University of Stuttgart
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Bachelor Thesis

Light propagation in Rydberg media

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Stuttgart, September 2, 2013 Jan Kumlin

Abstract

In the last decade the study of cold Rydberg atoms has caught more and more attention. Due to their strong dipole-dipole interactions they have become a prime candidate for realising a quantum computer [13]. Controlling and implementing Rydberg atoms as quantum gates is in this case strongly related to the interaction with light fields. For this interaction nonlinear optical effects arise like electromagnetically induced transparency (EIT) or the reduction of the group velocity to several meters per second [4].

The properties of dissipative and strongly interacting system, e.g. Rydberg media, give rise to applications like a single-photon-emitter as well as a single-photon-absorber [7], [10]. Especially the blockade mechanism arising in Rydberg media plays an important role in this case. This mechanism can be used, depending on the application, to isolate a single photon from an ensemble of photons as well as to absorb a single photon making the medium transparent for the other photons. The resulting state, however, is not a pure state any more which requires a description with the help of the density matrix formalism and the quantum optical master equation.

In this thesis we deal with the light propagation in Rydberg media. Therefore we first study the classical dynamics of the electromagnetic field in inhomogeneous media. In addition, we calculate the amplitude and intensity reflection coefficients for two selected functions of the refractive index, which have the form of Eckart potentials. In the second part we consider the quantum master equation for continuous media and present a general solution which we apply on Fock states as well as on coherent states. Additionally, we take account of the blockade-mechanism of strongly interacting Rydberg atoms that leads to deterministic single photon subtraction in a saturating absorber.

Zusammenfassung

Das Studium ultrakalter Quantengase hat in den letzten Jahren mehr und mehr an Bedeutung gewonnen. Durch ihre starke Dipol-Dipol-Wechselwirkung und die hohe Lebensdauer sind vor allem Rydberg-Gase, die als Kandidat zur Realisierung eines Quantencomputers gelten, in den Fokus gerückt [13]. Die Kontrolle und Implementierung der Rydberg-Gase als Quantengatter hängt dabei wesentlich mit ihrer Wechselwirkung mit Lichtfeldern zusammen. Im Rahmen dieser Wechselwirkung treten Effekte wie die elektromagnetisch induzierte Transparenz (EIT) und das Abbremsen der Gruppengeschwindigkeit des Lichts auf wenige Meter pro Sekunde auf [4].

Dissipative und stark wechselwirkende Systeme wie z.b. Rydberg-Gase besitzen Eigenschaften, die genutzt werden können, um Anwendungen wie Einzelphotonenquellen oder Einzelphotonendetektoren realisieren zu können [7], [10]. Dabei spielt der bei Rydberg-Medien auftretende Blockade-Effekt eine wichtige Rolle. Je nach Anwendung kann dieser Effekt dazu genutzt werden, ein einzelnes Photon aus einem Ensemble zu isolieren oder genau ein Photon zu absorbieren und das Medium für die verbleibenden Photonen transparent erscheinen zu lassen. Der resultierende Zustand ist dann nicht mehr rein, weshalb man zur Beschreibung der Dynamik solcher Prozesse auf den Dichtematrixformalismus und die quantenmechanische Master-Gleichung zurückgreifen muss.

In dieser Arbeit befassen wir uns mit der Lichtausbreitung in Rydberg-Gasen. Dazu untersuchen wir zuerst die klassische Dynamik des elektromagnetischen Feldes in inhomogenen Gasen und berechnen die Reflexionskoeffizienten für zwei ausgewählte Funktionen des Brechungsindex in Form der Eckart-Potentiale. Im zweiten Teil der Arbeit betrachten wir die quantenmechanische Master-Gleichung für kontinuierliche Medien und präsentieren eine allgemeine Lösung, die wir auf Fock-Zustände und kohärente Zustände anwenden werden. Zusätzlich berücksichtigen wir den Blockade-Mechanismus stark wechselwirkender Rydberg-Atome, der bei einem saturierenden Medium zu einer deterministischen Absorption eines einzelnen Photons führt.

Contents

1 Basic concepts

In this chapter, we present the basic concepts that are used throughout this thesis. This includes the classical description of electromagnetic fields as well as their quantum mechanical one. In addition, the concept of open quantum systems and quantum master equations is discussed. Since we are concerned with Rydberg atoms, we give a short overview of their properties, too.

1.1 Classical wave equation for inhomogeneous media

In classical electrodynamics, electromagnetic fields in the vacuum are described by the Maxwell equations (in Gaussian units) [11]

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \tag{1.1a}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c}\partial_t \mathbf{E} = 0. \tag{1.1b}$$

However, in order to treat electromagnetic fields in macroscopic media these equations have to be modified due to the response of the medium to the electromagnetic field. The electric field $\mathbf{E}$ is then transformed into the displacement field $\mathbf{D}$ because of the induced polarisation $\mathbf{P}$

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}. \tag{1.2}$$

The polarisation is related to the electric field by the dielectric susceptibility χ

$$\mathbf{P} = \chi\mathbf{E}. \tag{1.3}$$

In general, χ is anisotropic, inhomogeneous and depends on the frequency and the intensity of the electric field. Since we will later restrict our calculations to one-dimensional system, we can assume it to be isotropic. We further assume χ to be independent of the frequency and the intensity of the light field. Hence, eq. (1.2) reduces to

$$\mathbf{D} = (1 + 4\pi\chi(x))\,\mathbf{E} = \varepsilon(x)\mathbf{E} \tag{1.4}$$

where ε denotes the electric permittivity.

The magnetic induction $\mathbf{B}$ is modified by the magnetization $\mathbf{M}$ of the medium and thus yields the magnetic field

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}. \tag{1.5}$$

Since we treat systems whose magnetization is negligible, we assume the magnetic field to be equal to the magnetic induction. Thus, the macroscopic Maxwell equations for a non-conducting medium read

$$\nabla \cdot \mathbf{D} = 0 \tag{1.6a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.6b}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{B} = 0 \tag{1.6c}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\partial_t \mathbf{D} = 0. \tag{1.6d}$$

As for the vacuum [11], using these equations we can derive a wave equation in an inhomogeneous medium. Therefore, we take the curl of eq. (1.6c) and substitute eq. (1.6d). Noting that $\mathbf{D} = \varepsilon(x)\mathbf{E}$ we get

$$\nabla\left(\nabla \cdot \mathbf{E}\right) - \nabla^2 \mathbf{E} = -\frac{\varepsilon(x)}{c^2}\partial_t^2 \mathbf{E}. \tag{1.7}$$

Here we used the relation

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla\left(\nabla \cdot \mathbf{E}\right) - \nabla^2 \mathbf{E}. \tag{1.8}$$

Eq. (1.6a) leads to

$$\nabla \cdot \mathbf{E} = -\frac{1}{\varepsilon(x)}(\nabla \varepsilon(x)) \cdot \mathbf{E}. \tag{1.9}$$

We assume the electric field to be polarized perpendicular to the x-axis, i.e. $\mathbf{E} = E\hat{e}_z$, and therefore $(\nabla\varepsilon(x)) \cdot \mathbf{E} = 0$. The electric permittivity is related to the refractive index by Maxwell's relation

$$n^2(x) = \varepsilon(x) \tag{1.10}$$

and thus leads to the wave equation for one-dimensional inhomogeneous media

$$\partial_x^2 E - \frac{n^2(x)}{c^2}\partial_t^2 E = 0. \tag{1.11}$$

1.2 Quantization of the electromagnetic field

In order to quantize the electromagnetic field we use Maxwell's equation in the vacuum (eqs. (1.1)). The electric and magnetic fields can be derived by the use of a vector potential $\mathbf{A}$ that satisfies the wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\partial_t^2 \mathbf{A} = 0 \tag{1.12}$$

and the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0. \tag{1.13}$$

The electric and magnetic fields can be written as

$$\mathbf{E} = -\frac{1}{c}\partial_t \mathbf{A} \tag{1.14a}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.14b}$$

respectively. Further, we imagine the fields to be confined inside a cubic cavity of side length L of perfect reflecting walls where L is much larger than the wavelengths of the

fields. In the case of continuous mode fields, the limit $L \to \infty$ has to be taken afterwards. Imposing boundary conditions leads to discrete wave vectors

$$\mathbf{k} = \frac{2\pi}{L}(n_x,n_y,n_z) \tag{1.15}$$

where n_x, n_y and n_z are integer numbers. In order to simplify our calculations we chose $\mathbf{A}(\mathbf{r},t) = A(\mathbf{r},t)\hat{e}_z$. The Coulomb gauge (1.13) requires $\mathbf{k}$ to be perpendicular to the vector potential and we chose $\mathbf{k} = k\hat{e}_x$. The vector potential can then be expressed as a superposition of plane waves and takes the form

$$A(x,t) = \sum_k \left[A_k e^{i(kx-\omega_k t)} + A_k^* e^{-i(kx-\omega_k t)}\right]. \tag{1.16}$$

The energy of the electromagnetic field

$$H = \frac{1}{8\pi}\int_V \left(\mathbf{E}^2 + \mathbf{B}^2\right) \tag{1.17}$$

leads to [6]

$$H = \frac{V}{4\pi c^2}\sum_k \left(A_k A_k^* + A_k^* A_k\right). \tag{1.18}$$

Introducing the canonical variables q_k and p_k

$$A_k = \frac{1}{2\omega_k}\sqrt{\frac{4\pi c^2}{V}}\left(\omega_k q_k + i p_k\right) \tag{1.19a}$$

$$A_k^* = \frac{1}{2\omega_k}\sqrt{\frac{4\pi c^2}{V}}\left(\omega_k q_k - i p_k\right) \tag{1.19b}$$

the field energy takes the form of decoupled harmonic oscillators

$$H = \frac{1}{2}\sum_k \left(p_k^2 + \omega_k^2 q_k^2\right). \tag{1.20}$$

The quantization is performed by converting the canonical variables into their corresponding quantum-mechanical canonical operators $\hat{q}_k$ and $\hat{p}_k$ respectively. These operators obey the commutation relation

$$[\hat{q}_k, \hat{p}_{k'}] = i\hbar\delta_{k,k'}. \tag{1.21}$$

In analogy to eq. (1.19), we define the creation an annihilation operators

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k \hat{q}_k + i\hat{p}_k\right) \tag{1.22a}$$

$$\hat{a}_k^\dagger = \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k \hat{q}_k - i\hat{p}_k\right) \tag{1.22b}$$

that satisfy the bosonic commutation relations

$$[\hat{a}_k, \hat{a}_{k'}] = 0 = \left[\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger\right] \tag{1.23a}$$

$$\left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] = \delta_{k,k'}. \tag{1.23b}$$

The Hamiltonian of the quantized electromagnetic field thus reads

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right) = \sum_k \hbar\omega_k \left(\hat{n}_k + \frac{1}{2}\right). \tag{1.24}$$

where $\hat{n}_k$ is the number operator of the mode k. The term 1/2 takes into account an infinitely large vacuum energy that will be neglected in further calculations. Nevertheless, the vacuum energy gives rise to measurable effects like the Casimir force or the Lamb shift.

The quantized form of eq. (1.16) can be obtained by setting

$$\hat{A}_k = \sqrt{\frac{2\pi\hbar}{\omega_k V}}\hat{a}_k \tag{1.25}$$

and then reads

$$\hat{A}(x,t) = \sum_k \sqrt{\frac{2\pi\hbar}{\omega_k V}} \left[\hat{a}_k e^{i(kx-\omega_k t)} + \hat{a}_k^\dagger e^{-i(kx-\omega_k t)}\right]. \tag{1.26}$$

Thus the electric field operator, related to the vector potential by eq. (1.14a), takes the form

$$\hat{E}(x,t) = i\sum_k \sqrt{\frac{2\pi\hbar\omega_k}{V}} \left[\hat{a}_k e^{i(kx-\omega_k t)} + \hat{a}_k^\dagger e^{-i(kx-\omega_k t)}\right]. \tag{1.27}$$

One often splits the electric field operator into one part that contains the positive frequencies and another part containing the negative frequencies. The part of the electric field operator that contains the positive frequencies is given by

$$\hat{E}^+(x,t) = i\sum_k \sqrt{\frac{2\pi\hbar\omega_k}{V}}\hat{a}_k e^{i(kx-\omega_k t)}. \tag{1.28}$$

The negative-frequency part is then

$$\hat{E}^-(x,t) = \left[\hat{E}^+(x,t)\right]^\dagger. \tag{1.29}$$

In the following calculations, we omit the 'hat' since the operator character should be clear by the context.

1.2.1 Continuous-mode fields

In experiments, light beams are often time-dependent and therefore have to be described in terms of two or more modes. For systems studied in cavities it is sufficient to treat discrete modes according to the boundary conditions at the walls of the cavity. Nevertheless, in the absence of such cavities the light field has to be described in terms of continuous modes.

The spacing between two modes inside a cavity of length L is given by

$$\Delta k = \frac{2\pi}{L}. \tag{1.30}$$

The quantization in absence of cavities is best performed by assuming one axis (i.e x-axis) to be of infinite extent but retaining a finite cross-section area A perpendicular to this axis (c.f. [12]). In this case the spacing between to modes tends to zero and the sums over k can be converted into an integral with

$$\sum_k \quad \rightarrow \quad \frac{L}{2\pi}\int \mathrm{d}k. \tag{1.31}$$

The discrete Kronecker delta turns into the continuous Dirac delta-function with

$$\delta_{k,k'} \quad \rightarrow \quad \Delta k \delta\left(k-k'\right) \tag{1.32}$$

We can then relate the continuous-mode creation and annihilation operators, $a(k)$ and $a^\dagger(k)$ respectively, to their discrete counterparts by

$$a_k \rightarrow \sqrt{\Delta k}\, a(k) \qquad a_k^\dagger \rightarrow \sqrt{\Delta k}\, a^\dagger(k). \tag{1.33}$$

They now satisfy the commutation relation

$$\left[a(k), a^\dagger(k')\right] = \delta(k-k'). \tag{1.34}$$

With this results eq. (1.28) can be written as

$$E^+(x,t) = i\int_0^\infty \mathrm{d}k \sqrt{\frac{\hbar c k}{A}} a(k) e^{ik(x-ct)} \tag{1.35}$$

where we have set $\omega_k = ck$. The Hamiltonian of the electromagnetic field in terms of continuous modes then takes the form [12]

$$H = \int_0^\infty \mathrm{d}k\, \hbar c k a^\dagger(k) a(k) + \text{vacuum energy}. \tag{1.36}$$

The vacuum energy is ignored in the further discussions.

For systems where the bandwidth of the field excitation is much smaller than its central frequency $\omega_0 = ck_0$ we can perform the narrow-bandwidth approximation and put ω_0 in front of the integral. Furthermore, the range of integration can be extended over the negative frequencies as well since the will not contribute to the integral appreciably. We define the Fourier-transformed creation and annihilation operators by

$$a(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k a(k) e^{-ikx} \tag{1.37a}$$

$$a^\dagger(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k a^\dagger(k) e^{ikx}. \tag{1.37b}$$

These operators satisfy the commutation relation

$$\left[a(x), a^\dagger(y)\right] = \delta(x-y). \tag{1.38}$$

The electric field operator can then be expressed by

$$E^+(x,t) = i\sqrt{\frac{\hbar c k_0}{A}} a(x-ct). \tag{1.39}$$

Number states

In order to define continuous-mode number states we define the wave-packet creation operator

$$a_\phi^\dagger = \int \mathrm{d}k \phi(k,t) a^\dagger(k) = \int \mathrm{d}x \phi(x,t) a^\dagger(x) \tag{1.40}$$

where ϕ denotes a normalized pulse shape. A single photon state is then defined by

$$|1_\phi\rangle = a_\phi^\dagger |0\rangle. \tag{1.41}$$

General number states are therefore given by

$$|n_\phi\rangle = \frac{1}{\sqrt{n!}} \left(a_\phi^\dagger\right)^n |0\rangle. \tag{1.42}$$

For number states the expectation value of the electric field operator vanishes as in the case of single and discrete-mode fields

$$\langle n_\phi | E(x,t) | n_\phi \rangle = 0. \tag{1.43}$$

Coherent states

In analogy to single mode fields we define the coherent states

$$|\alpha\rangle = \exp\left(a_\phi^\dagger - a_\phi\right) |0\rangle. \tag{1.44}$$

In this case the pulse shape $\alpha(x)$ has not to be normalized but [12]

$$\int \mathrm{d}k |\alpha(k)|^2 = \int \mathrm{d}x |\alpha(x)|^2 = \langle n \rangle \tag{1.45}$$

where $\langle n \rangle$ denotes the mean photon number in this pulse. Using the commutation relation eq. (1.34) we can rewrite eq. (1.44) into

$$|\alpha\rangle = \exp\left(a_\phi^\dagger - \frac{1}{2}\langle n \rangle\right) |0\rangle. \tag{1.46}$$

Since the coherent states are eigenstates of the annihilation operator

$$a(k)|\alpha\rangle = \alpha(k)|\alpha\rangle \qquad a(x)|\alpha\rangle = \alpha(x)|\alpha\rangle \tag{1.47}$$

the expectation value of the electric field operator no longer vanishes but takes the form

$$\begin{aligned}\langle\alpha|E(x,t)|\alpha\rangle &= i\sqrt{\frac{\hbar c k_0}{A}}\int \mathrm{d}k\left(\alpha(k)e^{ik(x-ct)} - \alpha^*(k)e^{-ik(x-ct)}\right)\\ &= -2\int \mathrm{d}k\,\mathrm{Im}\left\{\alpha(k)e^{ik(x-ct)}\right\}\\ &= -2\,\mathrm{Im}\left\{\alpha(x-ct)\right\}. \end{aligned} \tag{1.48}$$

This is the same result we would obtain for the classical value of the electric field.

1.3 Open quantum systems and quantum master equation

The dynamics of closed and fully coherent quantum systems is governed by the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1.49}$$

where H is the Hamiltonian of this system. In open quantum systems where uncontrolled interactions with external degrees of freedom have to be taken into account the system cannot be described in terms of pure states any longer. We rather have to deal with a statistical mixture of states that can be described by the density operator

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \tag{1.50}$$

where p_i is the probability to find the system in the pure state $|\psi_i\rangle$.

Including the external degrees of freedom the evolution of the system is fully coherent and can be described by the von-Neumann equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\frac{i}{\hbar}\left[H,\rho\right]. \tag{1.51}$$

In this case, H is a composition of the Hamiltonian of the system H_S, the environment H_E and the interaction H_I between both of them (see Figure 1.1). Since we are not interested in the description of the evolution of the environment E, we have to consider the reduced density operator of the system S only

$$\rho_S = \mathrm{Tr_E}\left\{\rho\right\}. \tag{1.52}$$

In order to derive a quantum master equation we rewrite eq. (1.51) in the interaction picture

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\frac{i}{\hbar}\left[H_I(t),\rho(t)\right]. \tag{1.53}$$

Taking the trace over the environment this equation can be transformed into [2]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}s\,\mathrm{Tr_E}\left[H_I(t),\left[H_I(s),\rho(s)\right]\right]. \tag{1.54}$$

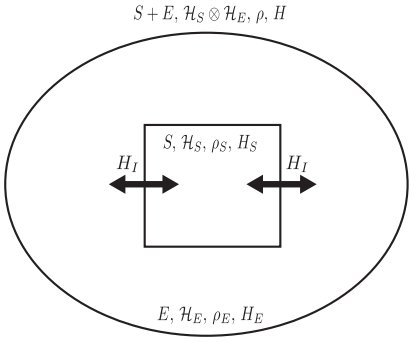

Figure 1.1: Schematic picture of an open quantum system (cf. [2]). The dynamics of the system S is given by the Hamiltonian H_S. The Hamiltonian H_E describes the evolution of the environment E. The interaction between those two subsystems is given by the interaction Hamiltonian H_I. The dynamics of the whole system $S+E$ is then governed by the Hamiltonian $H = H_S + H_E + H_I$.

This equation can be simplified by performing the Born-approximation and the Markov-approximation. The first one assumes the coupling between the environment and the system to be weak such that the environment is approximately unaffected by the system. Consequently, the density operator may be written as a tensor product of the density operators of the subsystems

$$\rho(t) \approx \rho_S(t) \otimes \rho_E. \tag{1.55}$$

The Markov-approximation is based on a coarse-grained time scale and the dynamics on a time scale comparable to the time scale over which the excitations of the environment decays is not resolved. Put simply, we assume the environment to have no memory in such a way that the change of the quantum system at a time t is only influenced by the state of the quantum system at a time t. Considering these approximations, eq. 1.54 can be written in the so-called Lindblad-form [2]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = \sum_k \Gamma_k \left(a_k \rho_S(t) a_k^\dagger - \frac{1}{2}\left\{ a_k^\dagger a_k, \rho_S(t) \right\} \right). \tag{1.56}$$

In the Schrödinger picture this reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\frac{i}{\hbar}\left[H_S, \rho_S(t)\right] + \sum_k \Gamma_k \left(A_k \rho_S(t) A_k^\dagger - \frac{1}{2}\left\{ A_k^\dagger A_k, \rho_S(t) \right\} \right). \tag{1.57}$$

The first term on the right side containing the Hamiltonian H_S describes the coherent evolution of the system S. The operators A_k and $A_k^\dagger$ are eigenoperators of H_S and induce non-coherent transitions from one state to another with the rate Γ_k due to the coupling to the environment.

1.4 Rydberg atoms

Rydberg atoms are atoms with one or more electrons excited into a state with a principal quantum number $n \gg 1$ [5]. Assuming only one electron to be excited into a higher state

we can treat the atom to behave approximately hydrogenicly since the positive charge of the core is shielded by the remaining electrons. In fact, the energy levels of a Rydberg atom can be calculated using the hydrogen formula by replacing the principle quantum number n by an effective principle quantum number n^*

$$E(n^*) = -\frac{2\pi\hbar c R}{(n^*)^2} \tag{1.58}$$

where R is the Rydberg constant and $n^* = n - \delta_l$ with the so-called quantum defect δ_l. As for the hydrogen atom the scaling of most of the important properties of Rydberg atoms are determined by the effective principle quantum number n^*. Table 1.4.1 shows some scaling laws for selected properties of Rydberg atoms.

Property	Scaling	^{87}Rb $\lvert 43s\rangle$
Binding energy	$(n^*)^{-2}$	8.56 meV
Level spacing	$(n^*)^{-3}$	109.99 GHz
Orbit-radius	$(n^*)^2$	2384.2 a_0
Polarizability	$(n^*)^7$	8.06 MHz/(V/cm)^2
Natural lifetime	$(n^*)^3$	99 μ s
Transition dipole moment	$(n^*)^{-3/2}$	-0.0176 a.u.
van der Waals C_6 coefficient	$(n^*)^{11}$	$-1.7 \cdot 10^{19}$ a.u.

Table 1.4.1: Scaling laws of selected properties of Rydberg atoms and exemplary values for the ^{87}Rb $\lvert 43s\rangle$ state [8]

1.4.1 Blockade Mechanism

The blockade mechanism of Rydberg atoms is based on the strong interaction between them. For a general two level system the interaction Hamiltonian reads

$$H_I = \begin{pmatrix} \Delta & V \\ V & 0 \end{pmatrix} \tag{1.59}$$

where Δ is the energy difference between the two states and V denotes the interaction. The eigenvalues of H_I are given by

$$E_\pm = \frac{\Delta}{2} \pm \frac{1}{2}\sqrt{\Delta^2 + 4V^2}. \tag{1.60}$$

For Rydberg atoms the dominating interaction is the dipole-dipole interaction that is

$$V = \frac{C_3}{r^3} \tag{1.61}$$

where angular dependencies have been dropped. For large interatomic distances the interaction is much smaller than the energy difference Δ and the negative eigenvalue is given in leading order by

$$V_I = E_- = -\frac{C_3^2}{\Delta r^6} \sim \frac{C_6}{r^6}. \tag{1.62}$$

This has the form of a van der Waals interaction. Since the van der Waals coefficient C_6 is proportional to $(n^*)^{11}$ this interaction is very strong for Rydberg atoms.

Strong interactions between excited atoms cause a blockade mechanism that allows one excitation only. The blockade-condition is given by

$$V_I > W \tag{1.63}$$

where W denotes the linewidth of the excitation laser. In order to discuss the blockade mechanism based on the strong interaction between Rydberg atoms we consider two atoms coupled resonantly to Rydberg states by the Rabi frequency Ω (see Figure 1.2). The linewidth is governed by power-broadening depending on the Rabi frequency and therefore

$$\frac{C_6}{r^6} > 2\hbar\Omega. \tag{1.64}$$

The blockade radius then is defined by

$$r_B = \left(\frac{C_6}{2\hbar\Omega}\right)^{\frac{1}{6}}. \tag{1.65}$$

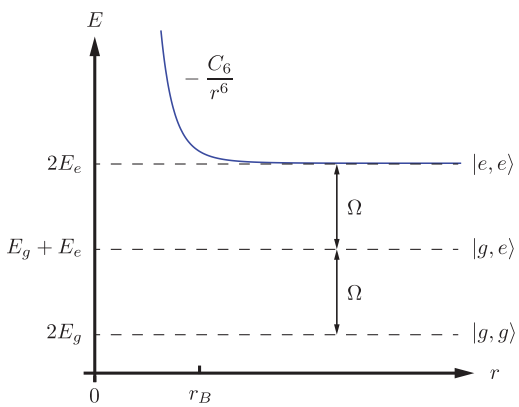

Figure 1.2: Due to the strong interaction between two excited states the doubly excited state is shifted out of resonance for spatial separations smaller than the blockade-radius r_B. Thus, a system confined in a region smaller than the blockade-volume $\sim r_B^3$ can only carry a single excitation. Cf. [9]

1.4.2 Collective States

Assuming that the ensemble of N Rydberg atoms is confined inside a volume smaller that the blockade-volume $\sim r_B^3$, the system cannot carry but a single excitation. In this case the Hilbert space of the system is spanned by the collective ground state

$$|G\rangle = |g_1, \ldots, g_N\rangle \tag{1.66}$$

and N excited states

$$|i\rangle = |g_1, \ldots, e_i, \ldots, g_N\rangle. \tag{1.67}$$

Since the system is invariant under permutation of atoms the light field couples to the collective excited state

$$|W\rangle = \frac{1}{\sqrt{N}} \sum_i |i\rangle. \tag{1.68}$$

In this case we can reduce the dynamics of the system of N atoms to a two-level system which gives rise to the term 'superatom'. The coupling of the collective ground state to the collective excited state is then given by the collective Rabi frequency

$$\Omega_{\text{coll}} = \sqrt{N}\Omega. \tag{1.69}$$

2 Classical wave propagation through inhomogeneous media

In this section we present a solution for the classical wave equation in inhomogeneous media. Further we derive the reflection coefficients for the anti-symmetric and symmetric Eckart potentials. In the last part we perform the slowly varying envelope approximation in order to reduce the wave equation to a first order partial differential equation.

2.1 Stationary solutions and reflection coefficients

The wave equation in inhomogeneous media with electric susceptibility $\chi(x)$ reads

$$\left(\partial_x^2 - \frac{n^2(x)}{c^2}\partial_t^2\right) E(x,t) = \left(\partial_x^2 - \frac{1+\chi(x)}{c^2}\partial_t^2\right) E(x,t). \tag{2.1}$$

We now make the separation ansatz $E(x,t) = E(x)e^{-ickt}$. In turn the wave equation reduces to a second order differential equation

$$\left(\partial_x^2 + k^2(1+\chi(x))\right) E(x) = 0. \tag{2.2}$$

Next, we assume particular shapes for the electric susceptibility $\chi(x)$ in order to find analytical solutions. These will have the forms of anti-symmetric and symmetric Eckart potentials, $\chi_a(x)$ and $\chi_s(x)$ [3] (see Figure 2.1), namely

$$\chi_a(x) = \chi_0 \frac{\exp\left(\frac{x}{x_0}\right)}{1+\exp\left(\frac{x}{x_0}\right)} \qquad \chi_s(x) = \chi_0 \frac{\exp\left(\frac{x}{x_0}\right)}{\left(1+\exp\left(\frac{x}{x_0}\right)\right)^2} \tag{2.3}$$

where χ_0 is related the hight of the potential and x_0 is related to the width over which the potential changes.

2.1.1 Anti-symmetric Eckart potential

We first concentrate on the anti-symmetric Eckart potential $\chi_a(x)$. With the substitution $\exp\left(\frac{x}{x_0}\right) = y$ we can rewrite eq. (2.2) into

$$\left(y^2\partial_y^2 + y\partial_y + (x_0k)^2\left(1+\chi_0\frac{y}{1+y}\right)\right) E(y) = 0. \tag{2.4}$$

Through $q = x_0k$ and $\sqrt{1+\chi_0} = n$ the solution can be expressed in terms of hypergeometric functions ${}_2F_1$

$$\begin{aligned} E(y) = {} & c_1 y^{iq}{}_2F_1\left(iq(1-n), ik(1+n), 1+2iq, -y\right) \\ & c_2 y^{-iq}{}_2F_1\left(-iq(1+n), -iq(1-n), 1-2iq, -y\right). \end{aligned} \tag{2.5}$$

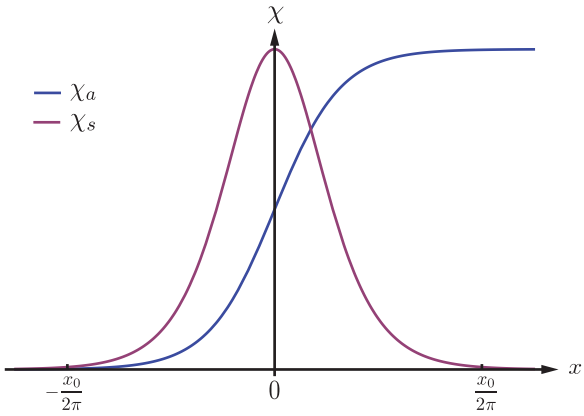

Figure 2.1: Plot of the anti-symmetric (purple line) and symmetric (blue line) Eckart potential 2.3 over the spatial coordinate x. The value of χ_0 of the symmetric Eckart potential in this plot is four times larger than the value of χ_0 of the anti-symmetric Eckart potential.

In order to calculate the reflection coefficient for the anti-symmetric potential, we expand the solution in the limit $y \to 0$ ($x \to -\infty$) and in the limit $y \to \infty$ ($x \to \infty$). In the first case, the hypergeometric functions reduce to 1 and the solution becomes

$$E(x \to -\infty) = c_1 e^{-ikx} + c_2 e^{ikx}. \tag{2.6}$$

The first term describes a reflected wave, which is due to the scattering at the medium, whereas the second term represents the incident wave. The expansion of eq. (2.5) for $x \to \infty$ then yields

$$\begin{aligned} E(x \to \infty) = e^{iknx} &\left(c_2 \frac{\Gamma(1-2iq)\Gamma(2iqn)}{\Gamma(iq(n-1))\Gamma(1+iq(n-1))} + c_1 \frac{\Gamma(1+2iq)\Gamma(2iqn)}{\Gamma(iq(n+1))\Gamma(1+iq(n+1))} \right) \\ + e^{-iknx} &\left(c_2 \frac{\Gamma(1-2iq)\Gamma(-2iq)}{\Gamma(-iq(n+1))\Gamma(1-iq(n+1))} + c_1 \frac{\Gamma(1+2iq)\Gamma(2iqn)}{\Gamma(-iq(n-1))\Gamma(1-iq(n-1))} \right) \end{aligned} \tag{2.7}$$

where $\Gamma(\ldots)$ denotes the Gamma function. The first term yields a wave propagating to the left with a modified wave vector $\bar{k} = kn$ which is due to the refractive medium. The second term gives a counterpropagating wave with the same modified wave vector. This term actually represents a wave scattered at $x = \infty$ and therefore its amplitude should vanish, which leads to the condition

$$c_2 \frac{\Gamma(1-2iq)\Gamma(-2iq)}{\Gamma(-iq(n+1))\Gamma(1-iq(n+1))} + c_1 \frac{\Gamma(1+2iq)\Gamma(2iqn)}{\Gamma(-iq(n-1))\Gamma(1-iq(n-1))} = 0. \tag{2.8}$$

The amplitude reflection coefficient $r(q,n)$ which is defined as the ratio of outgoing and incoming wave amplitude takes the form

$$r(q,n) = \frac{c_2}{c_1} = -\frac{\Gamma(1+2iq)\Gamma(2iqn)\Gamma(-iq(n+1))\Gamma(1-iq(n+1))}{\Gamma(-iq(n-1))\Gamma(1-iq(n-1))\Gamma(1-2iq)\Gamma(-2iqn)}. \tag{2.9}$$

The intensity reflection coefficient $R(q,n)$ is defined as the absolute square of eq. (2.9), which is valid for complex values of n as well. However, for real n this expression can be brought into a more elegant form by using the following identities

$$|\Gamma(ix)|^2 = \frac{\pi}{x\sinh(\pi x)} \tag{2.10a}$$

$$|\Gamma(1+ix)|^2 = \frac{\pi x}{\sinh(\pi x)}. \tag{2.10b}$$

Thus, the intensity reflection coefficient reads

$$R(q,n) = |r(q,n)|^2 = \frac{\sinh^2(\pi q(1-n))}{\sinh^2(\pi q(1+n))} \tag{2.11}$$

In the limit $x_0 \to 0$ ($q \to 0$), which represents a hard step, the intensity reflection coefficient remains finite and expanding eq. (2.11) for small x_0 up to leading order yields

$$R(0,n) = \left(\frac{n-1}{n+1}\right)^2. \tag{2.12}$$

This is exactly the same result that can be derived by imposing boundary conditions for the electric and magnetic fields at dielectric interfaces [11].

Figure 2.2 shows the intensity reflection coefficient $R(q,n)$, eq. (2.11), plotted over the dimensionless coordinate q for several values of n. It can be seen that for $q \to 0$ R remains finite and its value increases with increasing n. For large q, i.e. for a large ratio x_0/λ, the reflected wave vanishes. This can be interpreted physically by identifying k with the energy of the photon. For large q, this means large energies, the photon can penetrate the barrier easily whereas for small energies, it will be scattered. A more classical interpretation would be that for small wavelengths compared to the length over which the potential varies the photon does not notice the change of the barrier and therefore will not be scattered.

2.1.2 Symmetric Eckart potential

In this section, we concentrate on the symmetric Eckart potential $\chi_s(x)$. We follow basically the calculations of [3]. The wave equation for the symmetric Eckart potential is

$$\left[\partial_x^2 + k^2\left(1 + \chi_0 \frac{e^{x/x_0}}{\left(1+e^{x/x_0}\right)^2}\right)\right] E(x) = 0. \tag{2.13}$$

We now substitute $y = -e^{x/x_0}$ and with the notation $q^2 = k^2 x_0^2$ the wave equation becomes

$$\left[y^2\partial_y^2 + y\partial_y + q^2\left(1 - \chi_0 \frac{y}{(1-y)^2}\right)\right] E(y) = 0. \tag{2.14}$$

This hypergeometric differential equation can be solved in terms of hypergeometric functions just like above. Nevertheless, we impose the conditions that for $x \to \infty$ there should be only a transmitted wave $\sim e^{ikx} = (-y)^{iq}$. For $x \to -\infty$, we expect the solution to split

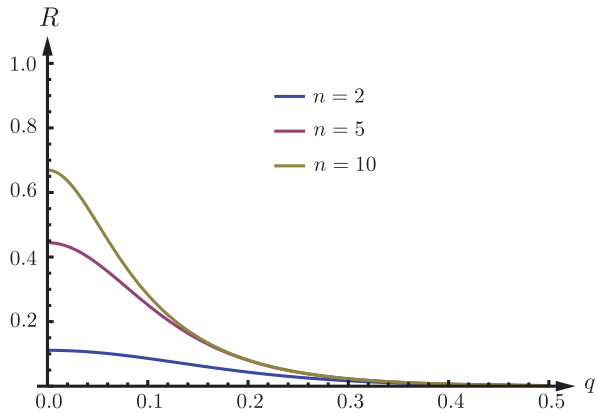

Figure 2.2: Intensity reflection coefficient R plotted over $q = kx_0$ for several values of n. R vanishes for large q and has a maximum for $q \to 0$ whose value increases for increasing n.

into two waves, the incident wave proportional to e^{ikx} and the reflected wave proportional to e^{-ikx}. It is shown [3] that the solution takes the form

$$E(y) = (-y)^{iq} {}_2F_1\left(\frac{1}{2} - \delta, -\frac{1}{2} + \delta, 1 - 2iq, \frac{1}{1-y}\right), \tag{2.15}$$

where $q = kx_0$ and $\delta = \frac{1}{2}\sqrt{1 + 4q^2\chi_0}$. For $x \to \infty$ ($-y \to \infty$) this expression converges to $(-y)^{iq}$ and therefore represents the transmitted wave. In the limit $x \to -\infty$ ($y \to 0$) the hypergeometric function used in eq. (2.15) does not converge. Thus the expression has to be analytically extended which is done by compounding two hypergeometric functions [3]. The analytically extended expression of eq. (2.15) then becomes

$$\begin{aligned} E(y) =& c_1(-y)^{iq} {}_2F_1\left(\frac{1}{2} - \delta, -\frac{1}{2} + \delta, 1 + 2iq, \frac{y}{y-1}\right) \\ &+ c_2\,(-y)^{-iq}\,{}_2F_1\left(\frac{1}{2} - \delta, -\frac{1}{2} + \delta, 1 - 2iq, \frac{y}{y-1}\right) \end{aligned} \tag{2.16}$$

with

$$c_1 = \frac{\Gamma(1-2iq)\Gamma(-2iq)}{\Gamma\left(\frac{1}{2} - 2iq + \delta\right)\Gamma\left(\frac{1}{2} - 2iq - \delta\right)}, \tag{2.17a}$$

$$c_2 = \frac{\Gamma(1-2iq)\Gamma(2iq)}{\Gamma\left(\frac{1}{2} + \delta\right)\Gamma\left(\frac{1}{2} - \delta\right)}. \tag{2.17b}$$

This extended form converges for $y \to 0$ ($x \to -\infty$) and yields the expression

$$E(y \to 0) = c_1(-y)^{iq} + c_2(-y)^{-iq}, \tag{2.18}$$

which satisfies the condition of an incident and a reflected wave. The intensity reflection coefficient $R(q,\chi_0)$ is thus the absolute square of the ratio c_2/c_1

$$R(q,\chi_0) = \left|\frac{c_2}{c_1}\right|^2 = \left|\frac{\Gamma\left(\frac{1}{2} - 2iq + \delta\right)\Gamma\left(\frac{1}{2} - 2iq - \delta\right)}{\Gamma\left(\frac{1}{2} + \delta\right)\Gamma\left(\frac{1}{2} - \delta\right)}\right|^2. \tag{2.19}$$

This expression holds for general χ_0 and therefore is valid for complex values of χ_0 in analogy to eq. (2.9). Nevertheless, for real values of χ_0, we can simplify this expression by making use of the following identities

$$\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)=\frac{\pi}{\cos(\pi s)}, \qquad \overline{\Gamma(z)}=\Gamma(\bar{z}), \tag{2.20a}$$

$$\cos(x-y)\cos(x+y)=\cos^2(x)-\sin^2(y), \qquad \sin(ix)=i\sinh(x). \tag{2.20b}$$

A straightforward calculation leads to

$$R(q,\chi_0)=\frac{\cos^2\left(\frac{\pi}{2}\sqrt{1+4q^2\chi_0}\right)}{\cos^2\left(\frac{\pi}{2}\sqrt{1+4q^2\chi_0}\right)+\sinh^2\left(\pi 2q\right)}. \tag{2.21}$$

Figures 2.3 and 2.4 show the intensity reflection coefficient $R(q,\chi_0)$, eq. (2.19), of the symmetric Eckart potential for several values of χ_0. It can be seen that the reflection coefficient vanishes for large q as in the case of the anti-symmetric potential. In contrast, for $q \to 0$ it tends to zero. This behaviour is obvious, since for non-vanishing k, $q \to 0$ means that there is no barrier and therefore no reflection. A very interesting observation can be made for large values of χ_0 (see Figure 2.3). There are values of q for which R is zero due to destructive interference of the scattered and incident wave. This arises whenever $\sqrt{1+4q^2\chi_0}=m$ with an integer number m. However, this is only true for real values of χ_0. For complex values of χ_0 these minima are not zero any more and vanish for higher values of the imaginary part of χ_0 (see Figure 2.4).

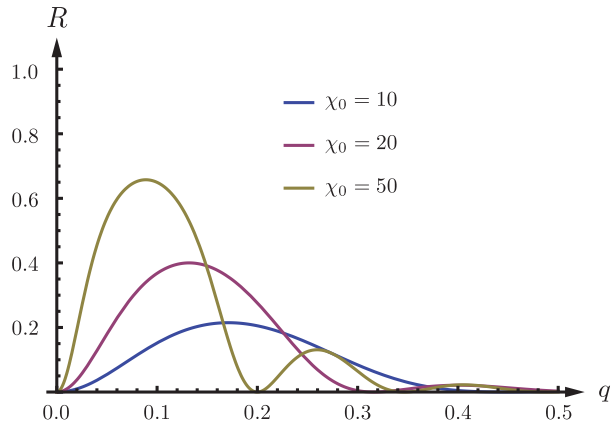

Figure 2.3: Intensity reflection coefficient for the symmetric Eckart potential plotted over q for real values of χ_0. R vanishes for large values of q. There are values of q for which R is equal to zero due to destructive interference.

2.2 Slowly varying envelope approximation

The previous discussions concentrated on plane waves of the form $e^{i(kx-\omega t)}$. However, in reality such plane waves never occur and are better described in terms of wave packets.

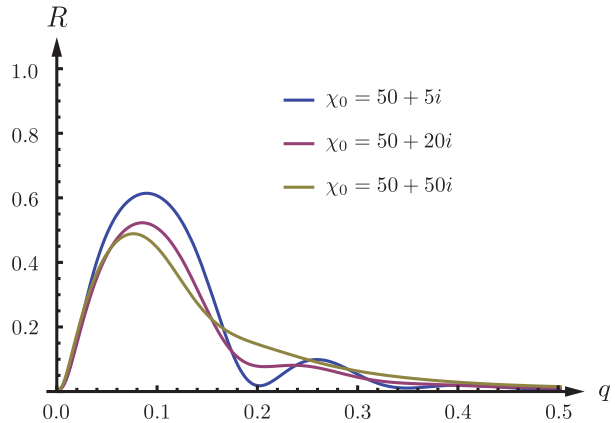

Figure 2.4: Intensity reflection coefficient of the symmetric Eckart potential plotted over q for complex values of χ_0. The minima of R are not zero any more and vanish for large values of the imaginary part of χ_0.

They are constructed by superposition of plane waves

$$\psi(x,t) = \int \mathrm{d}k e^{i(kx-\omega(k)t)}\phi(k), \tag{2.22}$$

where $\phi(k)$ is the distribution of wave vectors in k-space and $\omega(k) = ck$.

We now assume the bandwidth of $\phi(k)$ to be very small compared to the value k_0 around which ϕ is centered. This is a very reasonable assumption since in the optical regime, the bandwidth of the laser exceeds the frequency by many magnitudes ($\frac{\omega_0}{\Delta\omega} \approx 10^{10}$). We further assume that in configuration space the pulse has the form

$$\psi(x,t) = e^{ik_0(x-ct)}\phi(x,t). \tag{2.23}$$

Here we neglect the wave packet travelling in the opposite direction. Since the wave vectors are very sharply centered around k_0 for lasers for example we assume that $\phi(x,t)$ only varies slowly with x and t. This is called the slowly varying envelope approximation and we can drop the higher order derivatives

$$\left|\partial_x^2\phi\right| \ll |k_0\partial_x\phi|\,, \qquad \left|\partial_t^2\phi\right| \ll |\omega_0\partial_t\phi|\,. \tag{2.24}$$

Inserting the ansatz (2.23) into the wave equation and performing the slowly varying envelope approximation (2.24) we get

$$\partial_x\phi(x,t) + \frac{1}{c}\left(1+\chi(x)\right)\partial_t\phi(x,t) + \frac{k_0}{2}\chi(x)\phi(x,t) = 0. \tag{2.25}$$

This equation can be solved analytically assuming that $\chi(x \to -\infty) = 0$

$$\phi(x,t) = \exp\left(-\frac{k}{2}\int_{-\infty}^{x} \mathrm{d}x'\chi(x')\right) f\left(\int_{-\infty}^{x} \mathrm{d}x'(1+\chi(x')) - ct\right), \tag{2.26}$$

where we claim $f(\cdots)$ to be a normalized function for every t. The exponential factor gives rise to a decaying amplitude. The integral in the argument of f can be understood as a reduction of the group velocity. In the absence of a refractive medium the wave packet is centered around $x = ct$. In an inhomogeneous medium, the group velocity is not constant any longer but varies locally $c(x) = c/n(x)$. If $\chi(x)$ is very small compared to 1 we can neglect this effect and the solution eq. (2.26) just describes the decay of the amplitude.

Anti-symmetric Eckart potential

We now want to apply the results derived above and consider the Eckart potentials (2.3) that satisfy the condition $\chi(x \to -\infty) = 0$. Both potentials can be integrated analytically and for the anti-symmetric potential eq. (2.26) leads to

$$\begin{aligned}\phi(x,t) &= \exp\left\{-\frac{k}{2}\chi_0 x_0 \log\left(1+e^{x/x_0}\right)\right\} f\left(x+\chi_0 x_0 \log\left(1+e^{x/x_0}\right)-ct\right)\\ &= \left(1+e^{x/x_0}\right)^{-\frac{k\chi_0 x_0}{2}} f\left(x+\chi_0 x_0 \log\left(1+e^{x/x_0}\right)-ct\right). \end{aligned} \tag{2.27}$$

It can be easily seen that the wave packet will vanish for large x because the decay factor $\left(1+e^{x/x_0}\right)^{-\frac{k\chi_0 x_0}{2}}$ will tend to zero (see Figure 2.5).

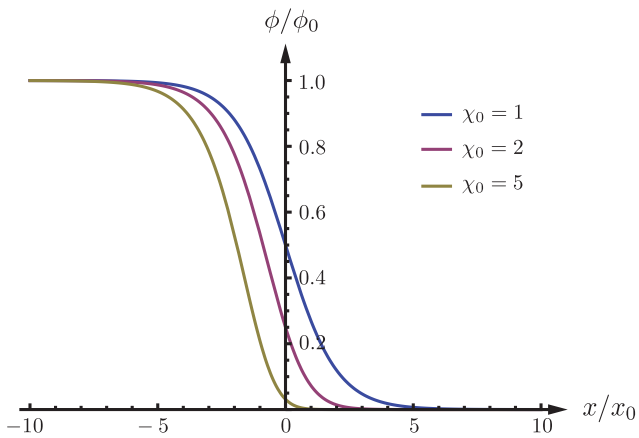

Figure 2.5: Ratio of the amplitude of the incident wave (ϕ_0) and the amplitude of $\phi(x)$ for several values of χ_0 ($n = \sqrt{1+\chi_0}$). Due to the decay factor the amplitude will vanish completely for sufficient large x. We chose $k = 1$.

Symmetric Eckart Potential

In the case of the symmetric Eckart potential we rewrite $\chi(x)$

$$\chi_s(x) = \chi_0 \frac{\exp\left(\frac{x}{x_0}\right)}{\left(1+\exp\left(\frac{x}{x_0}\right)\right)^2} = \frac{\chi_0}{4}\frac{1}{\cosh^2\left(\frac{x}{2x_0}\right)}. \tag{2.28}$$

Then the solution reads

$$\phi(x,t) = \exp\left\{-\frac{k\chi_0 x_0}{4}\left(\tanh\left(\frac{x}{2x_0}\right)+1\right)\right\} f\left(x+\frac{1}{2}\chi_0 x_0\left(\tanh\left(\frac{x}{2x_0}\right)+1\right)-ct\right). \tag{2.29}$$

A graphical illustration of this solution is shown in Figure 2.6. For $x/x_0 \ll 0$ the shape of the decay factor is similar to the anti-symmetric case whereas for large x it has a non-vanishing value due to the localized refractive medium. For large values of χ_0 the wave will vanish completely.

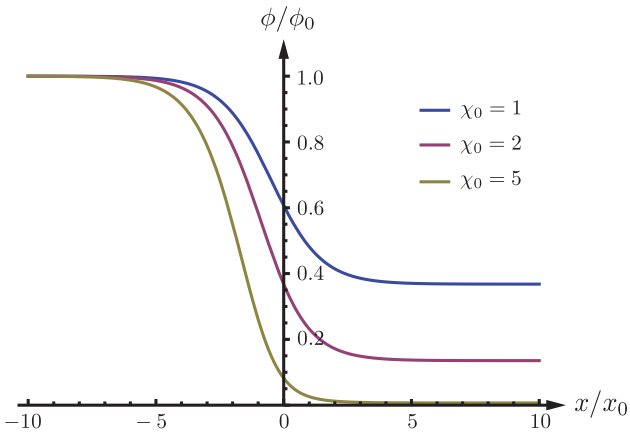

Figure 2.6: Ratio of the amplitude of the incident wave (ϕ_0) and the amplitude $\phi(x)$ plotted over x/x_0 for several parameters of χ_0. The amplitude does not vanish in general for $x \to \infty$ but remains constant after having passed through the medium and only vanishes completely in the case of $\chi_0 \to \infty$. We chose $k = 1$.

We want to keep this result in mind because we will come across it in the next chapter.

2.A Hypergeometric Series

The Gauss hypergeometric series ${}_2F_1(a,b;c;z)$ is defined as the infinite series

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{1\cdot 2c(c+1)}z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3c(c+1)(c+2)}z^3 + \cdots \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!} \end{aligned} \tag{2.30}$$

which converges only for $|z| < 1$ and for $\Re(c-a-b) > -1$ [1]. It is not defined if c is a negative integer. It is obvious that ${}_2F_1(a,b;c;0) = 1$. In order to extend eq.(2.30) for $|z| \geq 1$ there are several useful linear transformation formula but we only concentrate on the ones we used above. Since we are concerned with the case where $z = \pm\infty$ the following

transformation formula proves useful

$$
\begin{aligned}
{}_2F_1(a,b;c;z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1\left(a,1-c+a;1-b+a;\frac{1}{z}\right) \\
& \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1\left(b,1-c+b;1-a+b;\frac{1}{z}\right). \qquad (2.31)
\end{aligned}
$$

Another useful relation is given in [3]

$$
\begin{aligned}
& y^a(1-y)^c{}_2F_1(a+b+c,a+b'+c;1+a-a';y) \\
& \quad = \frac{\Gamma(1+a-a')\Gamma(c'-c)}{\Gamma(1-a'-b-c)\Gamma(-a'b'-c)}y^a(1-y)^c{}_2F_1(a+b+c,a+b'+c;1+c-c';1-y) \\
& \qquad + \frac{\Gamma(1+a-a')\Gamma(c-c')}{\Gamma(1-a'-b-c')\Gamma(-a'-b'-c')}y^a(1-y)^{c'}{}_2F_1(a+b+c',a+b'+c';1+c'-c;1-y). \\
& \qquad (2.32)
\end{aligned}
$$

3 Solution of the quantum master equation for continuous mode fields

In this chapter we will present a solution of the quantum optical master equation. In order to see how the quantum master equation looks like for continuous inhomogeneous media we will give a short derivation of it first. In the second section we will solve the master equation for a density matrix built up from a single-photon state as well as the vacuum state. Hence we will receive a differential equation for the pulse shape similar to eq.(2.25). This solution will then be generalized to arbitrary states and in the last section we will take account of the blockade radius by implementing a mechanism that only allows a single excitation.

3.1 Derivation of the quantum master equation for a continuous medium

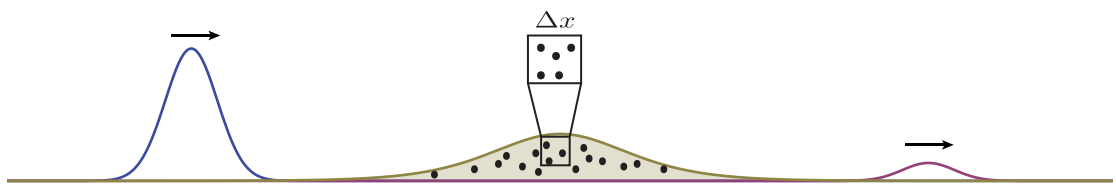

Figure 3.1: A single photon is quasi-localized in an interval I inside the medium (shaded). Inside this interval there are several atoms that interact with each other and with the photon. The number of atoms inside this interval has to be sufficiently large in order to behave like a bath. Due to the interaction with the medium the amplitude of the wave packet will decrease.

We first consider a reservoir of two-level systems (atoms). Then the interaction between photons and the reservoir can be described by the interaction Hamiltonian

$$H_I = g \sum_i \left(a_i \sigma_+^i + a_i^\dagger \sigma_-^i \right) \tag{3.1}$$

where $\sigma_-^i = (|g\rangle\langle e|)_i$ and $(\sigma_-^i)^\dagger = \sigma_+^i$ represent a two-level system and i denotes the specific location of photons and atoms. The coupling strength g is assumed to be equal for every atom. We further assume a coarse-grained structure of length with a characteristic lengthscale Δx so we do not want to distinguish between photons absorbed at a place i or j as long as $|i - j| \leq \Delta x$. Hence, a photon quasi-localized in an interval I interacts with

the atoms in an interval of length Δx and we can rewrite eq. (3.1) into

$$H_I = g \sum_I N_I \left(a_I \sum_{J \in \Delta x} \sigma_+^J + a_I^\dagger \sum_{J \in \Delta x} \sigma_-^J \right). \tag{3.2}$$

where N_I denotes the number of atoms in the interval I. Since the atoms represent a bath we have to keep in mind that the number of atoms in an interval of length Δx has to be sufficiently large. The property of representing a bath is connected to the fact that there are no 'memory' effects in the bath. As this behaviour relies on the interaction between the atoms there is an 'effective range' $\Delta x'$ over which the summation over J has to take place (see Figure 3.2).

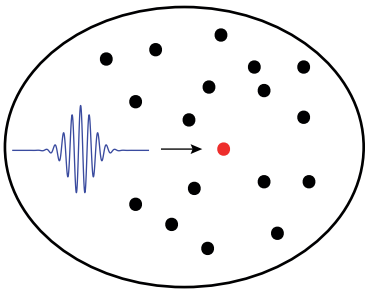

Figure 3.2: The photon is absorbed by an atom (red dot). This atom interacts with all atoms inside an effective range $\Delta x'$ (ellipse)

In the limit $\Delta x \to 0$ this range can be larger than Δx itself. Therefore eq. (3.2) takes the form

$$H_{\text{Int}} = \hbar \sum_I N_I \left(a_I \kappa^\dagger + a_I^\dagger \kappa \right) \tag{3.3}$$

with

$$\kappa = g \sum_{J \in \Delta x'} \sigma_-^J \tag{3.4}$$

The derivation of the quantum master equation for this type of interaction is well understood and can be found in several textbooks about quantum optics, i.e. [14]. Since we are interested in the limit of a continuous field we have to take the limit $a_I \to a(x)$ and hence rewrite eq. (3.3) into

$$H_{\text{Int}} = \hbar \sum_I \Delta x \frac{N_I}{\Delta x} \left(\frac{a_I}{\sqrt{\Delta x}} \sqrt{\Delta x} \kappa^\dagger + \frac{a_I^\dagger}{\sqrt{\Delta x}} \sqrt{\Delta x} \kappa \right). \tag{3.5}$$

In the limit $\Delta x \to 0$ the sum turns into an integral and the photon creation and annihilation operator are converted into their corresponding continuous operators. The atomic reservoir and its coupling strength g give rise to an absorption rate $\gamma = g^2 \rho^2(\omega_0)$, where $\rho(\omega_0)$ is the density of states for the carrier frequency ω_0 of the light field. Due to the continuous limit this absorption rate is modified by the factor Δx into $\Gamma = \gamma \Delta x$. This can be interpreted as a differential absorption rate that now is given over an infinitesimal interval

Δx and as a result it gets an additional dimension of length. Following the calculations of [14] then leads to the master equation in the interaction picture

$$\partial_t \rho = \Gamma \int \mathrm{d}x\, n(x) \left(a(x)\rho a^\dagger(x) - \frac{1}{2}\left\{ a^\dagger(x)a(x), \rho \right\} \right). \tag{3.6}$$

The master equation in the Schrödinger picture is achieved by simply including $-\frac{i}{\hbar}[H,\rho]$ and thus yields

$$\partial_t \rho = -\frac{i}{\hbar}[H,\rho] + \Gamma \int \mathrm{d}x\, n(x) \left(a(x)\rho a^\dagger(x) - \frac{1}{2}\left\{ a^\dagger(x)a(x), \rho \right\} \right). \tag{3.7}$$

3.2 Solving for a single photon state

After having derived the quantum master equation for a continuous medium in the previous section, we now want to solve it for a density matrix whose basis consist of a single-photon state and the vacuum state.

We consider a pulse that is built as a superposition of single-photon continuous modes $\{|k\rangle\}$ and the vacuum state.

$$|\psi(t)\rangle = \int \mathrm{d}k \phi(k,t)|k\rangle + c_0(t)|0\rangle = |1\rangle + c_0(t)|0\rangle \tag{3.8}$$

The coefficients $\phi(k,t)$ and $c_0(t)$ have to satisfy the initial values

$$\int \mathrm{d}k |\phi(k,t \to -\infty)|^2 = 1 \qquad c_0(t \to -\infty) = 0 \tag{3.9}$$

as the influence of the density distribution vanishes for $x \to \pm\infty$.

Therefore, the density operator for this system reads

$$\begin{aligned}
\rho(t) &= |\psi(t)\rangle\langle\psi(t)| \\
&= \int \mathrm{d}k \int \mathrm{d}k' \phi(k,t)\phi^*(k',t)|k\rangle\langle k'| + c_0^*(t) \int \mathrm{d}k \phi(k,t)|k\rangle\langle 0| \\
&\quad + c_0(t) \int \mathrm{d}k \phi^*(k,t)|0\rangle\langle k| + |c_0(t)|^2 |0\rangle\langle 0| \\
&= |1\rangle\langle 1| + |c_0(t)|^2 |0\rangle\langle 0|
\end{aligned} \tag{3.10}$$

In order to receive a set of differential equations for the coefficients $\phi(k,t)$ and $c_0(t)$ and their complex conjugates we need to calculate the matrix elements of the density operator defined in eq. (3.10). Hence, we project eq. (3.7) onto the substates $|k\rangle$ and $|0\rangle$ that build the base of the Hilbert space of the system. As the calculation of the matrix elements in k-space will be more convenient, we have to fourier transform the right part of eq. (3.7).

$$\int \mathrm{d}x n(x) a(x) \rho a^\dagger(x) = \int \mathrm{d}k \int \mathrm{d}k' \frac{1}{\sqrt{2\pi}} n(k-k') a(k') \rho a^\dagger(k) \tag{3.11a}$$

$$\int \mathrm{d}x n(x) \left\{ {}^{\dagger}(x) a(x), \rho \right\} = \int \mathrm{d}k \int \mathrm{d}k' \frac{1}{\sqrt{2\pi}} n(k-k') \left\{ a^{\dagger}(k) a(k'), \bullet \right\} \tag{3.11b}$$

These relations will transform eq. (3.7) into

$$\partial_t \rho = -\frac{i}{\hbar} [H, \rho] + \frac{\Gamma}{\sqrt{2\pi}} \int \mathrm{d}k \mathrm{d}k' n(k-k') \left(a(k') \rho a^{\dagger}(k) - \frac{1}{2} \left\{ a^{\dagger}(k) a(k'), \rho \right\} \right) \tag{3.12}$$

Since the commutator as well as the anti-commutator have the same order of creation and annihilation operators we can introduce an effective Hamiltonian

$$H_{\mathrm{eff}} = \int \mathrm{d}k \left[\hbar c k a^{\dagger}(k) a(k) - i \frac{\hbar \Gamma}{2\sqrt{2\pi}} \int \mathrm{d}k' n(k-k') a^{\dagger}(k) a(k') \right], \tag{3.13}$$

which in fact is not hermitian any more. The second term in this effective Hamiltonian gives rise to dissipation in the system and therefore leads to a decay of the amplitude $\phi(x,t)$. If we redefine the commutator $[A,B]_{\mathrm{h.c}} = AB - A^{\dagger}B^{\dagger}$, we can simplify the master equation

$$\partial_t \rho = -\frac{i}{\hbar} [H_{\mathrm{eff}}, \rho]_{\mathrm{h.c}} + \frac{\Gamma}{\sqrt{2\pi}} \int \mathrm{d}k \mathrm{d}k' n(k-k') a(k') \rho a^{\dagger}(k) \tag{3.14}$$

This very compact form allows for calculating the matrix elements very quickly, noting that

$$a^{\dagger}(k) a(k') |q\rangle\langle q'| = \delta(k'-q) |k\rangle\langle q'| \qquad a^{\dagger}(k) a(k') |0\rangle\langle q'| = 0 \tag{3.15a}$$

$$a(k') |q\rangle\langle q'| a^{\dagger}(k) = \delta(k'-q) |0\rangle\langle 0| \delta(k-q') \qquad a(k') |0\rangle\langle q'| a^{\dagger}(k) = 0 \tag{3.15b}$$

Finally, the matrix element $\langle q| \cdots |q'\rangle$ yields

$$\begin{aligned} (\partial_t \phi(q,t)) \phi^*(q',t) + \phi(q,t) (\partial_t \phi^*(q',t)) = & -icq \phi(q,t) \phi^*(q',t) + icq' \phi^*(q',t) \phi(q,t) \\ & - \frac{\Gamma}{2} (\phi(t) \star n)(q) \phi^*(q',t) \\ & - \frac{\Gamma}{2} (\phi^*(t) \star n)(q') \phi(q,t) \end{aligned} \tag{3.16}$$

where $\phi(t) \star n$ denotes the convolution of $\phi(k,t)$ and $n(k)$. The matrix element $\langle 0| \cdots |0\rangle$ does not depend on k and yields

$$\partial_t |c_0(t)|^2 = \Gamma \int \mathrm{d}k (\phi(t) \star n)(k) \phi^*(k,t) = \Gamma \int \mathrm{d}x n(x) |\phi(x,t)|^2 \tag{3.17}$$

At last the matrix element $\langle q| \cdots |0\rangle$ yields

$$\partial_t (c_0^*(t) \phi(q,t)) = c_0^*(t) \left(-icq \phi(q,t) - \frac{\Gamma}{2} (\phi(t) \star n)(q) \right) \tag{3.18}$$

as well as the complex conjugate equation.

Let us now take a closer look at the equations derived above. In fact, eq. (3.16) can be split into two differential equations, one for ϕ and another for its complex conjugate

$$\partial_t \phi(q,t) = -icq \phi(q,t) - \frac{\Gamma}{2} (\phi(t) \star n)(q) \tag{3.19a}$$

$$\partial_t \phi^*(q,t) = icq\phi^*(q,t) - \frac{\Gamma}{2}(\phi^*(q,t) \star n)(q) \tag{3.19b}$$

The first equation describes a propagating wave packet with a group velocity of $-c$ and a decaying amplitude due to the absorbing medium in the second term. Analogously, the second equation yields a wave packet propagating with a group velocity of c and a decay that is equal to the first one. Fourier transformation of eq. (3.19a) yields

$$\partial_t \phi(x,t) = c\partial_x \phi(x,t) - \frac{\Gamma}{2} n(x)\phi(x,t) \tag{3.20}$$

This equation is equivalent to eq. (2.25) we derived in section 2.2. For $n(x \to \pm\infty) = 0$ we can solve this equation

$$\phi(x,t) = \exp\left(-\frac{\Gamma}{2c}\int_x^{+\infty} \mathrm{d}x' n(x')\right) f(x+ct) \tag{3.21}$$

where f is the normalized pulse shape. Since the density is positive the exponential is always smaller than one and thus yields to a decay of the amplitude.

In order to receive an equation for the absolute square of ϕ, we multiply eqs. (3.19a) and (3.19b) with ϕ and ϕ^* respectively and add them. This yields

$$\partial_t |\phi(x,t)|^2 = -\Gamma n(x)|\phi(x,t)|^2. \tag{3.22}$$

Performing the integration over the space variable x leads to an equation similar to eq. (3.17).

$$\partial_t \int \mathrm{d}x |\phi(x,t)|^2 = -\Gamma \int \mathrm{d}x n(x)|\phi(x,t)|^2 \tag{3.23}$$

The negative sign takes the decreasing probability of finding a photon into account whereas the positive sign in eq. (3.17) increases the probability for a photon being absorbed. Furthermore, the total change of probability is zero as can be seen by taking the derivative of the trace of the density matrix with respect to time.

$$\partial_t \operatorname{Tr}\{\rho\} = \operatorname{Tr}\{\partial_t \rho\} = -\Gamma \int \mathrm{d}x n(x)|\phi(x,t)|^2 + \Gamma \int \mathrm{d}x n(x)|\phi(x,t)|^2 = 0 \tag{3.24}$$

For the sake of simplicity we have performed the calculation of the trace in space representation. Because of the vanishing time derivative the trace of the density matrix will remain constant. Its initial value is 1 because $c_0(t \to -\infty) = 0$ and $\int \mathrm{d}k |\phi(k,t \to -\infty)|^2 = 1$. Therefore the trace is preserved as it should be.

$$\operatorname{Tr}\{\rho(t)\} = 1 \tag{3.25}$$

3.2.1 Application on a given density distribution

As we will derive a more general solution in the next section the following discussion will be rather short and is just meant to give a foretaste of what will come in the next section.

Until now we have not assumed a particular density distribution except the condition that it should vanish in the limit $x \to \pm\infty$. A distribution satisfying this condition is the symmetric normalized Eckart potential

$$n(x) = \frac{a}{2\cosh^2(ax)}, \tag{3.26}$$

we used earlier. With this distribution eq. (3.21) becomes

$$\phi(x,t) = \exp\left\{\frac{-\Gamma}{4c}\left(1 - \tanh(ax)\right)\right\} f(x + ct) \tag{3.27}$$

with a normalized function f. In the limit $x \to \infty$ where the influence of the medium becomes negligible the exponential term converges to one and we gain the initial state wave packet. In the limit $x \to -\infty$ where the wave packet has passed through the medium, the exponential term converges to $\exp\left(-\frac{\Gamma}{2c}\right)$.

Integrating eq. (3.17) yields the probability of a photon being absorbed at a given time t. This is illustrated in Figure 3.3. The probability increases as the wave packet is passing through the medium and remains constant afterwards ($t \to \infty$). In the limit $t \to \infty$ the probability of a photon being absorbed depends on the ratio Γ/c. For a high absorption (high Γ/c) the probability is almost one whereas for a weak absorption (small Γ/c) the probability of a photon being absorbed is rather small.

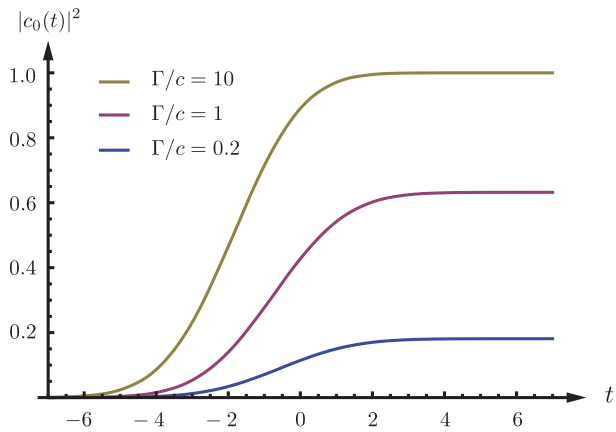

Figure 3.3: Probability of finding no photon at a time t. For a high absorption (high Γ/c) the photon is absorbed almost completely. For weak absorption (small Γ/c) the probability of a photon to be absorbed is very small.

3.3 General solution of the quantum master equation

So far we have studied the dynamics of a single-photon state passing through a dissipative medium. Since the calculations we performed above turn out to be very cumbersome, we derive a quite more powerful and general solution that actually holds true for arbitrary

states.

We can simplify the master equation by introducing 'superoperators' (the bullets are placeholders for other operators that stand on the right of these 'superoperators')

$$K = \int \mathrm{d}k(-ick)\left[a^\dagger(k)a(k), \bullet\right] \tag{3.28a}$$

$$J = \Gamma \int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}} n(k-k')a(k') \bullet a^\dagger(k) = \Gamma \int \mathrm{d}x n(x)a(x) \bullet a^\dagger(x) \tag{3.28b}$$

$$L = -\frac{1}{2}\Gamma \int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}} n(k-k')\left\{a^\dagger(k)a(k'), \bullet\right\} = -\frac{1}{2}\Gamma \int \mathrm{d}x n(x)\left\{a^\dagger(x)a(x), \bullet\right\}. \tag{3.28c}$$

With these 'superoperators' the master equation reduces to

$$\partial_t \rho = (K + J + L)\,\rho. \tag{3.29}$$

Obviously, this is a very useful reduction as this equation can be integrated and leads to a formal solution of our problem. Assuming that the initial state is defined at $t = 0$ the formal solution reads

$$\rho(t) = \exp\left(t(K + J + L)\right)\rho(0) = U(t)\rho(0) \tag{3.30}$$

From this equation, we see that the initial state $\rho(0)$ will evolve in time according to a density matrix propagator $U(t)$ that consists of the three 'superoperators' defined above. Since we know the action of each of these operators alone on a given density matrix, we are interested in a disentangled operator of the form $e^{tJ}e^{tL}e^{tK}$. The operator e^{tK} containing the Hamiltonian leads to a propagation of the wave packet. Since L has a similar structure as K but is , it also preserves the total photon number but leads to a decay of the amplitude. At last J lowers the number of photons by one and thus e^J leads to a decay into the vacuum state.

3.3.1 Splitting the time evolution operator

In order to disentangle the time evolution operator we will make use of the Baker-Campbell-Hausdorff formula (BCH formula) whose properties and corollary formulas are presented in section 3.A. Using the relation $\log\left(e^X e^Y\right) = Z$ with $Z = X + \frac{s}{1-e^{-s}}Y$ for $[X,Y] = sY$ we have to rescale the operator Y

$$Y \to \frac{1-e^{-s}}{s}Y. \tag{3.31}$$

This yields the BCH formula

$$e^{X+Y} = e^X e^{\frac{1-\exp(-s)}{s}Y}. \tag{3.32}$$

We now define a new operator $N(s)$ with $N(0) = J + L$ that will help us to apply the modified BCH formula. This operator reads

$$N(s) = \Gamma \int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}} n(k-k')e^{-ic(k-k')ts}\left(a(k') \bullet a^\dagger(k) - \frac{1}{2}\left\{a^\dagger(k)a(k'), \bullet\right\}\right) \tag{3.33}$$

The commutator of tK and $tN(s)$ then yields

$$[tK, tN(s)] = t\partial_s N(s) \tag{3.34}$$

and therefore we can disentangle the time evolution operator in a first step

$$e^{t(K+N(s))} = e^{tK} e^{\frac{1-\exp(-\partial_s)}{\partial_s} tN(s)}. \tag{3.35}$$

It will prove useful to put the operator K that contains the Hamiltonian of the system on the right since this leads to a propagating wave packet. Interchanging both exponentials again yields another exponential factor

$$e^{t(K+N(s))} = e^{tK} e^{\frac{1-\exp(-\partial_s)}{\partial_s} tN(s)} = e^{\frac{\exp(\partial_s)-1}{\partial_s} tN(s)} e^{tK}, \tag{3.36}$$

where we have used eq. (3.105).

Performing the derivatives and taking the limit $s \to 0$ yields

$$\begin{aligned}
\frac{\exp(\partial_s)-1}{\partial_s} tN(s) &= t\Gamma \int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}} n(k-k') \frac{e^{-ic(k-k')t}-1}{-ic(k-k')t} e^{-ic(k-k')ts} \\
&\quad \left(a(k') \bullet a^\dagger(k) - \frac{1}{2} \left\{ a^\dagger(k) a(k'), \bullet \right\} \right) \\
&\overset{s \to 0}{=} t\Gamma \int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}} n(k-k') \frac{e^{-ic(k-k')t}-1}{-ic(k-k')t} \\
&\quad \left(a(k') \bullet a^\dagger(k) - \frac{1}{2} \left\{ a^\dagger(k) a(k'), \bullet \right\} \right)
\end{aligned} \tag{3.37}$$

Now we have to split the exponential containing the modified operators $\bar{J}$ and $\bar{L}$ and therefore need their commutation relations. As both were defined originally in the space domain we perform our calculations there. For the sake of a clearer presentation we introduce

$$\Omega(k-k', t) = t\Gamma n(k-k') \frac{e^{-ic(k-k')t}-1}{-ic(k-k')t} \tag{3.38}$$

with which eq. (3.37) reduces to

$$\begin{aligned}
\bar{J} + \bar{L} &= \int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}} \Omega(k-k', t) \left(a(k') \bullet a^\dagger(k) \right. \\
&\quad \left. - \frac{1}{2} \left\{ a^\dagger(k) a(k'), \bullet \right\} \right) \\
&= \int \mathrm{d}x \Omega(x,t) \left(a(x) \bullet a^\dagger(x) - \frac{1}{2} \left\{ a^\dagger(x) a(x), \bullet \right\} \right).
\end{aligned} \tag{3.39}$$

The commutator for $\bar{J}$ and $\bar{L}$ yields

$$\left[\bar{L}, \bar{J} \right] = \int \mathrm{d}x \Omega^2(x,t) a(x) \bullet a^\dagger(x). \tag{3.40}$$

Using the same mathematical trick as mentioned above we can introduce a new operator $\bar{J}(r)$ with $\bar{J}(0) = \bar{J}$. This operator is defined by

$$\bar{J}(r) = t\Gamma \int \mathrm{d}x \Omega(x,t) e^{r\Omega(x,t)} a(x) \bullet a^\dagger(x). \tag{3.41}$$

The commutator (3.40) then reads

$$\left[\bar{L}, \bar{J}(r)\right] = \partial_r \bar{J}(r) \tag{3.42}$$

A straightforward calculation using the BCH formula like above yields

$$e^{\bar{L}+\bar{J}(r)} = e^{\frac{\exp(\partial_r)-1}{\partial_r}\bar{J}(r)} e^{\bar{L}}. \tag{3.43}$$

Simplifying all exponentials and performing the limit $r \to 0$, we get

$$\begin{aligned} \frac{\exp(\partial_r) - 1}{\partial_r} \bar{J}(r) &= \int \mathrm{d}x \frac{e^{\Omega(x,t)} - 1}{\Omega(x,t)} \Omega(x,t) e^{r\Omega(x,t)} a(x) \bullet a^\dagger(x) \\ &= \int \mathrm{d}x \left(e^{\Omega(x,t)} - 1\right) a(x) \bullet a^\dagger(x) = \hat{J} \end{aligned} \tag{3.44}$$

This leads to the final result

$$\begin{aligned} e^{t(J+L+K)} = \exp &\left[\int \mathrm{d}x \left(e^{\Omega(x,t)} - 1\right) a(x) \bullet a^\dagger(x)\right] \\ \exp &\left[-\frac{1}{2} \int \mathrm{d}x \Omega(x,t) \left\{a^\dagger(x) a(x), \bullet\right\}\right] \exp\left[tK\right] \end{aligned} \tag{3.45}$$

We now want to apply this solution to a given initial state of the density matrix $\rho(0)$.

3.3.2 Application on number states

In this section we apply the scheme we developed above on a general number state. We will find that the solution reproduces our results for a single photon state. Consider now a continuous-mode number state $|n_\phi\rangle$ that was already defined in eq. (1.42). We assume the photons to be all in the same mode and to be indistinguishable so they all have the same pulse shape and the same polarization. Thus the multimode number state with a normalized pulse shape ϕ reads

$$|n_\phi\rangle = \frac{1}{\sqrt{n!}} \int \mathrm{d}k_1 \cdots \mathrm{d}k_n \phi(k_1, \ldots, k_n, t = 0) a^\dagger(k_1) \cdots a^\dagger(k_n) |0\rangle. \tag{3.46}$$

This expression holds true for entangled photons as well. If we further assume them to be disentangled the general pulse shape ϕ factorizes and eq.(3.46)leads to

$$|n_\phi\rangle = \frac{1}{\sqrt{n!}} \int \mathrm{d}k_1 \cdots \mathrm{d}k_n \phi(k_1) \cdots \phi(k_n) a^\dagger(k_1) \cdots a^\dagger(k_n) |0\rangle. \tag{3.47}$$

The initial density matrix therefore reads

$$\rho(0) = |n_\phi\rangle\langle n_\phi|. \tag{3.48}$$

For a number state we can further simplify the general solution found above

$$\rho(t) = e^{\hat{J}} e^{-\frac{1}{2}\int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}}\Omega(k-k')a^\dagger(k')a(k)} e^{-\frac{i}{\hbar}Ht}|n_\phi\rangle\langle n_\phi| e^{\frac{i}{\hbar}Ht} e^{-\frac{1}{2}\int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}}\Omega(k-k')a^\dagger(k')a(k)} \tag{3.49}$$

Since $|n_\phi\rangle$ is an eigenstate of H and $a^\dagger(k')a(k)$ are number conserving they will just influence the pulse shape ϕ but not the total photon number n. The dynamics of these operators therefore are coherent. The Hamiltonian will lead to a propagation of the wave packet whereas the exponential containing Ω will lead to a decay of the amplitude. In addition, $e^{\hat{J}}$ will lead to a decay into the vacuum state. For the sake of simplicity, we will use a single photon state first in order to understand the dynamics of this state. Simplifying notation we set $|1_\phi\rangle = |1\rangle$ because we treat continuous-mode states only.

Single photon state

Application of H onto a single photon state yields

$$-\frac{i}{\hbar}tH|1\rangle = \int \mathrm{d}k(-ickt)\phi(k)a^\dagger(k)|0\rangle \tag{3.50}$$

and therefore

$$e^{-\frac{i}{\hbar}tH}|1\rangle = \int \mathrm{d}k e^{-ickt}\phi(k)a^\dagger(k)|0\rangle \tag{3.51}$$

that represents a propagating wave with group velocity $-c$. Further, application of $\bar{L}$ yields

$$-\frac{1}{2}\int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}}\Omega(k-k')a^\dagger(k)a(k')|1\rangle = -\frac{1}{2}\int \mathrm{d}k(\Omega(t)\star\phi)(k)a^\dagger(k)|0\rangle \tag{3.52}$$

Thus we get

$$\begin{aligned}-\frac{1}{2}\int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}}\Omega(k-k',t)a^\dagger(k)a(k')e^{-\frac{i}{\hbar}Ht}|1\rangle &= -\frac{1}{2}\int \mathrm{d}k\mathrm{d}k' \frac{1}{\sqrt{2\pi}}\Omega(k-k',t)e^{-ick't}\phi(k')a^\dagger(k)|0\rangle \\ &\overset{FT}{=} -\frac{1}{2}\int \mathrm{d}x\Omega(x,t)\phi(x+ct)a^\dagger(x)|0\rangle.\end{aligned} \tag{3.53}$$

Applying $e^{\bar{L}}$ then yields higher orders of $-\frac{1}{2}\Omega(x,t)$ and we can define a new pulse shape

$$\psi(x,t) = e^{-\frac{1}{2}\Omega(x,t)}\phi(x+ct). \tag{3.54}$$

The time dependent density matrix thus reduces to

$$\rho(t) = e^{\hat{J}}|1_\psi\rangle\langle 1_\psi|. \tag{3.55}$$

Until now the operator $\hat{J}$ has not been taken into account but will be important later when we will implement the Rydberg blockade mechanism into our system. To see the action of this operator we expand the exponential in a series

$$\exp(\hat{J}) = \sum_m \frac{1}{m!}\hat{J}^m = 1 + \hat{J} + \frac{1}{2}\hat{J}^2 + \cdots \tag{3.56}$$

If we apply $\hat{J}$ on the single photon state $|1_\psi\rangle$ we get

$$\begin{aligned}\hat{J}|1_\psi\rangle\langle 1_\psi| &= \int \mathrm{d}x \left(e^{\Omega(x,t)} - 1\right) a(x)|1_\psi\rangle\langle 1_\psi|a^\dagger(x) = \int \mathrm{d}x \left(e^{\Omega(x,t)} - 1\right) |\psi(x,t)|^2|0\rangle\langle 0| \\ &= \int \mathrm{d}x \left(e^{\Omega(x,t)} - 1\right) e^{-\Omega(x,t)}|\phi(x+ct)|^2|0\rangle\langle 0| \\ &= \int \mathrm{d}x \left(1 - e^{-\Omega(x,t)}\right) |\phi(x+ct)|^2|0\rangle\langle 0| \end{aligned} \tag{3.57}$$

This operator just decreases the number of photons in the system. They are absorbed by the medium thus leading to a increase of probability of having zero photons in the course of time. For a single photon state the series expansion will break after the first order because there are no more photons left. The full solution of the density matrix for a single photon state then reads

$$\rho(t) = |1_\psi\rangle\langle 1_\psi| + \int \mathrm{d}x \left(1 - e^{-\Omega(x,t)}\right) |\phi(x+ct)|^2|0\rangle\langle 0|. \tag{3.58}$$

The trace of the density matrix is preserved as well because $|\phi(x+ct)|^2$ is still normalized

$$\begin{aligned}\mathrm{Tr}\{\rho(t)\} &= \int \mathrm{d}x\langle x|\rho(t)|x\rangle + \langle 0|\rho(t)|0\rangle \\ &= \int \mathrm{d}x \left[e^{-\Omega(x,t)} + 1 - e^{-\Omega(x,t)}\right] |\phi(x+ct)|^2 = 1 \end{aligned} \tag{3.59}$$

General number state

The following calculations are similar to those already performed above so we give the results only. Analogously, the exponential containing the Hamiltonian leads to propagation of the wave packet

$$\begin{aligned} e^{-\frac{i}{\hbar}Ht}|n_\phi\rangle &= \int \mathrm{d}k_1 \cdots \mathrm{d}k_n \frac{1}{\sqrt{n!}} e^{-ic(k_1+\cdots k_n)t}\phi(k_1,\ldots,k_n)a^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle \\ &= \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{1}{\sqrt{n!}} \phi(x_1+ct,\ldots,x_n+ct)a^\dagger(x_1)\cdots a^\dagger(x_n)|0\rangle \end{aligned} \tag{3.60}$$

Furthermore, application of the factor that leads to a decay of the amplitude yields

$$\begin{aligned} e^{-\frac{1}{2}\int \mathrm{d}y\Omega(y,t)a^\dagger(y)a(y)}e^{-\frac{i}{\hbar}Ht}|n_\phi\rangle = \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{1}{\sqrt{n!}} \exp\left(-\frac{1}{2}(\Omega(x_1,t) + \cdots \Omega(x_n,t))\right) \\ \phi(x_1+ct,\ldots,x_n+ct)a^\dagger(x_1)\cdots a^\dagger(x_n)|0\rangle \end{aligned} \tag{3.61}$$

If the photons are not entangled, they will of course all evolve just like the single photon state we discussed above.

$$\begin{aligned} &\int \mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{1}{\sqrt{n!}} \exp\left(-\frac{1}{2}(\Omega(x_1,t) + \cdots \Omega(x_n,t))\right) \\ &\phi(x_1+ct,\ldots,x_n+ct)a^\dagger(x_1)\cdots a^\dagger(x_n)|0\rangle = \frac{1}{\sqrt{n!}}\left(\int \mathrm{d}x e^{-\frac{1}{2}\Omega(x,t)}\phi(x+ct)a^\dagger(x)\right)^n|0\rangle \end{aligned} \tag{3.62}$$

The exponential containing $\hat{J}$ will again decrease the number of photons in the initial number state. As there are only n photons, the exponential series will stop after the n-th order. If we assume the photons not to be entangled, the application of the m-th order of $e^{\hat{J}}$ on a number state $|n_\psi\rangle$ $(n \geq m)$ yields

$$\begin{aligned}\frac{1}{m!}\hat{J}^m &= \frac{1}{m!}\frac{n!}{(n-m)!}\left(\int \mathrm{d}x \left(1-e^{-\Omega(x,t)}\right)|\phi(x+ct)|^2\right)^m |(n-m)_\psi\rangle\langle(n-m)_\psi| \\ &= \binom{n}{m}\left(\int \mathrm{d}x \left(1-e^{-\Omega(x,t)}\right)|\phi(x+ct)|^2\right)^m |(n-m)_\psi\rangle\langle(n-m)_\psi| \end{aligned} \tag{3.63}$$

So the full solution for an initial number state $|n_\phi\rangle$ is

$$\rho(t) = \sum_{m=0}^{\infty}\binom{n}{m}\left(\int \mathrm{d}x \left(1-e^{-\Omega(x,t)}\right)|\phi(x+ct)|^2\right)^m |(n-m)_\psi\rangle\langle(n-m)_\psi|. \tag{3.64}$$

Of course, the trace of $\rho(t)$ is preserved and remains constant for all times.

$$\begin{aligned}\mathrm{Tr}\{\rho(t)\} &= \sum_{m=0}^{\infty}\binom{n}{m}\left(\int \mathrm{d}x \left(1-e^{-\Omega(x,t)}\right)|\phi(x+ct)|^2\right)^m \left(\int \mathrm{d}x e^{-\Omega(x,t)}|\phi(x+ct)|^2\right)^{n-m} \\ &= \left(\int \mathrm{d}x \left(1-e^{-\Omega(x,t)}\right)|\phi(x+ct)|^2 + \int \mathrm{d}x e^{-\Omega(x,t)}|\phi(x+ct)|^2\right)^n = 1 \end{aligned} \tag{3.65}$$

Coherent states

Since coherent states can be expressed in terms of number states, we can treat them similarly to above. In section 1.2.1 the coherent states for continuous-mode fields were defined as

$$|\alpha\rangle = \exp\left(a_\alpha^\dagger - a_\alpha\right)|0\rangle = \mathcal{N}\exp\left(a_\alpha^\dagger\right)|0\rangle \tag{3.66}$$

with the normalization factor $\mathcal{N} = e^{-\frac{1}{2}\langle n\rangle}$. Expanding the exponential creation operator, the coherent state can be written as a superposition of number states

$$|\alpha\rangle = \mathcal{N}\sum_n \frac{1}{n!}\left(a_\alpha^\dagger\right)|0\rangle = \mathcal{N}\sum_n \frac{1}{\sqrt{n!}}|n_\alpha\rangle. \tag{3.67}$$

Since we know the action of $e^{\bar{L}}$ on number states (eq. (3.62)), a straightforward calculation yields

$$\begin{aligned} e^{\bar{L}}e^{tK}|\alpha\rangle\langle\alpha| &= \mathcal{N}^2\exp\left\{\int \mathrm{d}x e^{\Omega(x,t)}|\alpha(x+ct)|^2\right\}|\alpha(x+ct)e^{-\frac{1}{2}\Omega(x,t)}\rangle\langle\alpha^*(x-ct)e^{-\frac{1}{2}\Omega(x,t)}| \\ &= \exp\left\{-\int \mathrm{d}x \left(1-e^{\Omega(x,t)}\right)|\alpha(x+ct)|^2\right\}|\alpha(x+ct)e^{-\frac{1}{2}\Omega(x,t)}\rangle\langle\alpha^*(x-ct)e^{-\frac{1}{2}\Omega(x,t)}|. \end{aligned} \tag{3.68}$$

The factor in front of the 'bra' and the 'ket' is due to normalization. In contrast to the number states, coherent states are eigenstates of $e^{\hat{J}}$ since $a(x)|\alpha\rangle = \alpha(x)|\alpha\rangle$. This leads

to

$$
\begin{aligned}
e^{\hat{J}} e^{\bar{L}} e^{tK} |\alpha\rangle\langle\alpha| = \sum_m \frac{1}{m!} &\left[\int \mathrm{d}x \left(e^{\Omega(x,t)} - 1 \right) |\alpha(x+ct)|^2 e^{-\Omega(x,t)} \right] \\
&\exp\left\{ - \int \mathrm{d}x \left(1 - e^{\Omega(x,t)} \right) |\alpha(x+ct)|^2 \right\} |\alpha(x+ct) e^{-\frac{1}{2}\Omega(x,t)}\rangle\langle\alpha^*(x-ct) e^{-\frac{1}{2}\Omega(x,t)}| \\
= &\left| \alpha(x+ct) e^{\frac{-\Omega(x,t)}{2}} \right\rangle \left\langle \alpha(x-ct) e^{-\frac{\Omega(x,t)}{2}} \right| . \qquad (3.69)
\end{aligned}
$$

As can be seen, the coherent state remains coherent but its amplitude decays as it passes through the inhomogeneous medium.

3.3.3 Application on a given density distribution

As we have seen so far, the general solution derived above is a very useful tool for general number states. Since we only gave a short discussion about the solution for a single photon state above, we will now apply the general solution on a given density distribution in the case of a single photon state as well. Therefore we use again the normalized symmetric Eckart-potential, which we now shift by x_0 along the x-axis.

$$
n(x - x_0) = \frac{a}{2\cosh^2(a(x - x_0))} \qquad (3.70)
$$

The shift will be convenient due to the normalization condition the wave packet has to satisfy for $t = 0$ when it is in its initial state.

The decay of the wave packet is governed by the function $\Omega(x,t)$ which is defined as the fourier transform of

$$
\Omega(k,t) = t\Gamma n(k) \frac{e^{-ickt} - 1}{-ickt} = \Gamma n(k) \frac{e^{-ickt} - 1}{-ick}, \qquad (3.71)
$$

where $n(k)$ is the fourier transform of $n(x)$. Actually, the fourier transform of a shifted function is just the fourier transform of the unshifted function multiplied with a factor e^{ikx_0} and thus yields

$$
\mathcal{F}\{n(x - x_0)\} = e^{ikx_0} \mathcal{F}\{n(x)\} = e^{ikx_0} \frac{k\sqrt{\frac{\pi}{2}}}{2a\cosh\left(\frac{k\pi}{2a}\right)}. \qquad (3.72)
$$

The fourier transformation of eq. (3.71) thus yields

$$
\Omega(x,t) = \frac{\Gamma}{2c} \left(\tanh(a(ct + x - x_0)) - \tanh(a(x - x_0)) \right) \qquad (3.73)
$$

This looks different from the solution (eq.(2.26) for the single photon state at first instance. The difference is due to the fact that we assume the wave packet to be in its initial normalized state at $t = 0$. For $t = 0$ the wave packet is centered around $x = 0$. In addition this means that the influence of the absorbing medium has to vanish at $x = 0$ and therefore strictly speaking the limit $x_0 \to -\infty$ has to be taken. As we are only

interested in how Ω influences the wave packet, we can restrict the discussion to the case in which $x + ct = 0$ due to the fact that this is the center of the wave packet. Therefore eq. (3.73) reduces to

$$\Omega(x,t) = \frac{\Gamma}{2c}\left(\tanh(-x_0 a) - \tanh(a(x - x_0))\right). \tag{3.74}$$

Since the wave packet propagates in the negative direction of the x-axis and starts at $x = 0$, we treat all x-values to be negative. Thus, for $x_0 \ll x$, where the packet has not yet passed the medium, eq. (3.74) reduces to

$$\Omega(x,t) \approx \frac{\Gamma}{2c}\left(\tanh(-x_0 a) - \tanh(-x_0 a)\right) = 0. \tag{3.75}$$

In the other case when the wave packet passed through the medium ($x_0 \gg x$), we get

$$\Omega(x,t) \approx \frac{\Gamma}{2c}\left(\tanh(-x_0 a) - \tanh(ax)\right) \overset{x_0 \to -\infty}{\approx} \frac{\Gamma}{2c}\left(1 - \tanh(ax)\right). \tag{3.76}$$

This is exactly the same solution we obtained above in the case of a single photon state. The pulse shape ψ then reads

$$\psi(x,t) = \exp\left\{ -\frac{\Gamma}{4c}\left(1 - \tanh(ax)\right)\right\} \phi(x + ct). \tag{3.77}$$

The ratio of incoming and transmitted amplitude is then given by the square of the exponential for $x \to -\infty$

$$\frac{|\psi_{\text{trans}}|^2}{|\psi_{\text{in}}|^2} = \exp\left(-\frac{\Gamma}{c}\right) \tag{3.78}$$

For the study of this equation we have to keep in mind that Γ is given as a differential absorption rate and thus has the dimension of length over the time so that the ratio Γ/c is dimensionless.

The physical interpretation of eq.(3.78) is that if the time that light needs to pass through an infinitesimal interval is small compared to the time scale on which absorption takes place over this interval,$\tau_{\text{pass}} \ll \frac{1}{\gamma}$, the decay will be suppressed and the amplitude of the transmitted wave packet remains almost the same as the one of the incident wave packet. In the reverse case, if $\frac{1}{\gamma} \ll \tau_{\text{pass}}$, the photon will be absorbed completely. In Figure 3.4 this is shown for three values of Γ/c.

3.4 Implementation of a saturating absorber

In this section we implement the Rydberg blockade mechanism and derive the dynamics of such a single photon absorber. We assume the spatial confinement of the Rydberg atoms to be smaller than the Rydberg blockade radius so the system can be described by a two-state system with collective ground and excited state denoted by $|G\rangle$ and $|W\rangle$, respectively (cf. 1.4.2). Furthermore, we introduce the new jump-operators $c(x) = a(x) \otimes |W\rangle\langle G| = a(x)|W\rangle\langle G|$ and $c^\dagger(x) = a^\dagger(x)|G\rangle\langle W|$ that satisfy $c(x)c(x') = 0$

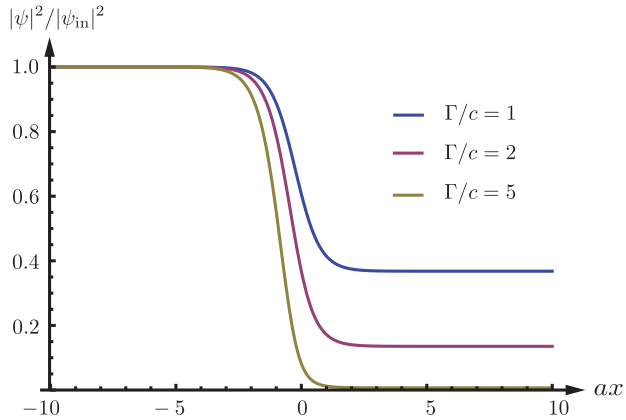

Figure 3.4: Amplitude ratio of the incident wave and the wave at a point x. If Γ is small compared to c, the amplitude will decrease very less, whereas for large Γ the photon will be absorbed almost completely.

and hence only allow a single excitation. The jump operators transfer excitations between the light field and the two level system. For example, if a photon is annihilated at a place x, the two-level system will go into its excited state $|W\rangle$. If the two-level system loses an excitation, a photon will be created.

Substituting the modified creation and annihilation operator into the quantum master equation (3.7) leads to

$$\partial_t \rho = -\frac{i}{\hbar}[H,\rho] + \Gamma \int \mathrm{d}x\, n(x) \left(a(x)|W\rangle\langle G|\rho|G\rangle\langle W|a^\dagger(x) - \frac{1}{2}\left\{a^\dagger(x)a(x)|G\rangle\langle G|, \rho\right\}\right). \tag{3.79}$$

With the help of sub-matrices of the density matrix defined by

$$\rho_G = \langle G|\rho|G\rangle, \tag{3.80a}$$

$$\rho_W = \langle W|\rho|W\rangle \tag{3.80b}$$

we can project eq. (3.79) onto the subspaces containing the ground state and the collective excited state, respectively. The equation thus splits into two equations

$$\partial_t \rho_G = -\frac{i}{\hbar}[H,\rho_G] + L\rho_G \tag{3.81a}$$

$$\partial_t \rho_W = -\frac{i}{\hbar}[H,\rho_W] + J\rho_G, \tag{3.81b}$$

where we use the notation introduced in the previous section. The first equation describes the well known propagation of a wave packet where L leads to decay of the amplitude of the wave packet. In contrast, the second equation leads to a differential equation for the wave packet in the case of an excited medium.

Assuming an incoming two-photon state

$$|2_\psi\rangle = \frac{1}{\sqrt{2}} \int \mathrm{d}x\mathrm{d}x'\, \psi(x,x',t) a^\dagger(x) a^\dagger(x')|0\rangle \tag{3.82}$$

and

$$\rho_G = |2_\psi\rangle\langle 2_\psi|, \tag{3.83}$$

eq. (3.81a) yields a differential equation for $\psi(x,x',t)$

$$\partial_t \psi(x,x',t) = c\partial_x \psi(x,x',t) + c\partial_{x'} \psi(x,x',t) - \frac{\Gamma}{2}\left(n(x) + n(x')\right)\psi(x,x',t). \tag{3.84}$$

This equation can be solved by making the separation ansatz $\psi(x,x',t) = \psi(x,t)\psi(x',t)$ and then yields the solution

$$\psi(x,x',t) = \phi(x+ct)\phi(x'+ct)\exp\left\{-\frac{\Gamma}{2c}\left(\int_x^\infty \mathrm{d}y\, n(y) + \int_{x'}^\infty \mathrm{d}y\, n(y)\right)\right\}, \tag{3.85}$$

where we imposed the boundary condition $\psi(x,x',t) = \phi(x+ct)\phi(y+ct)$ for $x,y \to \infty$. Actually, this is the same result that we have obtained in the sections 3.2 and 3.3.2.

We assume ρ_W to have the form of a single-photon density matrix since we expect one photon to be absorbed by the medium

$$\rho_W = \int \mathrm{d}x\mathrm{d}x' \zeta(x,x',t) a^\dagger(x)|0\rangle\langle 0|a(x'). \tag{3.86}$$

Substituting this form of ρ_W and the solution of ρ_G into eq. (3.81b) then yields a differential equation for $\zeta(x,x',t)$

$$\partial_t \zeta(x,x',t) = c\partial_x \zeta(x,x',t) + c\partial_{x'} \zeta(x,x',t) + 2\Gamma \int \mathrm{d}z\, n(z)\psi(z,x,t)\psi^*(z,x',t). \tag{3.87}$$

Since ψ can be separated (see eq. (3.85)), this equations takes the form

$$\partial_t \zeta(x,x',t) = c\partial_x \zeta(x,x',t) + c\partial_{x'} \zeta(x,x',t) + \underbrace{2\Gamma \int \mathrm{d}z\, n(z)|\psi(z,t)|^2}_{\gamma(t)} \psi(x,t)\psi^*(x',t). \tag{3.88}$$

This first order partial differential equation is inhomogeneous and the last term acts like a driving force that increases ζ to the overlapping of the density distribution and the two-photon wave packet at time t. Since the driving force vanishes for $t \to -\infty$, the homogeneous solution of this differential equation has to vanish due to the initial condition

$$\begin{aligned}\mathrm{Tr}\left\{\rho(t\to\infty)\right\} &= \mathrm{Tr}\left\{\rho_G(t\to\infty)\right\} + \mathrm{Tr}\left\{\rho_W(t\to\infty)\right\} \\ &= \left(\int \mathrm{d}x\, |\psi(x,t\to\infty)|^2\right)^2 + \int \mathrm{d}x\, \zeta(x,x,t\to\infty) = 1.\end{aligned} \tag{3.89}$$

The particular solution of eq. (3.88) is

$$\zeta(x,x',t) = \phi(x+ct)\phi(x'+ct) \int_{-\infty}^{t} \mathrm{d}s\, \gamma(s) \exp\left(N(x+c(t-s)) + N(x'+c(t-s))\right) \quad (3.90)$$

where $N(x) = -\frac{\Gamma}{2c} \int_{x}^{\infty} \mathrm{d}z\, n(z)$.

Strong absorption

We now assume the absorption to be very strong and therefore the absorption length to be very small. In this case, the photon is absorbed within a very short length scale and we can assume the remaining medium to be infinitely extended and the absorption length is much shorter than the pulse length. Thus we can assume the density to have the form

$$n(x) \sim 1 - \Theta(x), \quad (3.91)$$

where $\Theta(x)$ is the Heaviside step function

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}. \quad (3.92)$$

Since the absorption length is very short, we assume the decay to behave like a step function, too

$$\exp\left\{-\frac{\Gamma}{2c} \int_{x}^{\infty} \mathrm{d}z\, n(z)\right\} = \exp\left\{-\frac{\Gamma}{2c}\left(\Theta(x) - 1\right) x\right\} \approx \Theta(x). \quad (3.93)$$

In this case, eq. (3.88) takes the form [7]

$$\partial_t \zeta(x,x',t) = c\partial_x \zeta(x,x',t) + c\partial_{x'} \zeta(x,x',t) + 2|\phi(t)|^2 \phi(x+ct)\phi^*(x'+ct)\left(1-\Theta(x)\right)\left(1-\Theta(x')\right). \quad (3.94)$$

For $x,x' > 0$, this turns into a homogeneous differential equation where $\zeta(x,x',t) = 0$ due to the initial conditions we mentioned above. For $x,y \leq 0$, we can integrate eq. (3.94) which yields

$$\zeta(x,x',t) = 2\phi(x+ct)\phi^*(x'+ct) \int_{-\infty}^{t} \mathrm{d}s\, |\phi(s)|^2. \quad (3.95)$$

This represents the case in which both photons have entered the medium. In general, the solution reads

$$\zeta(x,x',t) = 2\phi(x+ct)\phi^*(x'+ct) \int_{-\infty}^{t+\min(x,x')/c} \mathrm{d}s\, |\phi(s)|^2. \quad (3.96)$$

The dependency of x and x' in the range of integration in the last integral provides knowledge of the timing of the absorption whose consequences will be discussed below. This result also can be obtained by using eq. (3.90). Thus, the complete density matrix reads

$$\rho = \rho_G + \rho_W = |2_\psi\rangle\langle 2_\psi| + \int \mathrm{d}x\mathrm{d}y\, \phi(x+ct)\phi(y+ct) \int\limits_{-\infty}^{t+\min(x,y)/c} \mathrm{d}s\, |\phi(s)|^2 a^\dagger(x)|0\rangle\langle 0|a(y), \tag{3.97}$$

where $|2_\psi\rangle$ and ψ are defined by eq. (3.82) and (3.85), respectively and ϕ is a normalized function, e.g. a Gaussian function. It can be seen that the outgoing field consists of one photon since $\mathrm{Tr}\{\rho_W\} = \int \mathrm{d}x\, \zeta(x,x,t) = 1$. This can be interpreted as follows. As soon as the first photon enters, the medium is saturated and therefore is transparent for the following photon. However, due to the knowledge of the timing of the absorption of the first photon, the resulting state cannot be pure any more since it carries information about the second photon. This can be verified mathematically by tracing over ρ_W

$$\mathrm{Tr}\left\{\rho^2\right\} = \int\limits_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\, 4\phi(x+ct)^2\phi(y+ct)^2 \left[\int\limits_{-\infty}^{t+\min(x,y)/c} \mathrm{d}s\, \phi(s)^2\right]^2 = \frac{2}{3} < 1. \tag{3.98}$$

This can be generalized to a n-photon Fock state, where the purity is given by [7]

$$\mathrm{Tr}\left\{\rho^2\right\} = \frac{n}{2n-1}. \tag{3.99}$$

Weak absorbtion

In the case of weak absorption, the absorption length is much larger than the pulse length. We can therefore assume $n(x)$ to be a slowly varying function of x whereas $\phi(x+ct)$ is sharply centered around $x = -ct$. In this approximation, the decay factors N in eq. (3.90) can be assumed to be centered around $x = -ct$ as well so that we get

$$\zeta(x,x',t) \approx \phi(x+ct)\phi(x'+ct) \int\limits_{-\infty}^{t} \mathrm{d}s\gamma(s)e^{2N(-cs)} = \phi(x+ct)\phi(x'+ct)G(t). \tag{3.100}$$

This represents a pure state since

$$\rho_W(t) = G(t) \int \mathrm{d}x\mathrm{d}x'\, \phi(x+ct)\phi(x'+ct)a^\dagger(x)|0\rangle\langle 0|a(x') = G(t)|1_\phi\rangle\langle 1_\phi|. \tag{3.101}$$

Since a weakly absorbing medium does not provide information about the time when a photon is absorbed, the outgoing state remains pure.

3.A Baker-Campbell-Hausdorff formula

The Baker-Campbell-Hausdorff formula is the solution of

$$Z(X,Y) = \log\left(e^X e^Y\right) \tag{3.102}$$

if X and Y do not commute in general. The function $Z(X,Y)$ can be represented as

$$Z(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] - \frac{1}{24}[Y,[X,[X,Y]]] + \cdots \quad (3.103)$$

where the higher orders are nested commutators of X and Y. If the commutator of X and Y takes the form $[X,Y] = sY$ for a non-zero number s, the formula above will reduce to

$$Z(X,Y) = X + \frac{s}{1 - e^{-s}}Y. \quad (3.104)$$

In addition, commutation of $e^X e^Y$ therefore reduces to

$$e^X e^Y = e^{\exp(s)Y} e^X. \quad (3.105)$$

Conclusion

In the first part of this thesis we discussed the classical behaviour of the electromagnetic field in inhomogeneous media. We saw that for very smoothly shaped surfaces the intensity of the reflected wave vanishes. Furthermore, for a localized medium the intensity of the reflected wave is exactly zero for certain combinations of the wavelength, the width of the medium and the dielectric susceptibility of the medium. In addition, we performed the slowly varying envelope approximation in order to transform the wave equation into a second order partial differential equation that can be solved analytically for certain shapes of the susceptibility.

In the second part, we stressed our discussions on the behaviour of the quantized electromagnetic field in dissipative media in terms of continuous modes. Therefore we considered the quantum master equation in continuous media. It was shown that the time evolution of the pulse shape that defines the photon state evolves the same way it does in classical electrodynamics. However, dissipative effects lead to incoherent dynamics that were discussed for number states. We also considered the dynamics of continuous-mode coherent states. Additionally, we studied the implementation of a saturating absorber that consists of an ensemble of Rydberg atoms confined in a volume smaller than its blockade-volume and therefore only allows for a single excitation. We saw that mixed states can be created by propagating through a strongly absorbing medium.

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