1}{2}\left\langle(\phi_i(\hat{X})-\phi_i(\hat{X}'))^2\right\rangle = \frac{1}{2\pi s_0}\sum_{\pm}\alpha_{ii}^{\pm}\ln\left(\sqrt{\left(1+\hat{v}_{\pm}\left|\Delta\hat{\tau}\right|/\alpha\right)^2+\left(\Delta\hat{x}/\alpha\right)^2}\right) \tag{A.32}$$

B Auxiliary Calculation for the Hamiltonian Treatment

B.1 Scaling

In the first three sections of the appendix we construct a transformation to diagonalize the Hamiltonian

$$\mathcal{H} = \frac{n}{2m}\begin{pmatrix} \Pi_\theta & p/n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1/\gamma \end{pmatrix}\begin{pmatrix} \Pi_\theta \\ p/n \end{pmatrix} + \frac{\kappa}{2}\begin{pmatrix} \partial_x\theta & \partial_x(nu) \end{pmatrix}\begin{pmatrix} 1 & \epsilon_1 \\ \epsilon_1 & \epsilon_2 \end{pmatrix}\begin{pmatrix} \partial_x\theta \\ \partial_x(nu) \end{pmatrix} \tag{B.1}$$

$$= \frac{1}{2}\frac{\hbar^2 n^3}{m}\begin{pmatrix} \hat{\Pi}_\theta & \hat{p} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1/\gamma \end{pmatrix}\begin{pmatrix} \hat{\Pi}_\theta \\ \hat{p} \end{pmatrix} + \frac{1}{2}\kappa n^2 \frac{m\kappa}{n\hbar^2}\begin{pmatrix} \partial_{\hat{x}}\hat{\theta} & \partial_{\hat{x}}\hat{u} \end{pmatrix}\begin{pmatrix} 1 & \epsilon_1 \\ \epsilon_1 & \epsilon_2 \end{pmatrix}\begin{pmatrix} \partial_{\hat{x}}\hat{\theta} \\ \partial_{\hat{x}}\hat{u} \end{pmatrix} \tag{B.2}$$

$$= \frac{1}{2}\kappa n^2\left[\frac{n\hbar^2}{m\kappa}v_1^T M_1 v_1 + \frac{m\kappa}{n\hbar^2}v_2^T M_2 v_2\right]. \tag{B.3}$$

In this context $\hat{u} = nu$, $\Pi_\theta = \hbar n\hat{\Pi}_\theta$ and $p/n = \hbar n\hat{p}$ (also $\hat{\theta} = \theta$ for aesthetic consistency). In this section we apply the first scaling transformation

$$\hat{\Pi}'_\theta = \frac{1}{s_1}\hat{\Pi}_\theta \qquad \hat{\theta}' = s_1\hat{\theta}$$
$$\hat{p}' = \frac{1}{s_2}\hat{p} \qquad \hat{u}\prime = s_2\hat{u}.$$

In matrix form this reads

$$S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1/s_1 & 0 \\ 0 & 1/s_2 \end{pmatrix}. \tag{B.4}$$

We apply the transformation to the matrices

$$M'_1 = \frac{n\hbar^2}{m\kappa}SM_1S^T = \begin{pmatrix} s_1^2 & -s_1s_2 \\ -s_1s_2 & s_2^2/\gamma \end{pmatrix} = s_1s_2\frac{n\hbar^2}{m\kappa}\begin{pmatrix} s & -1 \\ -1 & 1/(s\gamma) \end{pmatrix}, \tag{B.5}$$

$$M'_2 = \frac{m\kappa}{n\hbar^2}S^{-1}M_2(S^{-1})^T = \frac{m\kappa}{n\hbar^2}\frac{1}{s_1s_2}\begin{pmatrix} 1/s & \epsilon_1 \\ \epsilon_1 & s\epsilon_2 \end{pmatrix}, \tag{B.6}$$

where $s = s_1/s_2$. As said, we need to check for which values of s the new matrices commute. For this purpose we calculate the commutator $[M'_1,M'_2]$. As the matrices are symmetric this can be done efficiently by first evaluating

$$M'_1M'_2 \sim \begin{pmatrix} 1-\epsilon_1 & s(\epsilon_1-\epsilon_2) \\ (-1+\epsilon_1/\gamma)/s & -\epsilon_1+\epsilon_2/\gamma \end{pmatrix}. \tag{B.7}$$

Now the commutator is given by

$$[M'_1,M'_2] = M'_1M'_2 - \left(M'_1M'_2\right)^T \sim \left[s(\epsilon_1-\epsilon_2) - (-1+\epsilon_1/\gamma)/s\right] i\sigma_y, \tag{B.8}$$

and vanishes for

$$s^2 = \frac{1}{\gamma}\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}. \tag{B.9}$$

Note that for certain parameters we require an imaginary s in the scaling transformation. This seems problematic at first as it would lead to imaginary fields but we will later see that we can correct this. It is an artifact of the way we build the transformation. For now we must dutifully carry this imaginary s with us until we can get rid of it. We define now

$$Q = \begin{cases} i & \text{if } \operatorname{sign}\left((\gamma-\epsilon_1)(\epsilon_2-\epsilon_1)\right) = -1 \\ 1 & \text{if } \operatorname{sign}\left((\gamma-\epsilon_1)(\epsilon_2-\epsilon_1)\right) = 1 \end{cases}, \tag{B.10}$$

so we have

$$s = Q\sqrt{\frac{1}{\gamma}\frac{|\gamma-\epsilon_1|}{|\epsilon_2-\epsilon_1|}} \quad \text{and} \quad Q^2 = \operatorname{sign}\left((\gamma-\epsilon_1)(\epsilon_2-\epsilon_1)\right). \tag{B.11}$$

There is still freedom in the exact choice of $s_1 s_2$. To keep things simple we choose a phase and get rid of the prefactors with $s_1 s_2 = \mathrm{e}^{2i\varphi} Q^{-1} \frac{m\kappa}{n\hbar^2}$.

B.2 Diagonalization

We now attempt to simultaneously diagonalize the matrices

$$M_1' = \mathrm{e}^{2i\varphi} Q^* \begin{pmatrix} |s|\,Q & -1 \\ -1 & \frac{Q^*}{|s|\gamma} \end{pmatrix}, \quad M_2' = \mathrm{e}^{-2i\varphi} Q \begin{pmatrix} \frac{1}{|s|Q} & \epsilon_1 \\ \epsilon_1 & |s|\,Q\epsilon_2 \end{pmatrix} \tag{B.12}$$

where

$$s = Q\sqrt{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|}, \tag{B.13}$$

and

$$Q^2 = \operatorname{sign}\left((\gamma-\epsilon_1)(\epsilon_2-\epsilon_1)\right). \tag{B.14}$$

Beginning with M_1 we first calculate its Eigenvalues as

$$\lambda_{1\pm}' = e^{2i\varphi} Q^* \frac{1}{2}\left[sQ + \frac{1}{sQ\gamma} \pm \sqrt{\left(sQ - \frac{1}{sQ\gamma}\right)^2 + 4}\right] \tag{B.15}$$

$$= \frac{\mathrm{e}^{2i\varphi}}{2\sqrt{\gamma\,|(\epsilon_2-\epsilon_1)(\gamma-\epsilon_1)|}}\left[\operatorname{sign}(\gamma-\epsilon_1)(\gamma+\epsilon_2-2\epsilon_1) \pm \sqrt{(\gamma-\epsilon_2)^2 + 4\gamma(\epsilon_2-\epsilon_1)(\gamma-\epsilon_1)}\right] \tag{B.16}$$

$$= \frac{\mathrm{e}^{2i\varphi}}{2\sqrt{s'}}\left[\operatorname{sign}(\gamma-\epsilon_1)\alpha_0 \pm \sqrt{\Delta}\right]. \tag{B.17}$$

The definitions here are given as

$$s' = \gamma\,|(\epsilon_2-\epsilon_1)(\gamma-\epsilon_1)|, \quad \Delta = (\gamma-\epsilon_2)^2 + 4\gamma(\epsilon_2-\epsilon_1)(\gamma-\epsilon_1) = (\gamma-\epsilon_2)^2 + 4Q^2 s', \tag{B.18}$$

and

$$\alpha_0 = \frac{1}{2}\left(s + \frac{Q^2}{s\gamma}\right)\sqrt{s'}\operatorname{sign}(\gamma-\epsilon_1) = (\gamma+\epsilon_2-2\epsilon_1) \tag{B.19}$$

$$\beta_0 = \frac{1}{2}\left(s - \frac{Q^2}{s\gamma}\right)\sqrt{s'}\operatorname{sign}(\gamma-\epsilon_1) = (\gamma-\epsilon_2) \tag{B.20}$$

To find the corresponding eigenvectors one calculates the matrix

$$M_1' - \lambda_{1\pm}\mathcal{I} = \frac{\mathrm{e}^{2i\varphi}}{2\sqrt{s'}}\begin{pmatrix} \operatorname{sign}(\gamma-\epsilon_1)\beta_0 \mp \sqrt{\Delta} & -Q^*\sqrt{s'} \\ -Q^*\sqrt{s'} & -\operatorname{sign}(\gamma-\epsilon_1)\beta_0 \mp \sqrt{\Delta} \end{pmatrix} \tag{B.21}$$

Eigenvectors

$$w'_\pm = \begin{pmatrix} -Q\left[\mathrm{sign}(\gamma-\epsilon_1)\beta_0 \pm \sqrt{\Delta}\right] \\ \sqrt{s'} \end{pmatrix} \tag{B.22}$$

Notice that

$$\Delta - \beta_0^2 = 4Q^2 s' \implies \mathrm{sign}\left[\sqrt{\Delta} \pm \mathrm{sign}(\gamma-\epsilon_1)\beta_0\right] = \begin{cases} \pm\mathrm{sign}\,(\gamma-\epsilon_2)\,\mathrm{sign}(\gamma-\epsilon_1) & \text{if } Q^2=-1 \\ 1 & \text{if } Q^2 = 1 \end{cases}. \tag{B.23}$$

When $Q^2=-1$ then either $\epsilon_2<\epsilon_1$ and $\epsilon_1<\gamma$ which yields $\epsilon_2<\gamma$. Or $\epsilon_2>\epsilon_1$ and $\epsilon_1>\gamma$, in which case $\epsilon_2>\gamma$. In either case we get $(\gamma-\epsilon_1)\,(\gamma-\epsilon_2)>0$, which means

$$\mathrm{sign}\left[\sqrt{\Delta}\pm\beta_0\right] = \begin{cases} \pm 1 & \text{if } Q^2=-1 \\ 1 & \text{if } Q^2=1 \end{cases} = Q^{1\mp 1} \quad \text{when } Q^2=-1. \tag{B.24}$$

Now see that the eigenvectors are orthogonal and calculate their current norm

$$w'^T_+ w'_- = Q^2\left[\beta_0^2 - \Delta + Q^2 s'\right] = 0, \tag{B.25}$$

$$w'^T_- w'_- = Q^2\left[\beta_0^2 + \Delta + Q^2 s' - 2\mathrm{sign}(\gamma-\epsilon_1)\beta_0\sqrt{\Delta}\right] = 2\sqrt{\Delta}Q^2\left[\sqrt{\Delta} - \mathrm{sign}(\gamma-\epsilon_1)\beta_0\right] \tag{B.26}$$

$$= 2\sqrt{\Delta}\left|\sqrt{\Delta} - \mathrm{sign}(\gamma-\epsilon_1)\beta_0\right|, \tag{B.27}$$

$$w'^T_+ w'_+ = Q^2\left[\beta_0^2 + \Delta + Q^2 s' + 2\mathrm{sign}(\gamma-\epsilon_1)\beta_0\sqrt{\Delta}\right] = 2\sqrt{\Delta}Q^2\left[\sqrt{\Delta} + \mathrm{sign}(\gamma-\epsilon_1)\beta_0\right] \tag{B.28}$$

$$= 2\sqrt{\Delta}Q^2\left|\sqrt{\Delta} + \mathrm{sign}(\gamma-\epsilon_1)\beta_0\right|. \tag{B.29}$$

Appropriately normalizing the Eigenvectors gives us

$$w'_\pm = \frac{Q^{\frac{1\pm 1}{2}}}{\sqrt{2\sqrt{\Delta}}}\begin{pmatrix} \mp Q^{2\mp 1}\sqrt{\left|\sqrt{\Delta}\pm\mathrm{sign}(\gamma-\epsilon_1)\beta_0\right|} \\ \sqrt{\left|\sqrt{\Delta}\mp\mathrm{sign}(\gamma-\epsilon_1)\beta_0\right|} \end{pmatrix}. \tag{B.30}$$

The above representation is a bit convoluted, so here are the two vectors separately

$$w'_+ = \frac{1}{\sqrt{2\sqrt{\Delta}}}\begin{pmatrix} -Q^2\sqrt{\left|\mathrm{sign}(\gamma-\epsilon_1)\beta_0+\sqrt{\Delta}\right|} \\ Q\sqrt{\left|\mathrm{sign}(\gamma-\epsilon_1)\beta_0-\sqrt{\Delta}\right|} \end{pmatrix}, \quad w'_- = \frac{1}{\sqrt{2\sqrt{\Delta}}}\begin{pmatrix} Q^3\sqrt{\left|\mathrm{sign}(\gamma-\epsilon_1)\beta_0-\sqrt{\Delta}\right|} \\ \sqrt{\left|\mathrm{sign}(\gamma-\epsilon_1)\beta_0+\sqrt{\Delta}\right|} \end{pmatrix}. \tag{B.31}$$

At this point we have both Eigenvectors and Eigenvalues of the first matrix, so we could move on to the second one. However, we can notice something about our choice here: It is not nicely continuous in the parameter space. Both the Eigenvectors and Eigenvalues have discontinuities even in their absolute values.

This seems not desirable and we can fix this by altering the choice of nomenclature. This is done by choosing one Eigenvector left of the line given by $\epsilon_1=\gamma$ and the other Eigenvector on the right. A sketch of this can be found in figure 12. Im mathematical language this is expressed as

$$w_\delta = \begin{cases} w'_\delta & \text{if } \mathrm{sign}\,(\gamma-\epsilon_1)=1 \\ w'_{-\delta} & \text{if } \mathrm{sign}\,(\gamma-\epsilon_1)=-1 \end{cases}. \tag{B.32}$$

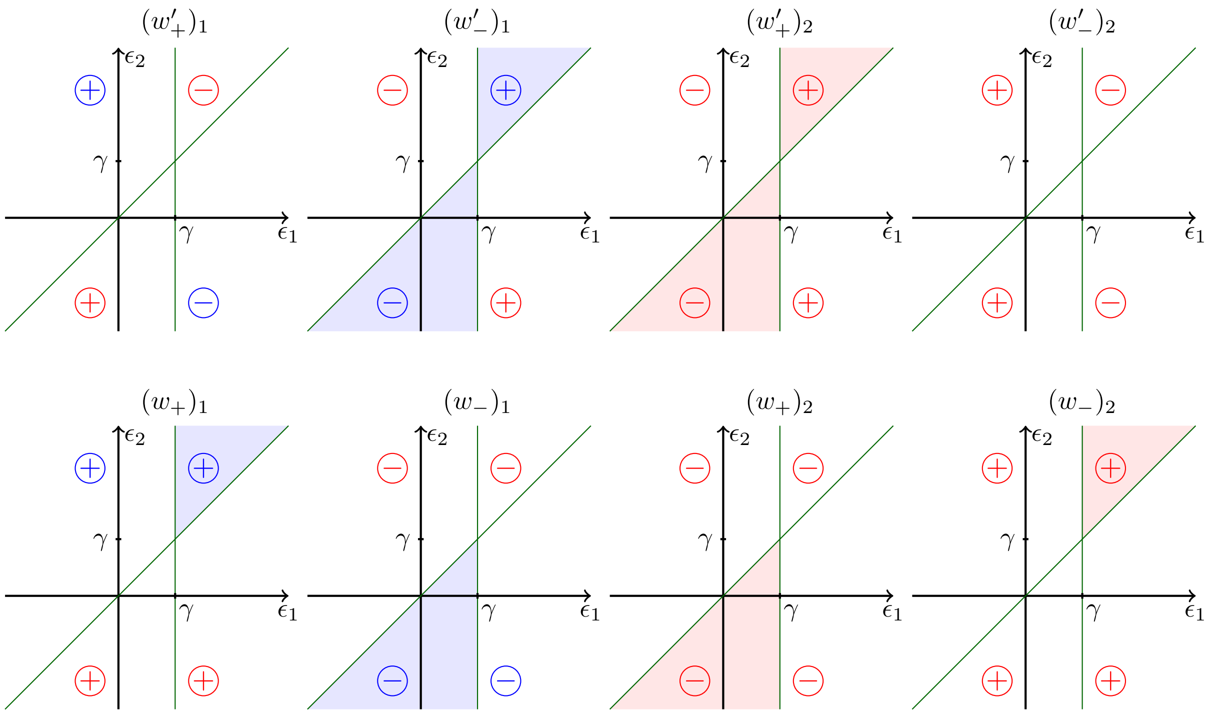

Figure 12: Sketch of the absolute value and sign of the old and new Eigenvectors. The color indicates the sign, blue for minus red for plus. The plus and minus sign give the absolute value of $\sqrt{\left|\sqrt{\Delta} \pm \beta_0\right|}$ respectively. A shaded region indicates an i as prefactor. One can see that the absolute value of the new Eigenvectors is continuous while the old one is not. On the border of the shaded region of the components of the new vectors the absolute value becomes zero such that there is no abrupt transition between real and purely imaginary.

One can then compute how these new Eigenvectors look (or simply piece it together via the sketches)

$$w_\delta = \frac{1}{\sqrt{2\sqrt{\Delta}}} \begin{pmatrix} -\text{sign}\left(\epsilon_2 - \epsilon_1\right) \delta Q_\delta \sqrt{\left|\sqrt{\Delta} + \delta\beta_0\right|} \\ Q_{-\delta}\sqrt{\left|\sqrt{\Delta} - \delta\beta_0\right|}. \end{pmatrix} \tag{B.33}$$

where

$$Q_\delta = \begin{cases} i & \text{if } \text{sign}\left(\gamma - \epsilon_1\right) = -\delta \wedge \text{sign}\left(\epsilon_2 - \epsilon_1\right) = \delta \\ 1 & \text{else} \end{cases}, \tag{B.34}$$

which means also

$$Q_\delta^2 = \begin{cases} -1 & \text{if } \text{sign}\left(\gamma - \epsilon_1\right) = -\delta \wedge \text{sign}\left(\epsilon_2 - \epsilon_1\right) = \delta \\ 1 & \text{else} \end{cases} = \begin{cases} \delta\text{sign}\left(\gamma - \epsilon_1\right) & \text{if } Q^2 = -1 \\ 1 & \text{else} \end{cases}. \tag{B.35}$$

This trivially leads also to new Eigenvalues

$$\lambda_{1\delta} = \frac{e^{2i\varphi}\text{sign}\left(\gamma - \epsilon_1\right)}{2\sqrt{\gamma\left|\left(\epsilon_2 - \epsilon_1\right)\left(\gamma - \epsilon_1\right)\right|}}\left[\left(\gamma + \epsilon_2 - 2\epsilon_1\right) + \delta\sqrt{\left(\gamma - \epsilon_2\right)^2 + 4\gamma\left(\epsilon_2 - \epsilon_1\right)\left(\gamma - \epsilon_1\right)}\right]. \tag{B.36}$$

There is now a new expression in the Eigenvectors of which we would like to know the sign

$$\mathrm{sign}\left[\sqrt{\Delta}\pm(\gamma-\epsilon_2)\right]=\begin{cases}\pm\mathrm{sign}\left(\gamma-\epsilon_2\right) & \text{if } Q^2=-1\\ 1 & \text{if } Q^2=1\end{cases}$$
$$=\begin{cases}-Q^2_{\mp 1} & \text{if } Q^2=-1\\ 1 & \text{if } Q^2=1\end{cases}=Q^2_{\mp 1}Q^2=Q^2_{\pm 1}. \tag{B.37}$$

Consider now the Eigenvalue to the second matrix M_2

$$\lambda'_{2\pm}=\mathrm{e}^{-2i\varphi}Q\frac{1}{2}\left[\epsilon_2\hat{s}Q+\frac{1}{\hat{s}Q}\pm\sqrt{\left(\epsilon_2\hat{s}Q-\frac{1}{\hat{s}Q}\right)^2+4\epsilon_1^2}\right] \tag{B.38}$$
$$=\frac{Q^2e^{-2i\varphi}}{2\sqrt{s'}}\left[\mathrm{sign}\left(\gamma-\epsilon_1\right)\left(2\epsilon_2\gamma-\epsilon_1\left(\epsilon_2+\gamma\right)\right)\pm\left|\epsilon_1\right|\sqrt{\Delta}\right]. \tag{B.39}$$

For finding Eigenvectors, calculate again the following matrix

$$M'_2-\lambda'_{2\pm}\mathcal{I}=Q\frac{\mathrm{e}^{-2i\varphi}}{2\sqrt{S}}\begin{pmatrix}-\mathrm{sign}(\gamma-\epsilon_1)\beta_0\epsilon_1\mp\left|\epsilon_1\right|\sqrt{\Delta} & \epsilon_1 Q^*\sqrt{s'}\\ \epsilon_1 Q^*\sqrt{s'} & \mathrm{sign}(\gamma-\epsilon_1)\beta_0\epsilon_1\mp\left|\epsilon_1\right|\sqrt{\Delta}\end{pmatrix}. \tag{B.40}$$

Now we simply check the old Eigenvector $w'_\pm$ against this new matrix and see what the corresponding Eigenvalue is

$$0\overset{!}{=}\begin{pmatrix}-\mathrm{sign}(\gamma-\epsilon_1)\beta_0\mp\mathrm{sign}\left(\epsilon_1\right)\sqrt{\Delta} & Q^*\sqrt{s'}\\ Q^*\sqrt{s'} & \mathrm{sign}(\gamma-\epsilon_1)\beta_0\mp\mathrm{sign}\left(\epsilon_1\right)\sqrt{\Delta}\end{pmatrix}\begin{pmatrix}-Q\left[\mathrm{sign}(\gamma-\epsilon_1)\beta_0+d\sqrt{\Delta}\right]\\ \sqrt{s'}\end{pmatrix}$$
$$=\begin{pmatrix}Q\left[\left(\mathrm{sign}(\gamma-\epsilon_1)\beta_0+d\sqrt{\Delta}\right)\left(\mathrm{sign}(\gamma-\epsilon_1)\beta_0\pm\mathrm{sign}\left(\epsilon_1\right)\sqrt{\Delta}\right)+Q^2s'\right]\\ \sqrt{s'}\left[-\mathrm{sign}(\gamma-\epsilon_1)\beta_0-d\sqrt{\Delta}+\mathrm{sign}(\gamma-\epsilon_1)\beta_0\mp\mathrm{sign}\left(\epsilon_1\right)\sqrt{\Delta}\right]\end{pmatrix}$$
$$=\begin{pmatrix}Q\left[\Delta\left(1\pm d\;\mathrm{sign}\left(\epsilon_1\right)\right)+\sqrt{\Delta}\mathrm{sign}(\gamma-\epsilon_1)\beta_0\left(d\pm\mathrm{sign}\left(\epsilon_1\right)\right)\right]\\ -\sqrt{s'}\left[d\pm\mathrm{sign}\left(\epsilon_1\right)\right]\sqrt{\Delta}\end{pmatrix}.$$

This holds for $\mp\mathrm{sign}\left(\epsilon_1\right)=d$. This means the Eigenvalue corresponding to $w'_\pm$ is

$$\lambda'_{2\pm}=\frac{Q^2e^{-2i\varphi}}{2\sqrt{S}}\left[\mathrm{sign}\left(\gamma-\epsilon_1\right)\left(2\epsilon_2\gamma-\epsilon_1\left(\epsilon_2+\gamma\right)\right)\mp\epsilon_1\sqrt{\Delta}\right]. \tag{B.41}$$

By the same logic as before, the Eigenvalue to the vector w_δ is given by

$$\lambda_{2\delta}=\frac{e^{-2i\varphi}\mathrm{sign}\left(\epsilon_2-\epsilon_1\right)}{2\sqrt{S}}\left[\left(2\epsilon_2\gamma-\epsilon_1\left(\epsilon_2+\gamma\right)\right)-\delta\epsilon_1\sqrt{\Delta}\right]. \tag{B.42}$$

Summary In total now we have found (almost continuous) Eigenvectors and Eigenvalues. Eigenvectors:

$$w_\delta=\frac{1}{\sqrt{2\sqrt{\Delta}}}\begin{pmatrix}-\mathrm{sign}\left(\epsilon_2-\epsilon_1\right)\delta Q_\delta\sqrt{\left|\sqrt{\Delta}+\delta\beta_0\right|}\\ Q_{-\delta}\sqrt{\left|\sqrt{\Delta}-\delta\beta_0\right|}\end{pmatrix}. \tag{B.43}$$

Eigenvalues:

$$\lambda_{1\delta}=\frac{\mathrm{e}^{2i\varphi}\mathrm{sign}\left(\gamma-\epsilon_1\right)}{2\sqrt{\gamma\left|\left(\epsilon_2-\epsilon_1\right)\left(\gamma-\epsilon_1\right)\right|}}\left[\left(\gamma+\epsilon_2-2\epsilon_1\right)+\delta\sqrt{\left(\gamma-\epsilon_2\right)^2+4\gamma\left(\epsilon_2-\epsilon_1\right)\left(\gamma-\epsilon_1\right)}\right], \tag{B.44}$$
$$\lambda_{2\delta}=\frac{e^{-2i\varphi}\mathrm{sign}\left(\epsilon_2-\epsilon_1\right)}{2\sqrt{\gamma\left|\left(\epsilon_2-\epsilon_1\right)\left(\gamma-\epsilon_1\right)\right|}}\left[\left(2\epsilon_2\gamma-\epsilon_1\left(\epsilon_2+\gamma\right)\right)-\delta\epsilon_1\sqrt{\Delta}\right]. \tag{B.45}$$

B.3 Complex Scaling

We can now begin to write down the diagonalized Hamiltonian. With

$$W = (w_+ \; w_-) \tag{B.46}$$

one obtains

$$W^T M_i' W = \begin{pmatrix} \tilde{\lambda}_{i+} & 0 \\ 0 & \tilde{\lambda}_{i-} \end{pmatrix}. \tag{B.47}$$

Defining new fields as

$$\begin{pmatrix} \tilde{\phi}_+ \\ \tilde{\phi}_- \end{pmatrix} = W^T \begin{pmatrix} \hat{\theta}' \\ \hat{u}' \end{pmatrix} = W^T S \begin{pmatrix} \hat{\theta} \\ \hat{u} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\Pi}_+ \\ \tilde{\Pi}_- \end{pmatrix} = W^T \begin{pmatrix} \hat{\Pi}_\theta' \\ \hat{p}' \end{pmatrix} = W^T S^{-1} \begin{pmatrix} \hat{\Pi}_\theta \\ \hat{p} \end{pmatrix}, \tag{B.48}$$

yields a diagonal Hamiltonian

$$\mathcal{H} = \frac{1}{2}\kappa n^2 \sum_{\pm} \left[\tilde{\lambda}_{1\pm} \tilde{\Pi}_\pm^2 + \tilde{\lambda}_{2\pm} \left(\partial_{\hat{x}} \tilde{\phi}_\pm \right)^2 \right]. \tag{B.49}$$

Here is now where we deal with the imaginary terms we so far have simply carried along. To get a better sense for the new fields we write down the transformation matrices once:

$$W^T S = \sqrt{\frac{\kappa m}{\hbar^2 n}} \frac{\mathrm{e}^{i\varphi}}{\sqrt{2\sqrt{\Delta}}} \begin{pmatrix} -\mathrm{sign}\left(\epsilon_2 - \epsilon_1\right) Q_{+1} \sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|} \sqrt{\left|\sqrt{\Delta}+\beta_0\right|} & Q_{+1}^* \sqrt[4]{\gamma\left|\frac{\epsilon_2-\epsilon_1}{\gamma-\epsilon_1}\right|} \sqrt{\left|\sqrt{\Delta}-\beta_0\right|} \\ \mathrm{sign}\left(\epsilon_2 - \epsilon_1\right) Q_{-1} \sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|} \sqrt{\left|\sqrt{\Delta}-\beta_0\right|} & Q_{-1}^* \sqrt[4]{\gamma\left|\frac{\epsilon_2-\epsilon_1}{\gamma-\epsilon_1}\right|} \sqrt{\left|\sqrt{\Delta}+\beta_0\right|} \end{pmatrix} \tag{B.50}$$

$$W^T S^{-1} = \sqrt{\frac{\hbar^2 n}{\kappa m}} \frac{\mathrm{e}^{-i\varphi}}{\sqrt{2\sqrt{\Delta}}} \begin{pmatrix} -\mathrm{sign}\left(\epsilon_2 - \epsilon_1\right) Q_{+1} \sqrt[4]{\gamma\left|\frac{\epsilon_2-\epsilon_1}{\gamma-\epsilon_1}\right|} \sqrt{\left|\sqrt{\Delta}+\beta_0\right|} & Q_{-1} \sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|} \sqrt{\left|\sqrt{\Delta}-\beta_0\right|} \\ \mathrm{sign}\left(\epsilon_2 - \epsilon_1\right) Q_{-1} \sqrt[4]{\gamma\left|\frac{\epsilon_2-\epsilon_1}{\gamma-\epsilon_1}\right|} \sqrt{\left|\sqrt{\Delta}-\beta_0\right|} & Q_{+1} \sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|} \sqrt{\left|\sqrt{\Delta}+\beta_0\right|} \end{pmatrix} \tag{B.51}$$

We notice that this Hamiltonian has imaginary fields for certain parameters. However, these factors of i cancel out nicely since the Hamiltonian is quadratic. The imaginary units were really only signs all along. It is therefore sensible to introduce another "scaling" transformation

$$Q = \mathrm{e}^{-i\varphi} \begin{pmatrix} Q_{+1} & 0 \\ 0 & Q_{-1} \end{pmatrix}, \tag{B.52}$$

such that with

$$W_\phi = Q W^T S, \quad W_\Pi = Q^{-1} W^T S^{-1}, \tag{B.53}$$

we obtain entirely real fields

$$\begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = W_\phi \begin{pmatrix} \hat{\theta} \\ \hat{u} \end{pmatrix}, \quad \begin{pmatrix} \Pi_+ \\ \Pi_- \end{pmatrix} = W_\Pi \begin{pmatrix} \hat{\Pi}_\theta \\ \hat{p} \end{pmatrix}. \tag{B.54}$$

Doing this makes the Eigenvalues real

$$\lambda_{1\delta} = \frac{Q_\delta^2 \mathrm{sign}\left(\gamma - \epsilon_1\right)}{2\sqrt{\gamma\left|\left(\epsilon_2 - \epsilon_1\right)\left(\gamma - \epsilon_1\right)\right|}} \left[\left(\gamma + \epsilon_2 - 2\epsilon_1\right) + \delta\sqrt{\Delta} \right], \tag{B.55}$$

$$\lambda_{2\delta} = \frac{Q_\delta^2 \mathrm{sign}\left(\epsilon_2 - \epsilon_1\right)}{2\sqrt{\gamma\left|\left(\epsilon_2 - \epsilon_1\right)\left(\gamma - \epsilon_1\right)\right|}} \left[\left(2\epsilon_2\gamma - \epsilon_1\left(\epsilon_2 + \gamma\right)\right) - \delta\epsilon_1\sqrt{\Delta} \right]. \tag{B.56}$$

The final - now entirely real - Hamiltonian reads

$$\mathcal{H} = \frac{1}{2}\kappa n^2 \sum_{\pm} \left[\lambda_{1\pm}\Pi_{\pm}^2 + \lambda_{2\pm}\left(\partial_{\hat{x}}\phi_{\pm}\right)^2\right]. \tag{B.57}$$

We can see now that there is no need to worry about the complex scaling. In total there is an entirely real canonical (see App. B.4) transformation that maps the original real Hamiltonian to the new real and diagonal one. The intermittent complex fields are an artifact of the way the transformation was constructed.

B.4 Full Transformation

We have our full transformation given by the two matrices

$$W_\phi = QW^TS, \quad W_\Pi = Q^{-1}W^TS^{-1}. \tag{B.58}$$

Generally for a canonical transformation, we want to conserve the commutator. Say for instance we have $\Pi_i' = U_{ij}\Pi_j$ and $\phi_k' = P_{kl}\phi_l$. Then the commutator would read

$$[\Pi_i',\phi_k'] = U_{ij}P_{kl}\left[\Pi_j,\phi_l\right] = U_{ij}(P^T)_{jk} \overset{!}{=} \delta_{ik}. \tag{B.59}$$

This condition in matrix form is $UP = I$. In our case we calculate

$$W_\phi W_\Pi^T = QW^TS(S^{-1})^TW(Q^{-1})^T = I. \tag{B.60}$$

The calculation is very simple since Q and S are symmetric and W is orthogonal. We thus see that our transformation is canonical. Also it is nice to have

$$W_\phi^{-1} = W_\Pi^T, \quad W_\Pi^{-1} = W_\phi^T. \tag{B.61}$$

The full matrices are given by

$$W_\phi = \sqrt{\frac{\kappa m}{\hbar^2 n}}\frac{1}{\sqrt{2\sqrt{\Delta}}}\begin{pmatrix} -\mathrm{sign}\left(\epsilon_2-\epsilon_1\right)Q_{+1}^2\sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|}\sqrt{\left|\sqrt{\Delta}+\beta_0\right|} & \sqrt[4]{\gamma\left|\frac{\epsilon_2-\epsilon_1}{\gamma-\epsilon_1}\right|}\sqrt{\left|\sqrt{\Delta}-\beta_0\right|} \\ \mathrm{sign}\left(\epsilon_2-\epsilon_1\right)Q_{-1}^2\sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|}\sqrt{\left|\sqrt{\Delta}-\beta_0\right|} & \sqrt[4]{\gamma\left|\frac{\epsilon_2-\epsilon_1}{\gamma-\epsilon_1}\right|}\sqrt{\left|\sqrt{\Delta}+\beta_0\right|} \end{pmatrix}, \tag{B.62}$$

$$W_\Pi = \sqrt{\frac{\hbar^2 n}{\kappa m}}\frac{1}{\sqrt{2\sqrt{\Delta}}}\begin{pmatrix} -\mathrm{sign}\left(\epsilon_2-\epsilon_1\right)\sqrt[4]{\gamma\left|\frac{\epsilon_2-\epsilon_1}{\gamma-\epsilon_1}\right|}\sqrt{\left|\sqrt{\Delta}+\beta_0\right|} & Q_{-1}^2\sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|}\sqrt{\left|\sqrt{\Delta}-\beta_0\right|} \\ \mathrm{sign}\left(\epsilon_2-\epsilon_1\right)\sqrt[4]{\gamma\left|\frac{\epsilon_2-\epsilon_1}{\gamma-\epsilon_1}\right|}\sqrt{\left|\sqrt{\Delta}-\beta_0\right|} & Q_{+1}^2\sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|}\sqrt{\left|\sqrt{\Delta}+\beta_0\right|} \end{pmatrix}. \tag{B.63}$$

The transformation and its inverse are then

$$\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix} = \tilde{W}_\phi\begin{pmatrix}\hat{\theta}\\ \hat{u}\end{pmatrix}, \qquad \begin{pmatrix}\Pi_+\\ \Pi_-\end{pmatrix} = \tilde{W}_\Pi\begin{pmatrix}\hat{\Pi}_\theta\\ \hat{p}\end{pmatrix}, \tag{B.64}$$

$$\begin{pmatrix}\hat{\theta}\\ \hat{u}\end{pmatrix} = \tilde{W}_\Pi^T\begin{pmatrix}\phi_+\\ \phi_-\end{pmatrix}, \qquad \begin{pmatrix}\hat{\Pi}_\theta\\ \hat{p}\end{pmatrix} = \tilde{W}_\phi^T\begin{pmatrix}\Pi_+\\ \Pi_-\end{pmatrix}. \tag{B.65}$$

We can write down more specifically that

$$\hat{\Pi}_\theta = \sqrt{\frac{\kappa m}{\hbar^2 n}}\sum_{\pm}(-a_\pm)\Pi_\pm, \quad \hat{u} = \sqrt{\frac{\hbar^2 n}{\kappa m}}\sum_{\pm} b_\pm\phi_\pm, \quad \theta = \sqrt{\frac{\hbar^2 n}{\kappa m}}\sum_{\pm} c_\pm\phi_\pm \tag{B.66}$$

where

$$a_\pm = \pm\text{sign}(\epsilon_2-\epsilon_1)Q^2_{\pm1}\sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|}\sqrt{\frac{\left|\sqrt{\Delta}\pm\beta_0\right|}{2\sqrt{\Delta}}} \tag{B.67}$$

$$b_\pm = Q^2_{\mp1}\sqrt[4]{\frac{1}{\gamma}\left|\frac{\gamma-\epsilon_1}{\epsilon_2-\epsilon_1}\right|}\sqrt{\frac{\left|\sqrt{\Delta}\mp\beta_0\right|}{2\sqrt{\Delta}}} \tag{B.68}$$

$$c_\pm = \mp\text{sign}\left(\epsilon_2-\epsilon_1\right)\sqrt[4]{\gamma\left|\frac{\epsilon_2-\epsilon_1}{\gamma-\epsilon_1}\right|}\sqrt{\frac{\left|\sqrt{\Delta}\pm\beta_0\right|}{2\sqrt{\Delta}}} \tag{B.69}$$

B.5 Expectation Values

Here we calculate all the expectation values that go into $\langle\rho\rho'\rangle$ where

$$\rho \approx [n+\delta n(x,t)]\left[1+\sum_{l=1}2\cos\left(2\pi l\left(\theta+nx\right)\right)\right]+2c_1\cos\left(k_0(x-u)\right). \tag{B.70}$$

Many of these expectation values are zero simply because $\langle\phi_\pm^2\rangle=\infty$ and $\langle\phi_\pm\rangle=0$:

$$\langle\cos\left(k_0x-s_0k_0/n\boldsymbol{b\phi}\right)\rangle = \cos(k_0x)\mathrm{e}^{-\frac{1}{2}(s_0k_0/n)^2\left\langle(\boldsymbol{b\phi})^2\right\rangle}=0 \tag{B.71}$$

$$\langle\cos\left(2\pi lnx+2\pi ls_0\boldsymbol{c\phi}\right)\rangle = \cos(2\pi lnx)\mathrm{e}^{-\frac{1}{2}(2\pi ls_0)^2\left\langle(\boldsymbol{c\phi})^2\right\rangle}=0 \tag{B.72}$$

$$\langle\delta n/n\rangle = \langle\boldsymbol{c}\partial_{\hat{x}}\boldsymbol{\phi}\rangle=0 \tag{B.73}$$

$$\begin{aligned}\left\langle\cos\left(k_0x-s_0\frac{k_0}{n}\boldsymbol{b\phi}\right)\cos\left(2\pi lnx'-2\pi ls_0\boldsymbol{c\phi}'\right)\right\rangle &= \sum_\pm\cos\left(k_0x\pm2\pi lnx'\right)\mathrm{e}^{-\frac{1}{2}s_0^2\left\langle\left[\frac{k_0}{n}\boldsymbol{b\phi}\pm2\pi l\boldsymbol{c\phi}'\right]^2\right\rangle}\\ =\sum_\epsilon\cos\left(k_0x+\epsilon2\pi lnx'\right)e^{-\frac{1}{2}s_0^2\sum_\pm\left\langle\left[\frac{k_0}{n}b_\pm\phi_\pm+\epsilon2\pi lc_\pm\phi'_\pm\right]^2\right\rangle} &= 0 \text{ except if } \left|k_0/nb_\pm\right|=\left|2\pi lc_\pm\right|\end{aligned} \tag{B.74}$$

This expectation value is also zero for the same reason, but it is harder to see

$$\left\langle\boldsymbol{c}\partial_{\hat{x}}\boldsymbol{\phi}\ \mathrm{e}^{i\epsilon\boldsymbol{m\phi}'}\right\rangle = \sum_\pm c_\pm\iint\frac{d\hat{Q}}{4\pi^2}\hat{k}\mathrm{e}^{i\hat{Q}\hat{X}}\left\langle\overline{\phi}_\pm\ \mathrm{e}^{i\epsilon\sum_\pm m_\pm\iint\frac{d\hat{Q}'}{4\pi^2}\mathrm{e}^{i\hat{Q}'\hat{X}'}\overline{\phi}'_\pm}\right\rangle \tag{B.75}$$

Consider only this expectation value. It is best calculated by

$$\left\langle\overline{\phi}_\pm\ \mathrm{e}^{i\epsilon\sum_\pm m_\pm\iint\frac{d\hat{Q}'}{4\pi^2}\mathrm{e}^{i\hat{Q}'\hat{X}'}\overline{\phi}'_\pm}\right\rangle = \frac{1}{Z}\left[4\pi^2\frac{\delta}{\delta\overline{J}_{\phi\pm}}\right]Z[\overline{J}_{\phi\pm},\overline{J}_{\Theta\pm}]\Bigg|_{\substack{\overline{J}_{\phi\pm}=-i\epsilon m_\pm\mathrm{e}^{-i\hat{Q}\hat{X}'}\\ \overline{J}_{\Theta\pm}=0}} \tag{B.76}$$

$$=\left[-\frac{1}{s_0}(M_\pm)_{22}\right]\left[-i\epsilon m_\pm\mathrm{e}^{-i\hat{Q}\hat{X}'}\right]\exp\left(-\frac{1}{2}\iint\frac{d\hat{Q}}{4\pi^2}\sum_\pm m_\pm^2\left[-\frac{1}{s_0}(M_\pm)_{22}\right]\right) \tag{B.77}$$

$$=-i\epsilon m_\pm\mathrm{e}^{-i\hat{Q}\hat{X}'}\left\langle\phi_\pm(\hat{Q}\phi_\pm(-\hat{Q})\right\rangle\mathrm{e}^{-\frac{1}{2}\left\langle(\boldsymbol{m\phi})^2\right\rangle}=0 \tag{B.78}$$

The expectation values that do not vanish are calculated as:

$$\left\langle \cos\left(k_0x - s_0\frac{k_0}{n}\boldsymbol{b\phi}\right)\cos\left(k_0x' - s_0\frac{k_0}{n}\boldsymbol{b\phi}'\right)\right\rangle = \sum_\pm \cos\left(k_0(x\pm x')\right)\mathrm{e}^{-\frac{1}{2}(s_0\frac{k_0}{n})^2\left\langle[\boldsymbol{b}(\boldsymbol{\phi}\pm\boldsymbol{\phi}')]^2\right\rangle}$$

$$= \cos\left(k_0(x-x')\right)\mathrm{e}^{-\frac{1}{2}(s_0\frac{k_0}{n})^2\left\langle[\boldsymbol{b}(\boldsymbol{\phi}-\boldsymbol{\phi}')]^2\right\rangle} \quad (B.79)$$

$$\left\langle\cos\left(2\pi lnx + 2\pi ls_0\boldsymbol{c\phi}\right)\cos\left(2\pi lnx' + 2\pi ls_0\boldsymbol{c\phi}'\right)\right\rangle = \sum_\pm \cos\left(2\pi ln(x\pm x')\right)\mathrm{e}^{-\frac{1}{2}(2\pi ls_0)^2\left\langle[\boldsymbol{c}(\boldsymbol{\phi}\pm\boldsymbol{\phi}')]^2\right\rangle}$$

$$= \cos\left(2\pi ln(x-x')\right)\mathrm{e}^{-\frac{1}{2}(2\pi ls_0)^2\left\langle[\boldsymbol{c}(\boldsymbol{\phi}-\boldsymbol{\phi}')]^2\right\rangle} \quad (B.80)$$

$$\left\langle\delta n\delta n'/n^2\right\rangle = \partial_{\hat{x}}\partial_{\hat{x}'}\left\langle(\boldsymbol{c\phi})(\boldsymbol{c\phi}')\right\rangle = s_0\sum_\pm \hat{K}_\pm c_\pm^2 \iint \frac{d\hat{q}}{2\pi}\frac{d\hat{k}}{2\pi}\frac{\hat{k}^2\hat{v}_\pm}{\hat{q}^2+\hat{v}_\pm^2\hat{k}^2}\mathrm{e}^{i(\hat{k}\Delta\hat{x}+\hat{q}\Delta\hat{\tau})} \quad (B.81)$$

$$= s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{4\pi^2\hat{v}_\pm^2}\iint d\hat{q}d\hat{k}'\frac{\hat{k}'^2}{\hat{q}^2+\hat{k}'^2}\cos\left(\hat{k}'\Delta\hat{x}/\hat{v}_\pm\right)\cos\left(\hat{q}\Delta\hat{\tau}\right) \quad (B.82)$$

$$= s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{4\pi^2\hat{v}_\pm^2}\iint da d\hat{k}'\frac{|\hat{k}'|}{a^2+1}\cos\left(\hat{k}'\Delta\hat{x}/\hat{v}_\pm\right)\cos\left(a|\hat{k}'|\Delta\hat{\tau}\right) \quad (B.83)$$

$$= s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{4\pi^2\hat{v}_\pm^2}\int d\hat{k}'|\hat{k}'|\cos\left(\hat{k}'\Delta\hat{x}/\hat{v}_\pm\right)\int da\sum_\epsilon\left(\frac{i\epsilon}{2}\right) \quad (B.84)$$

$$\times\frac{\cos\left((a+\epsilon i)|\hat{k}'|\Delta\hat{\tau}\right)\cosh(|\hat{k}'|\Delta\hat{\tau}) + i\epsilon\sin\left((a+\epsilon i)|\hat{k}'|\Delta\hat{\tau}\right)\sinh(|\hat{k}'|\Delta\hat{\tau})}{a+\epsilon i} \quad (B.85)$$

$$= s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{4\pi\hat{v}_\pm^2}\int d\hat{k}'|\hat{k}'|\cos\left(\hat{k}'\Delta\hat{x}/\hat{v}_\pm\right)\left[\cosh(|\hat{k}'|\Delta\hat{\tau}) - \sinh(|\hat{k}'|\Delta\hat{\tau})\right] \quad (B.86)$$

$$= s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{4\pi\hat{v}_\pm^2}\int d\hat{k}'|\hat{k}'|\cos\left(\hat{k}'\Delta\hat{x}/\hat{v}_\pm\right)\mathrm{e}^{-|\hat{k}'||\Delta\hat{\tau}|} \quad (B.87)$$

$$= s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{4\pi\hat{v}_\pm^2}\int_0^\infty d\hat{k}'\hat{k}'\sum_\epsilon \mathrm{e}^{\hat{k}'[-|\Delta\hat{\tau}|-\alpha+\epsilon i|\Delta\hat{x}|/\hat{v}_\pm]} \quad (B.88)$$

$$= -s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{4\pi\hat{v}_\pm^2}\partial_\alpha\int_0^\infty d\hat{k}'\sum_\epsilon \mathrm{e}^{\hat{k}'[-|\Delta\hat{\tau}|-\alpha+\epsilon i|\Delta\hat{x}|/\hat{v}_\pm]} \quad (B.89)$$

$$= s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{4\pi\hat{v}_\pm^2}\partial_\alpha\sum_\epsilon\frac{1}{-|\Delta\hat{\tau}|-\alpha+\epsilon i|\Delta\hat{x}|/\hat{v}_\pm} \quad (B.90)$$

$$= s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{2\pi}\partial_\alpha\frac{-|\Delta\hat{\tau}|-\alpha}{\hat{v}_\pm^2(|\Delta\hat{\tau}|+\alpha)^2+|\Delta\hat{x}|^2} \quad (B.91)$$

$$= s_0\sum_\pm \frac{\hat{K}_\pm c_\pm^2}{2\pi}\frac{\hat{v}_\pm^2\left(|\Delta\hat{\tau}|+\alpha\right)^2-|\Delta\hat{x}|^2}{\left(\hat{v}_\pm^2(|\Delta\hat{\tau}|+\alpha)^2+|\Delta\hat{x}|^2\right)^2} \quad (B.92)$$

B.6 Parameter Calculations

Critical Exponents Here we are interested in the expressions

$$\frac{a_\pm^2}{\hat{K}_\pm},\quad b_\pm^2\hat{K}_\pm,\quad c_\pm^2\hat{K}_\pm, \quad (B.93)$$

as well as their sums. We will see how these calculations work at the example of $b_\pm^2 \hat{K}_\pm$, the others are very similar and their results are also given here. We begin by writing down the definition

$$b_\pm^2 \hat{K}_\pm = \frac{Q_\mp^2}{2\sqrt{\Delta}} \sqrt{\frac{1}{\gamma} \left| \frac{\gamma - \epsilon_1}{\epsilon_2 - \epsilon_1} \right|} \left(\sqrt{\Delta} \mp (\gamma - \epsilon_2) \right) \frac{\left| \gamma + \epsilon_2 - 2\epsilon_1 \pm \sqrt{\Delta} \right|}{2\hat{v}_\pm \sqrt{\gamma \left| \gamma - \epsilon_1 \right| \left| \epsilon_2 - \epsilon_1 \right|}} \tag{B.94}$$

$$= \frac{1}{\sqrt{\Delta} \hat{v}_\pm} \frac{1}{4\gamma(\epsilon_2 - \epsilon_1)} \left(\sqrt{\Delta} \mp (\gamma - \epsilon_2) \right) \left(\gamma + \epsilon_2 - 2\epsilon_1 \pm \sqrt{\Delta} \right). \tag{B.95}$$

The complicated part is the multiplication of the two brackets. This yields

$$\left(\sqrt{\Delta} \mp (\gamma - \epsilon_2) \right) \left(\gamma + \epsilon_2 - 2\epsilon_1 \pm \sqrt{\Delta} \right) = \mp \left[(\gamma - \epsilon_2)(\gamma - \epsilon_2 - 2\epsilon_1) - \Delta \right] + \sqrt{\Delta} \left[2\epsilon_2 - 2\epsilon_1 \right] \tag{B.96}$$

$$= 2\sqrt{\Delta} \left[\epsilon_2 - \epsilon_1 \right] \mp 2 \left[\epsilon_1 \epsilon_2 + \gamma \epsilon_2 - \gamma \epsilon_1 - \epsilon_2^2 - 2\gamma(\epsilon_2 - \epsilon_1)(\gamma - \epsilon_1) \right] \tag{B.97}$$

$$= \pm 2(\epsilon_2 - \epsilon_1) \left[\left[\epsilon_2 - \gamma + 2\gamma(\gamma - \epsilon_1) \right] \pm \sqrt{\Delta} \right] \tag{B.98}$$

$$= \pm 2(\epsilon_2 - \epsilon_1) \left[2\gamma \hat{v}_\pm - 2\gamma(1 - \gamma) \right] = \pm 4\gamma(\epsilon_2 - \epsilon_1) \left[\hat{v}_\pm - (1 - \gamma) \right]. \tag{B.99}$$

Inserting this result gives us the final expression

$$b_\pm^2 \hat{K}_\pm = \pm \frac{\hat{v}_\pm^2 - (1 - \gamma)}{\hat{v}_\pm \sqrt{\Delta}}. \tag{B.100}$$

In a similar way we find the other two expressions

$$\frac{a_\pm^2}{\hat{K}_\pm} = \pm \frac{\gamma v_\pm^2 - (\epsilon_2 - \epsilon_1^2)}{v_\pm \sqrt{\Delta}}, \quad c_\pm^2 \hat{K}_\pm = \pm \frac{\gamma \hat{v}_\pm^2 - \epsilon_2 (1 - \gamma)}{\hat{v}_\pm \sqrt{\Delta}}. \tag{B.101}$$

Calculating the sum is straight forward

$$B = \sum_\pm b_\pm^2 \hat{K}_\pm = \sum_\pm \left[\pm \frac{\hat{v}_\pm^2 - (1 - \gamma)}{\hat{v}_\pm \sqrt{\Delta}} \right] = \frac{1}{\sqrt{\Delta} \hat{v}_+ \hat{v}_-} \left(\hat{v}_+ \hat{v}_- + (1 - \gamma) \right) \left(\hat{v}_+ - \hat{v}_- \right) \tag{B.102}$$

$$= \frac{1}{\sqrt{\gamma} \left(\hat{v}_+ + \hat{v}_- \right)} \left[\frac{1}{\sqrt{\gamma}} + \frac{1 - \gamma}{\sqrt{\gamma} \hat{v}_+ \hat{v}_-} \right] = \frac{1}{\sqrt{\gamma} \left(\hat{v}_+ + \hat{v}_- \right)} \left[\frac{1}{\sqrt{\gamma}} + \sqrt{\frac{1 - \gamma}{\epsilon_2 - \epsilon_1^2}} \right]. \tag{B.103}$$

In the same way we can evaluate

$$C = \frac{1}{\left(\hat{v}_+ + \hat{v}_- \right) \sqrt{\gamma}} \left[\sqrt{\gamma} + \epsilon_2 \sqrt{\frac{1 - \gamma}{\epsilon_2 - \epsilon_1^2}} \right], \quad A = \frac{1}{\left(\hat{v}_+ + \hat{v}_- \right) \sqrt{\gamma}} \left[\sqrt{\gamma} + \sqrt{\frac{\epsilon_2 - \epsilon_1^2}{1 - \gamma}} \right]. \tag{B.104}$$

Equipotential lines Here we calculate the equipotential lines for A,B and C.

B: We begin with

$$B = \frac{1}{\sqrt{\gamma}(\hat{v}_+ - \hat{v}_-)} \left[\frac{1}{\sqrt{\gamma}} + \sqrt{\frac{1 - \gamma}{\epsilon_2 - \epsilon_1^2}} \right]. \tag{B.105}$$

A closer look at the expression in front of the brackets reveals

$$\sqrt{\gamma}(\hat{v}_+ - \hat{v}_-) = \sqrt{\epsilon_2 + \gamma - 2\gamma\epsilon_1 + 2\sqrt{\gamma(1 - \gamma)(\epsilon_1 - \epsilon_2)}} \tag{B.106}$$

$$= \sqrt{(\epsilon_2 - \epsilon_1^2) + \gamma(1 - \gamma) + (\gamma - \epsilon_1)^2 + 2\sqrt{\gamma(1 - \gamma)(\epsilon_1 - \epsilon_2)}}. \tag{B.107}$$

Define now

$$a = \sqrt{\epsilon_2 - \epsilon_1^2} + \sqrt{\gamma(1-\gamma)}, \quad b = \epsilon_1 - \gamma, \tag{B.108}$$

where we have the condition $a > \sqrt{\gamma(1-\gamma)}$. Then we see a convenient definition

$$\sqrt{\gamma}(\hat{v}_+ - \hat{v}_-) = \sqrt{a^2 + b^2} =: r \tag{B.109}$$

and subsequently also define

$$a = r\cos(\varphi), \quad b = r\sin(\varphi), \tag{B.110}$$

where $\varphi \in \{-\pi/2, \pi/2\}$ since $a > 0$. With this we obtain

$$B = \frac{1}{r}\left[\frac{1}{\sqrt{\gamma}} + \frac{\sqrt{1-\gamma}}{r\cos(\varphi) - \sqrt{\gamma(1-\gamma)}}\right]. \tag{B.111}$$

Solving this equation for r gives

$$r = \frac{1}{\sqrt{\gamma}B} + \frac{\sqrt{\gamma(1-\gamma)}}{\cos(\varphi)}. \tag{B.112}$$

With this the condition

$$a = r\cos(\varphi) = \frac{\cos(\varphi)}{\sqrt{\gamma}B} + \sqrt{\gamma(1-\gamma)} > \sqrt{\gamma(1-\gamma)} \tag{B.113}$$

is fulfilled. Finally, by combining the results obtained so far, we obtain the equipotential lines

$$\epsilon_1 = \gamma + \frac{\sin(\varphi)}{\sqrt{\gamma}B} + \sqrt{\gamma(1-\gamma)}\tan(\varphi), \tag{B.114}$$

$$\epsilon_2 = \epsilon_1^2 + \frac{\cos^2(\varphi)}{\gamma B^2}. \tag{B.115}$$

C: The same approach also works for finding equipotential lines for

$$C = \frac{1}{\sqrt{\gamma}(\hat{v}_+ - \hat{v}_-)}\left[\sqrt{\gamma} + \epsilon_2\sqrt{\frac{1-\gamma}{\epsilon_2 - \epsilon_1^2}}\right]. \tag{B.116}$$

Using the same definitions we obtain

$$C = \frac{1}{r}\left[\sqrt{\gamma} + \left(\left(r\cos(\varphi) - \sqrt{\gamma(1-\gamma)}\right)^2 + (r\sin(\varphi) + \gamma)^2\right)\frac{\sqrt{1-\gamma}}{r\cos(\varphi) - \sqrt{\gamma(1-\gamma)}}\right] \tag{B.117}$$

$$= \frac{1}{r}\left[\sqrt{\gamma} + \left(r^2 + \gamma + 2r\left[\gamma\sin(\varphi) - \sqrt{\gamma(1-\gamma)}\cos(\varphi)\right]\right)\frac{\sqrt{1-\gamma}}{r\cos(\varphi) - \sqrt{\gamma(1-\gamma)}}\right] \tag{B.118}$$

Obtaining an equation for r is a bit more difficult here. We first multiply the equation by the denominator

$$[rC - \sqrt{\gamma}]\left[r\cos(\varphi) - \sqrt{\gamma(1-\gamma)}\right] = \sqrt{1-\gamma}\left(r^2 + \gamma + 2r\left[\gamma\sin(\varphi) - \sqrt{\gamma(1-\gamma)}\cos(\varphi)\right]\right). \tag{B.119}$$

Here we see that one term cancels and we can the divide by r. This gives us the equation

$$[rC - \sqrt{\gamma}]\cos(\varphi) - C\sqrt{\gamma(1-\gamma)} = \sqrt{1-\gamma}\left(r + 2\left[\gamma\sin(\varphi) - \sqrt{\gamma(1-\gamma)}\cos(\varphi)\right]\right), \tag{B.120}$$

which we can then solve for r

$$r\left[C\cos(\varphi)-\sqrt{1-\gamma}\right]=2\gamma\sqrt{1-\gamma}\sin(\varphi)+\sqrt{\gamma}\left[1-2(1-\gamma)\right]\cos(\varphi)+C\sqrt{\gamma(1-\gamma)}. \quad \text{(B.121)}$$

We need to again check if $a>\sqrt{\gamma(1-\gamma)}$. To do that we consider

$$a\left[C\cos(\varphi)-\sqrt{1-\gamma}\right]=2\gamma\sqrt{1-\gamma}\sin(\varphi)\cos(\varphi)-\sqrt{\gamma}\left[1-2\gamma\right]\cos^2(\varphi)+C\sqrt{\gamma(1-\gamma)}\cos(\varphi). \quad \text{(B.122)}$$

After moving some terms around we see that

$$\left[a-\sqrt{\gamma(1-\gamma)}\right]\left[C\cos(\varphi)-\sqrt{1-\gamma}\right]=2\gamma\sqrt{1-\gamma}\sin(\varphi)\cos(\varphi)-\sqrt{\gamma}\left[1-2\gamma\right]\cos^2(\varphi)+\sqrt{\gamma}(1-\gamma) \quad \text{(B.123)}$$

$$=\gamma\sqrt{1-\gamma}\sin(2\varphi)-\frac{1}{2}\sqrt{\gamma}\left[1-2\gamma\right]\cos(2\varphi)+\frac{1}{2}\sqrt{\gamma} \quad \text{(B.124)}$$

$$=\frac{\sqrt{\gamma}}{2}\left[1+2\sqrt{\gamma(1-\gamma)}\sin(2\varphi)-\left[1-2\gamma\right]\cos(2\varphi)\right]. \quad \text{(B.125)}$$

At this point we can write this in a more compact manner

$$1+2\sqrt{\gamma(1-\gamma)}\sin(2\varphi)-\left[1-2\gamma\right]\cos(2\varphi)=1+\sin(2\varphi-\vartheta)>0, \quad \text{(B.126)}$$

where

$$\cos(\vartheta)=2\sqrt{\gamma(1-\gamma)},\quad \sin(\vartheta)=1-2\gamma. \quad \text{(B.127)}$$

We now know that we have

$$\left[a-\sqrt{\gamma(1-\gamma)}\right]\left[C\cos(\varphi)-\sqrt{1-\gamma}\right]>0, \quad \text{(B.128)}$$

which leads to the condition

$$C\cos(\varphi)>\sqrt{1-\gamma}. \quad \text{(B.129)}$$

We define

$$T=\frac{\sqrt{\gamma}}{2}\frac{1+2\sqrt{\gamma(1-\gamma)}\sin(2\varphi)-\left[1-2\gamma\right]\cos(2\varphi)}{C\cos(\varphi)-\sqrt{1-\gamma}} \quad \text{(B.130)}$$

This leads us to the equation for the equipotential lines

$$\epsilon_1=\gamma+\left[T+\sqrt{\gamma(1-\gamma)}\right]\tan(\varphi),\quad \epsilon_2=\epsilon_1^2+T^2, \quad \text{(B.131)}$$

with $\varphi\in[-\arccos(\sqrt{1-\gamma}/C),\arccos(\sqrt{1-\gamma}/C)]$.

A: The simplest of the three cases is

$$A=\sum\frac{a_\pm^2}{\hat{K}_\pm}=\frac{1}{(v_++v_-)\sqrt{\gamma}}\left[\sqrt{\gamma}+\sqrt{\frac{\epsilon_2-\epsilon_1^2}{1-\gamma}}\right]. \quad \text{(B.132)}$$

We begin by multiplying the expression with the denominator

$$A\sqrt{1-\gamma}\sqrt{\epsilon_2+\gamma-2\epsilon_1\gamma+2\sqrt{\gamma(1-\gamma)(\epsilon_2-\epsilon_1^2)}}=\sqrt{\gamma(1-\gamma)}+\sqrt{\epsilon_2-\epsilon_1^2}. \quad \text{(B.133)}$$

This expression we square and identify some terms on the left hand side to obtain

$$A^2(1-\gamma)\left[(\gamma-\epsilon_1)^2+\left[\sqrt{\gamma(1-\gamma)}+\sqrt{\epsilon_2-\epsilon_1^2}\right]^2\right]=\left[\sqrt{\gamma(1-\gamma)}+\sqrt{\epsilon_2-\epsilon_1^2}\right]^2. \quad \text{(B.134)}$$

We can sort these terms and with the constraint $1-A^2(1-\gamma)>0$ we can take the square root to arrive at

$$\left[\sqrt{\gamma(1-\gamma)}+\sqrt{\epsilon_2-\epsilon_1^2}\right]\sqrt{1-A^2(1-\gamma)}=A\sqrt{1-\gamma}\,|\gamma-\epsilon_1|\,. \quad \text{(B.135)}$$

Rearranging these terms we get the expression

$$\sqrt{\epsilon_2-\epsilon_1^2}=\frac{A\sqrt{1-\gamma}\,|\gamma-\epsilon_1|}{\sqrt{1-A^2(1-\gamma)}}-\sqrt{\gamma(1-\gamma)}. \quad \text{(B.136)}$$

From this we obtain the equipotential lines

$$\epsilon_2=\epsilon_1^2+(1-\gamma)\left[\frac{A\,|\gamma-\epsilon_1|}{\sqrt{1-A^2(1-\gamma)}}-\sqrt{\gamma}\right]^2, \quad \text{(B.137)}$$

with the conditions

$$1-A^2(1-\gamma)>0, \quad |\gamma-\epsilon_1|>\sqrt{\gamma/A^2-\gamma(1-\gamma)}. \quad \text{(B.138)}$$

The second condition we can also rewrite as

$$A^2>\frac{\gamma}{(\gamma-\epsilon_1)^2+\gamma(1-\gamma)}. \quad \text{(B.139)}$$

C Perturbations: Second Order

C.1 Impurity

Consider again the impurity perturbation

$$\mathcal{S}' = g \iint d\hat{x}d\hat{\tau}\ V(\hat{x})\rho(\hat{x},\hat{\tau}) = g \int d\hat{\tau}\ n(0,\hat{\tau}) \left[1 + \sum_l 2\cos(2\pi l\theta(0,\hat{\tau}))\right] + 2\cos(k_0 u(0,\hat{\tau})). \quad \text{(C.1)}$$

The second order turns out to not be as trivial as the first. If we were to follow the same procedure which gave the solution to the first order, we would run in to trouble when later trying to identify terms during rescaling since the original action is two dimensional while the perturbation has only one. To work around this problem, we will try to obtain an effective one dimensional action for $\phi_{0\pm}(\tau) = \phi_\pm(0,\tau)$.

We begin by splitting $\phi_\pm$ into a symmetric $\phi_{s\pm}$ and antisymmetric $\phi_{a\pm}$ part. The (anti)symmetry is with respect to position and as such

$$\phi_{a\pm}(0,\tau) = 0, \quad \phi_{s\pm}(0,\tau) = \phi_{0\pm}(\tau). \quad \text{(C.2)}$$

Again, due to its form, the action then splits as

$$S_0 = S_a + S_s. \quad \text{(C.3)}$$

We can now again consider the term Z_g/Z_0, which simplifies to

$$Z_g/Z_0 = \frac{1}{Z_0} Z_a \iint D\phi_{s\pm}\ \mathrm{e}^{\frac{S_s}{\hbar}} \mathrm{e}^{\frac{S'}{\hbar}}, \quad \text{(C.4)}$$

with

$$\mathcal{S}_s = s_0 \iint_D d\hat{x}d\hat{\tau} \sum_\pm \frac{1}{\hat{K}_\pm} \left[\frac{1}{\hat{v}_\pm} \left(\partial_{\hat{\tau}}\phi_{s\pm}\right)^2 + \hat{v}_\pm \left(\partial_{\hat{x}}\phi_{s\pm}\right)^2\right], \quad \text{(C.5)}$$

and

$$D = \{(x,\tau)|\ x > 0; \tau,x \in \mathbb{R}\}. \quad \text{(C.6)}$$

The Lagrangian for this action we can directly read of as

$$\mathcal{L}_s \propto \sum_\pm \frac{1}{\hat{K}_\pm} \left[\frac{1}{\hat{v}_\pm} \left(\partial_{\hat{\tau}}\phi_{s\pm}\right)^2 + \hat{v}_\pm \left(\partial_{\hat{x}}\phi_{s\pm}\right)^2\right]. \quad \text{(C.7)}$$

This gives the Euler-Lagrange equations

$$0 = \frac{1}{\hat{v}_\pm}\partial^2_{\hat{\tau}}\phi_{s\pm} + \hat{v}_\pm\partial^2_{\hat{x}}\phi_{s\pm}. \quad \text{(C.8)}$$

Substituting $\hat{y}_\pm = \hat{v}_\pm\hat{\tau}$ this equation becomes

$$0 = \left(\partial^2_{\hat{y}_\pm} + \partial^2_{\hat{x}}\right)\phi_{s\pm} = \Delta_\pm\phi_{s\pm}. \quad \text{(C.9)}$$

The fundamental solution of the two dimensional Laplace operator is known to be

$$u(\hat{x},\hat{y}_\pm,\hat{x}',\hat{y}'_\pm) = -\frac{1}{2\pi}\ln\left(r_\pm\right), \quad r_\pm = \sqrt{\left(\hat{x}-\hat{x}'\right)^2 + \left(\hat{y}_\pm - \hat{y}'_\pm\right)^2}. \quad \text{(C.10)}$$

Our boundary conditions in this case are

$$\phi_{s\pm}(0,\hat{y}_\pm) = \phi_{0\pm}(\hat{y}_\pm), \quad \text{(C.11)}$$

as well as that $\phi_{s\pm}$ and its first derivatives vanish as either of its arguments tends to infinity. This now means that for the greens function we need a fundamental solution which is zero for $\hat{x} = 0$. This is achieved by

$$G = -\frac{1}{4\pi} \ln\left((\hat{x} - \hat{x}')^2 + (\hat{y}_\pm - \hat{y}'_\pm)^2 \right) + \frac{1}{4\pi} \ln\left((\hat{x} + \hat{x}')^2 + (\hat{y}_\pm - \hat{y}'_\pm)^2 \right). \tag{C.12}$$

Then the solution is given by

$$\phi_{s\pm}(\hat{x},\hat{y}_\pm) = \int d\hat{y}'_\pm (\partial_{\hat{x}'} G)(\hat{x},\hat{y}_\pm,0,\hat{y}'_\pm)\phi_{0\pm}(\hat{y}'_\pm), \tag{C.13}$$

where

$$(\partial_{\hat{x}'} G)(\hat{x},\hat{y}_\pm,0,\hat{y}'_\pm) = \frac{1}{\pi} \frac{\hat{x}}{\hat{x}^2 + (\hat{y}_\pm - \hat{y}'_\pm)^2} = \partial_{\hat{x}} \left[\frac{1}{2\pi} \ln\left(\hat{x}^2 + (\hat{y}_\pm - \hat{y}'_\pm)^2 \right) \right] = \partial_{\hat{x}} f. \tag{C.14}$$

We also directly see that

$$\lim_{\hat{x}\to 0} \partial_{\hat{x}} f = \delta(\hat{y}_\pm - \hat{y}'_\pm). \tag{C.15}$$

We are now interested in

$$\begin{aligned}
\iint_{D_\pm} d\hat{x} d\hat{y}_\pm \, (\partial_{\hat{x}}\phi_{s\pm})^2 + (\partial_{\hat{y}_\pm}\phi_{s\pm})^2 &= -\iint_{D_\pm} d\hat{x} d\hat{y}_\pm \phi_{s\pm}\Delta_\pm\phi_{s\pm} - \int d\hat{y}_\pm \left[\phi_{s\pm}\partial_{\hat{x}}\phi_{s\pm}\right]|_{\hat{x}=0} \\
&= -\iiint d\hat{y}_\pm d\hat{y}'_\pm d\hat{y}''_\pm (\partial^2_{\hat{x}} f)(0,\hat{y}_\pm,\hat{y}''_\pm)\phi_{0\pm}(\hat{y}''_\pm)\delta(\hat{y}_\pm - \hat{y}''_\pm)\phi_{0\pm}(\hat{y}'_\pm) \\
&= -\frac{1}{\pi} \iint d\hat{y}_\pm d\hat{y}'_\pm \frac{\phi_{0\pm}(\hat{y}_\pm)\phi_{0\pm}(\hat{y}'_\pm)}{(\hat{y}_\pm - \hat{y}'_\pm)^2}.
\end{aligned} \tag{C.16}$$

Fourier transforming this yields

$$\begin{aligned}
S_e &= \sum_\pm \frac{s_0}{\hat{K}_\pm} \left[-\frac{1}{\pi} \iint d\hat{\tau} d\hat{\tau}' \frac{\phi_{0\pm}(\hat{\tau})\phi_{0\pm}(\hat{\tau}')}{(\hat{\tau} - \hat{\tau}')^2} \right] && \text{(C.17)} \\
&= \sum_\pm \frac{s_0}{\hat{K}_\pm} \left[-\frac{1}{\pi} \iint \frac{d\hat{q}}{2\pi} \frac{d\hat{q}'}{2\pi} \iint da db \frac{1}{a^2} \mathrm{e}^{i\frac{a}{2}(\hat{q}-\hat{q}')} \mathrm{e}^{ib(\hat{q}+\hat{q}')} \overline{\phi}_{0\pm}(\hat{q})\overline{\phi}_{0\pm}(\hat{q}') \right] && \text{(C.18)} \\
&= \sum_\pm \frac{s_0}{\hat{K}_\pm} \left[-\frac{1}{\pi} \int \frac{d\hat{q}}{2\pi} \int da \frac{1}{a^2} \mathrm{e}^{ia\hat{q}} \overline{\phi}_{0\pm}\overline{\phi}^\dagger_{0\pm} \right] = \sum_\pm \frac{s_0}{\hat{K}_\pm} \int \frac{d\hat{q}}{2\pi} |\hat{q}| \, \overline{\phi}_{0\pm}\overline{\phi}^\dagger_{0\pm} && \text{(C.19)} \\
&= \sum_\pm \frac{1}{2} \int \frac{d\hat{q}}{2\pi} \frac{s_0}{\hat{K}_\pm} |\hat{q}| \, \overline{\phi}_{0\pm}\overline{\phi}^\dagger_{0\pm} = \sum_\pm \frac{1}{2} \int \frac{d\hat{q}}{2\pi} \hat{\eta}_\pm |\hat{q}| \, \overline{\phi}_{0\pm}\overline{\phi}^\dagger_{0\pm}, && \text{(C.20)}
\end{aligned}$$

where

$$\hat{\eta}_\pm = 2\frac{s_0}{\hat{K}_\pm} = 2\hat{m}_\pm \hat{v}_\pm. \tag{C.21}$$

As before we see that to calculate correlation functions with this action we will need a high frequency cutoff. This canbe done by introducing a mass term into the action. The action with the cutoff then has the form

$$\sum_\pm \frac{1}{2} \int \frac{d\hat{q}}{2\pi} \left[\hat{m}_\pm \hat{q}^2 + \hat{\eta}_\pm |\hat{q}| \right] \overline{\phi}_{0\pm}\overline{\phi}^\dagger_{0\pm}. \tag{C.22}$$

This action now describes the sum of two dissipative, one dimensional systems with particles of mass $\hat{m}$ and friction coefficient $\hat{\eta}$. Such systems were treated by Schmid[13] and Bulgadaev[14].

For actual renormalization group calculations would like to know the new correlation functions. Trivial now are

$$\left\langle\overline{\phi}_{0\pm}(\hat{q})\overline{\phi}_{0\pm}(\hat{q}')\right\rangle = \frac{1}{\hat{m}_\pm \hat{q}^2 + \hat{\eta}_\pm |\hat{q}|} 2\pi\delta(\hat{q}+\hat{q}'). \tag{C.23}$$

Interesting are correlation functions like

$$\frac{1}{2}\left\langle\left[\phi_{0\pm}(\hat{\tau}) - \phi_{0\pm}(\hat{\tau}')\right]^2\right\rangle = \int \frac{dq}{2\pi}\frac{1-\cos\left(\hat{q}\Delta\hat{\tau}\right)}{\hat{m}_\pm q^2 + \hat{\eta}_\pm |q|} = \frac{2}{\hat{\eta}_\pm}\int_0^\infty \frac{dq}{2\pi}\frac{1-\cos(qr)}{q^2+q}, \tag{C.24}$$

where

$$q = \frac{\hat{m}_\pm}{\hat{\eta}_\pm}\hat{q}, \quad r = \frac{\hat{\eta}_\pm}{\hat{m}_\pm}\Delta\hat{\tau}. \tag{C.25}$$

Now with some clever redistributing

$$\int_0^\infty dq \frac{1-\cos(qr)}{q^2+q} = \int_0^\infty dq\left[\frac{1}{q^2+q} - \frac{\cos(qr)}{q} + \frac{\cos(qr)}{q+1}\right]. \tag{C.26}$$

One needs to be careful with transformations here, as one is toying with infinities. We see that

$$\int_0^\infty dq\frac{1}{q^2+q} = \int_0^1 dq\frac{1}{q} - \int_0^1 dq\frac{1}{q+1} + \int_1^\infty dq\frac{1}{q^2+q} = \int_0^1 dq\frac{1}{q}. \tag{C.27}$$

One last transformation is needed, and then we see by the definitions of cosine integrals that

$$\int_0^1 dq\frac{1}{q} - \int_0^\infty dq\frac{\cos(qr)}{q} = \int_0^1 dq\frac{1-\cos(qr)}{q} - \int_1^\infty dq\frac{\cos(qr)}{q} \tag{C.28}$$

$$= \int_0^r dq\frac{1-\cos(q)}{q} - \int_r^\infty dq\frac{\cos(q)}{q} = \ln(r) + \gamma. \tag{C.29}$$

with the Euler-Mascheroni constant γ. In total this yields

$$\frac{1}{2}\left\langle\left[\phi_{0\pm}(\hat{\tau}) - \phi_{0\pm}(\hat{\tau}')\right]^2\right\rangle = \frac{1}{\hat{\eta}_\pm\pi}\left[\ln(r) + \gamma + \int_0^\infty dq\frac{\cos(qr)}{q+1}\right] = \begin{cases} \frac{1}{\hat{\eta}_\pm\pi}\left[\ln(r)+\gamma\right] & r \gg 1 \\ \frac{1}{2\hat{\eta}_\pm} r & r \ll 1 \end{cases} \tag{C.30}$$

Let us first consider

$$\mathrm{e}^{g\int\cos(\boldsymbol{d\phi_0})} = \sum_n \frac{1}{n!}\left[\prod_i^n \int d\hat{\tau}_i\right]\left(\frac{g}{2}\right)^n \sum_{\{\epsilon_i=\pm1\}} \mathrm{e}^{i\sum_i \epsilon_i \boldsymbol{d\phi_0}(\hat{\tau}_i)}. \tag{C.31}$$

Taking the expectation value we see that only terms with even n survive as we need charge neutrality $\sum_i \epsilon_i = 0$ for the expression not to become zero. Also, since the specific configuration of $\{\epsilon_i\}$ does not matter as one can rearrange the integrals, we can choose one specific configuration.

$$\left\langle \mathrm{e}^{g\int\cos(\boldsymbol{d\phi_0})}\right\rangle = \sum_n \frac{1}{(n!)^2}\left(\frac{g}{2}\right)^{2n}\left[\prod_i^{2n}\int d\hat{\tau}_i\right]\mathrm{e}^{-\frac{1}{2}\left\langle\left[\sum_i \epsilon_i \boldsymbol{d\phi_0}(\hat{\tau}_i)\right]^2\right\rangle}. \tag{C.32}$$

A closer look at the expectation value in the exponent reveals

$$-\frac{1}{2}\left\langle\left[\sum_i \epsilon_i \boldsymbol{d\phi_0}(\hat{\tau}_i)\right]^2\right\rangle = \frac{1}{2}\sum_{i<j}\epsilon_i\epsilon_j\left\langle\left[\boldsymbol{d}(\boldsymbol{\phi_0}(\hat{\tau}_i) - \boldsymbol{\phi_0}(\hat{\tau}_j))\right]^2\right\rangle, \tag{C.33}$$

since with charge neutrality we have

$$\Big(\sum \epsilon_i A_i\Big)^2 = \sum_{i,j} \epsilon_i\epsilon_j A_i A_j = 2\sum_{i<j} \epsilon_i\epsilon_j A_i A_j + \sum_i \epsilon_i^2 A_i^2 = 2\sum_{i<j} \epsilon_i\epsilon_j A_i A_j - \sum_i \epsilon_i A_i^2 \sum_{j\neq i} \epsilon_j$$
$$= 2\sum_{i<j} \epsilon_i\epsilon_j A_i A_j - \sum_{i<j} \epsilon_i\epsilon_j \left(A_i^2 + A_j^2\right) = -\sum_{i<j} \epsilon_i\epsilon_j \left(A_i - A_j\right)^2 .$$

We know these expectation values, when we consider the long range case this yields

$$\frac{1}{2}\sum_{i<j} \epsilon_i\epsilon_j \sum_{\pm} \frac{d_\pm^2}{\hat{\eta}_\pm \pi}\left[\ln\left(\frac{\hat{\eta}_\pm}{\hat{m}_\pm}\left|\hat{\tau}_i - \hat{\tau}_j\right|\right) + \gamma\right]. \tag{C.34}$$

Separating this a bit further gives

$$\frac{1}{2}\sum_{i<j} \epsilon_i\epsilon_j \left[\ln\left(\left|\hat{\tau}_i - \hat{\tau}_j\right|\right)\right]\left(\sum_{\pm}\frac{d_\pm}{\hat{\eta}_\pm \pi}\right) + \frac{1}{2}\sum_{i<j} \epsilon_i\epsilon_j \sum_{\pm} \frac{d_\pm}{\hat{\eta}_\pm \pi}\left[\ln\left(\frac{\hat{\eta}_\pm}{\hat{m}_\pm}\right) + \gamma\right], \tag{C.35}$$

where we nicely see the logarithmic dependence on $\hat{\tau}_i$. After some restructuring we get

$$\frac{1}{2}\sum_{i<j} \epsilon_i\epsilon_j \frac{1}{\pi\eta}\ln\left(r'\right). \tag{C.36}$$

with

$$\frac{1}{\eta} = \sum_{\pm} \frac{d_\pm^2}{\hat{\eta}_\pm}, \quad r' = \frac{\eta}{m}\left|\hat{\tau}_i - \hat{\tau}_j\right|, \quad m = \frac{1}{e^\gamma \eta}\prod_{\pm} \hat{\eta}_\pm^{-\frac{\eta d_\pm^2}{\hat{\eta}_\pm}} \hat{m}_\pm^{\frac{\eta d_\pm^2}{\hat{\eta}_\pm}}. \tag{C.37}$$

In full this means again that

$$\left\langle e^{g\int \cos(\boldsymbol{d\phi_0})}\right\rangle = \sum_n \frac{1}{(n!)^2}\left(\frac{g}{2}\right)^{2n}\left[\prod_i^{2n}\int d\hat{\tau}_i\right] e^{\frac{1}{2}\sum_{i<j}\epsilon_i\epsilon_j \frac{1}{\pi\eta}\ln(r')}. \tag{C.38}$$

This is the problem Bulgadaev treats in [14], where he obtains the renormalization flow equations

$$\frac{d\left(\frac{m}{a}\right)}{dl} = -\frac{m}{a} + \frac{2}{3\pi\eta}(ga)^2, \quad \frac{d(ga)}{dl} = \left(1 - \frac{1}{2\pi\eta}\right)(ga) - \frac{16}{3}(ga)^3, \quad dl = \frac{da}{a}, \quad a(l=0) = \frac{m}{\eta}. \tag{C.39}$$

We can see that there exists a critical point $\eta^* = \frac{1}{2\pi}$ which, since

$$\frac{1}{2\pi\eta} = \frac{1}{2\pi}\sum_{\pm}\frac{d_\pm^2}{\hat{\eta}_\pm} = \frac{1}{s_0}\sum_{\pm}\frac{d_\pm^2 \hat{K}_\pm}{4\pi}, \tag{C.40}$$

corresponds to the critical point we have already found in first order. In the case of $1/(2\pi\eta) > 1/(2\pi\eta^*) = 1$ the behavior of the system is governed by dissipation. The results from [14] then give the same logarithmic dependence in the correlation functions as before (with a short range cutoff here)

$$\sum_{\pm} d_\pm^2 \left\langle \phi_{0\pm}(\hat{\tau})\phi_{0\pm}(\hat{\tau}')\right\rangle \sim -\frac{1}{\pi\eta}\ln\left(\left|\hat{\tau} - \hat{\tau}'\right|\right). \tag{C.41}$$

This expected as it is the case where our perturbation is irrelevant. The case of $1/(2\pi\eta) < 1$ is more interesting. There we then get correlation functions of the form

$$\sum_{\pm} d_\pm^2 \left\langle \phi_{0\pm}(\hat{\tau})\phi_{0\pm}(\hat{\tau}')\right\rangle \sim -\frac{1}{\pi\eta}\left[\mathrm{Si}(\tau M)\sin(\tau M) - \mathrm{Ci}(\tau M)\cos(\tau M)\right], \tag{C.42}$$

where M is a constant related to the correlation length $\xi \sim (m/\eta)e^{l^*}$[14]. This correlation oscillates while decreasing in a power-law fashion. An interesting thing is the regime of strong coupling $mg \gg 1$. There an expansion in kink/antikink solutions [13] yields the same action as before after a replacement $\frac{1}{2\pi\eta} \to 2\pi\eta$ (and some other replacements, but that's not that important here). Thus we observe the same phase transition in the strong coupling case, but with relevant and irrelevant regimes swapped.