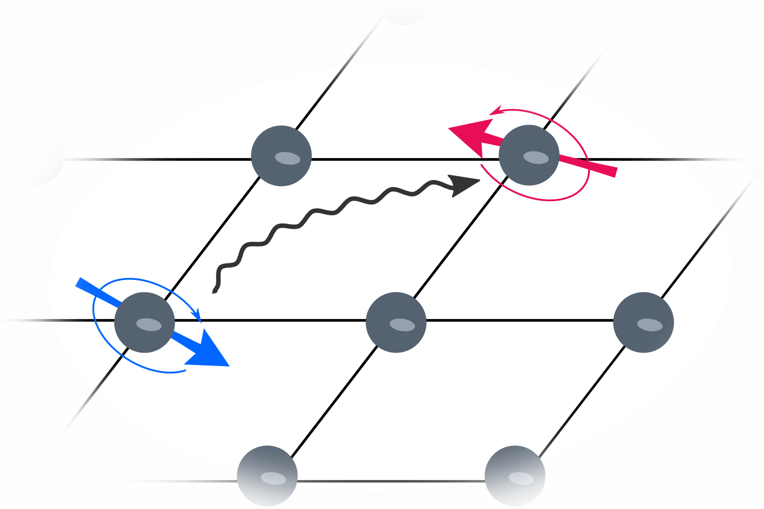


Topological flat bands with Chern number $C = 2$ by dipolar exchange interactions

David Peter — Palaiseau — November 6th, 2014

team

Hans Peter Büchler
Nicolai Lang

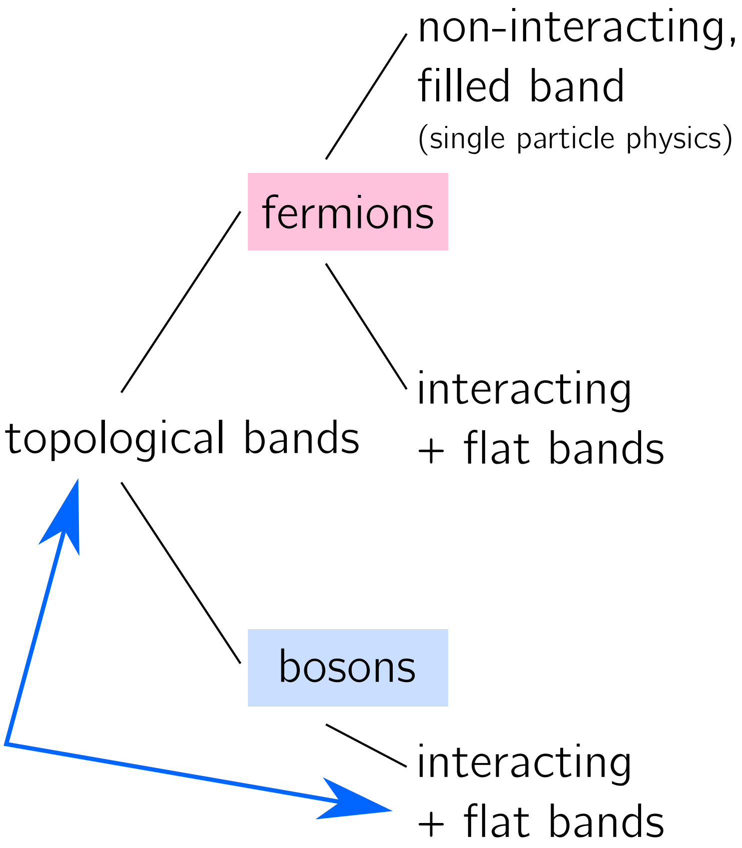
Mikhail Lukin
Norman Yao

Sebastian Huber

motivation: topological states of matter

- topological bands
  - **fermions**
    - non-interacting, filled band (single particle physics) — topological insulators; integer quantum Hall effect (QHE) (edge states, quantized σ_{xy})
    - interacting + flat bands — FQHE with anyonic excitations; open question: non-Abelian states?
  - **bosons**
    - interacting + flat bands — bosonic FQHE states

motivation: topological states of matter

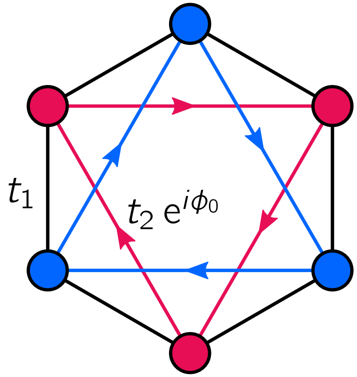


topological bands

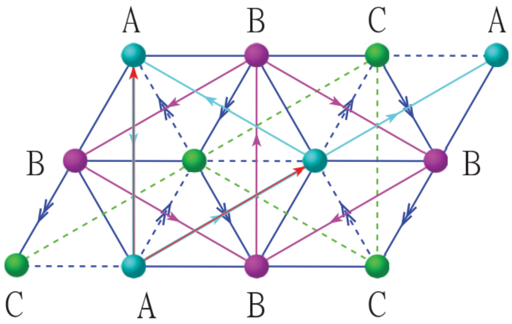
- **fermions**
  - non-interacting, filled band (single particle physics)
    - topological insulators
    - integer quantum Hall effect (QHE)
    - (edge states, quantized σ_{xy})
  - interacting + flat bands
    - FQHE with anyonic excitations
    - open question: non-Abelian states?
- **bosons**
  - interacting + flat bands
    - bosonic FQHE states

topological bands

tunneling phases for neutral particles?



F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988)

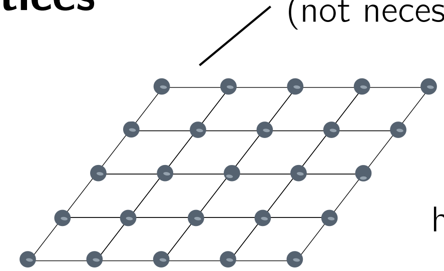


Y. Wang. Phys. Rev. B **86**, 201101 (2012)

motivation: experimental requirements

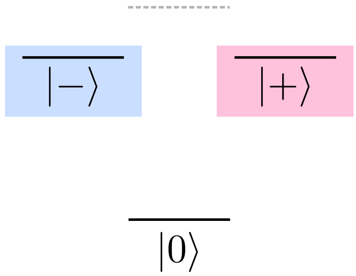
"natural" realization with dipoles on lattices

▷ Two-dimensional lattice


(not necessarily perfect)

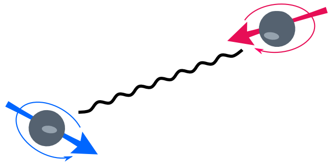
square
triangular
honeycomb
…

▷ Three-level dipole

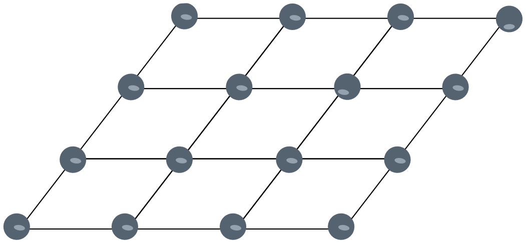


polar molecules
Rydberg atoms
NV centers

▷ Strong dipole-dipole interactions



setup with polar molecules



one molecule
pinned at each lattice site

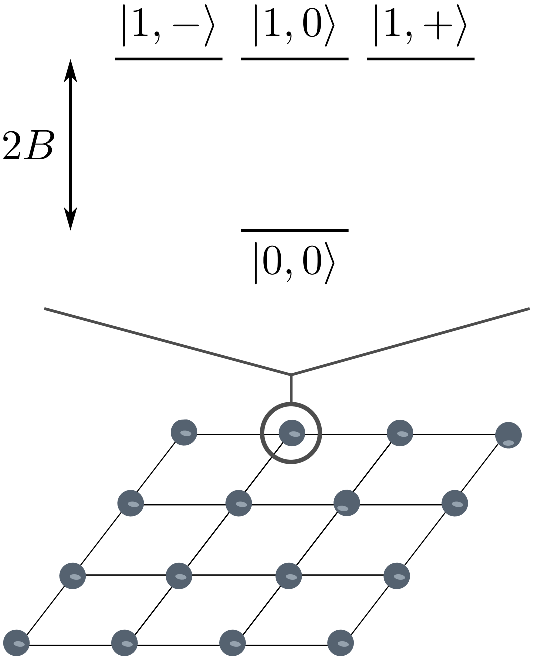
setup with polar molecules

Three level dipole from rotational structure:

$$H_i^{\mathrm{rot}} = \underbrace{B}_{\text{rotational constant}} \underbrace{\boldsymbol{J}_i^2}_{\text{angular momentum}} - \underbrace{\boldsymbol{d}_i}_{\text{dipole moment}} \cdot \underbrace{\boldsymbol{E}}_{\text{external fields}}$$

eigenstates without electric field:

$$|J, m\rangle$$



one molecule
pinned at each lattice site

see also:
N. Yao, Phys. Rev. Lett. **109**, 266804 (2012)
N. Yao, Phys. Rev. Lett. **110**, 185302 (2013)
S. Syzranov, arXiv:1406.0570

setup with polar molecules

Three level dipole from rotational structure:

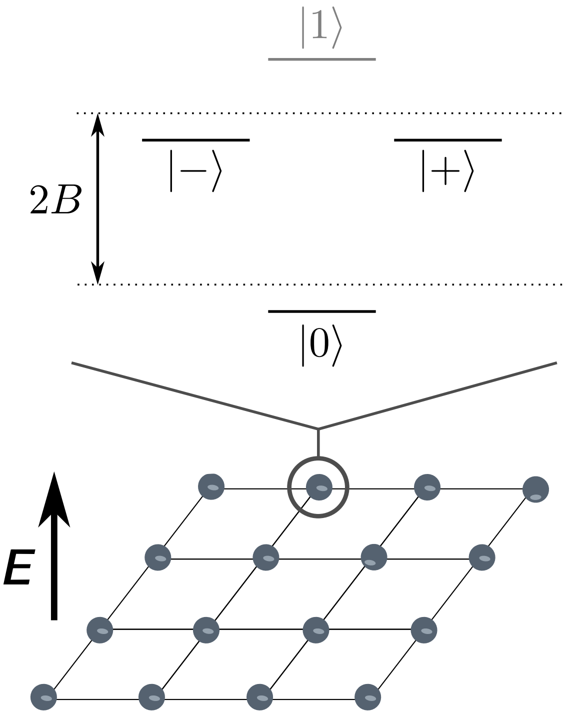
$$H_i^{\mathrm{rot}} = B\boldsymbol{J}_i^2 - \boldsymbol{d}_i \cdot \boldsymbol{E}$$

rotational constant

angular momentum

dipole moment

external fields



one molecule
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eigenstates with electric field:

m is still a good quantum number

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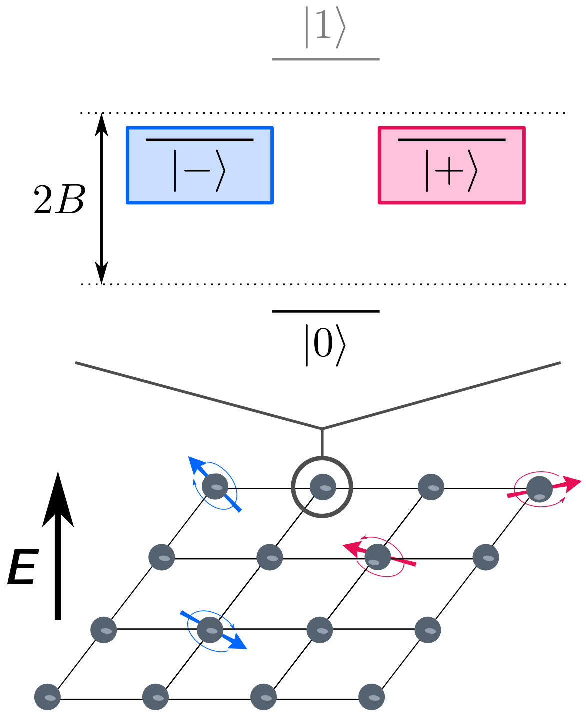

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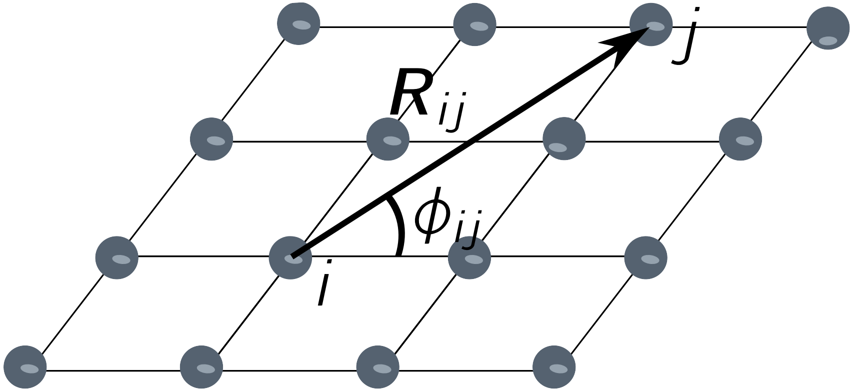
dipole-dipole interactions in the plane

$$H_{ij}^{\mathrm{dd}} = \frac{1}{|\boldsymbol{R}_{ij}|^3}\left[d_i^0 d_j^0 + \frac{1}{2}\left(d_i^- d_j^+ + d_i^+ d_j^-\right) - \frac{3}{2}\left(d_i^- d_j^- \, \mathrm{e}^{2i\phi_{ij}} + d_i^+ d_j^+ \, \mathrm{e}^{-2i\phi_{ij}}\right)\right]$$

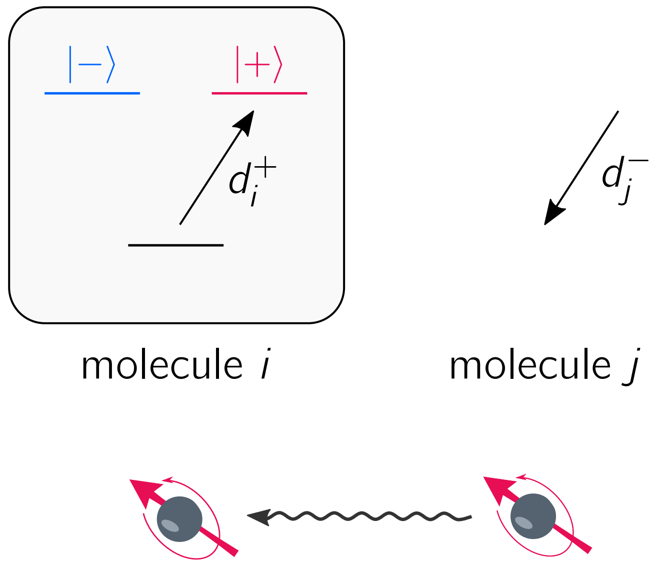


dipolar exchange interactions

dipole-dipole interactions in the plane

$$H_{ij}^{\mathrm{dd}} = \frac{1}{|\boldsymbol{R}_{ij}|^3}\left[d_i^0 d_j^0 + \frac{1}{2}\left(d_i^- d_j^+ + d_i^+ d_j^-\right) - \frac{3}{2}\left(d_i^- d_j^- \, \mathrm{e}^{2i\phi_{ij}} + d_i^+ d_j^+ \, \mathrm{e}^{-2i\phi_{ij}}\right)\right]$$

energy conserving processes:



dipolar exchange interactions

dipole-dipole interactions in the plane

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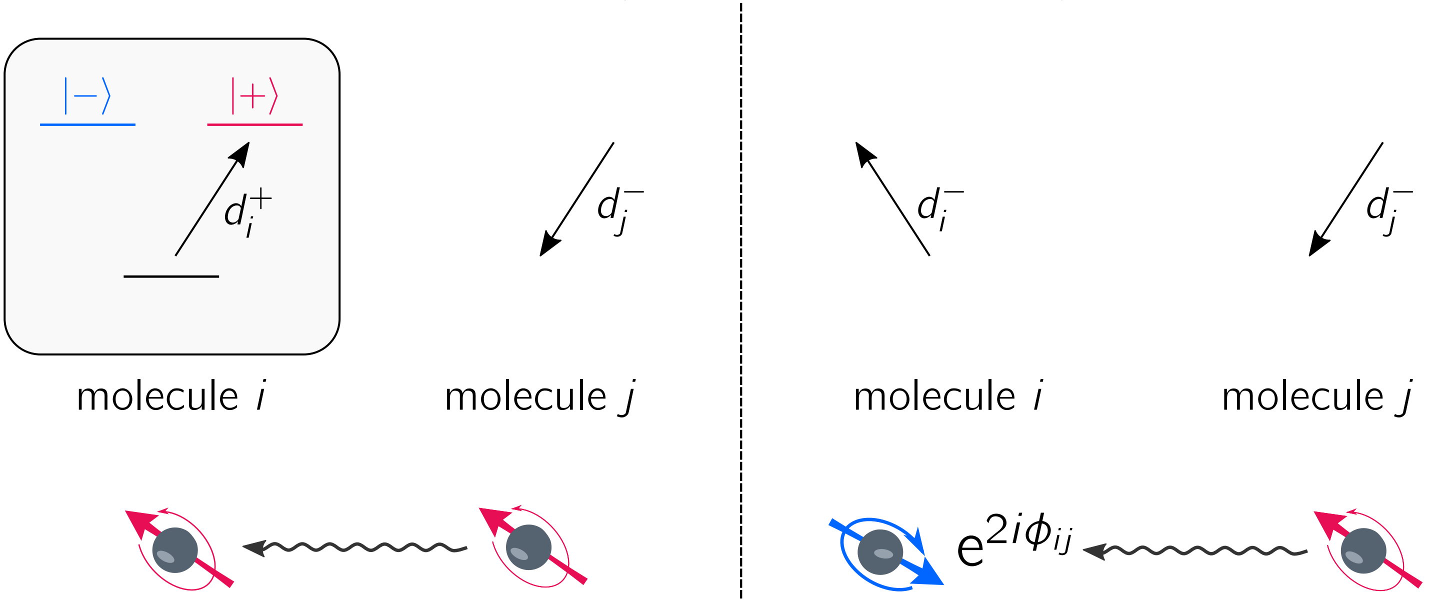
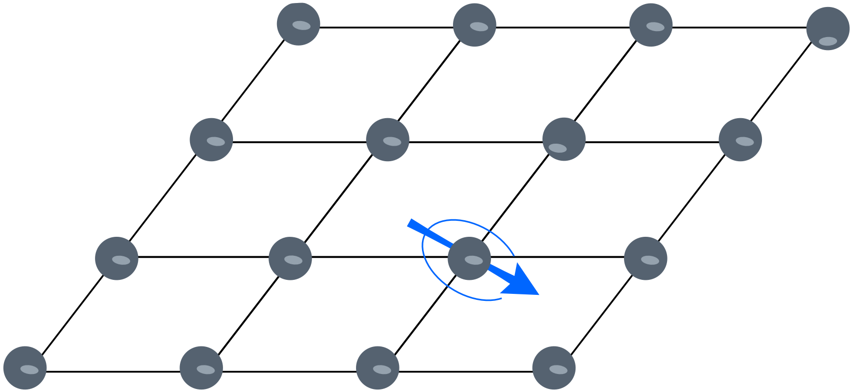


tunneling model



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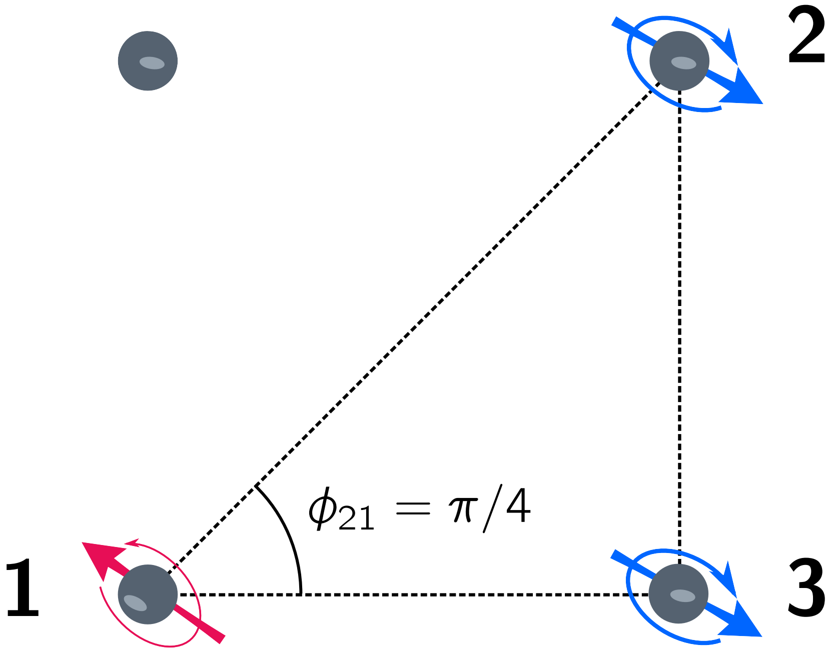
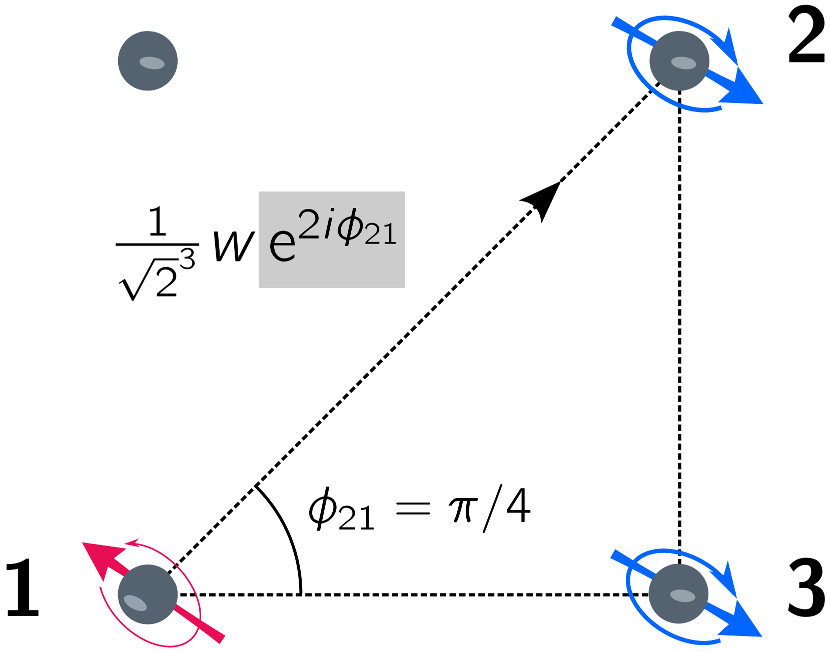
interaction term, not relevant
for single excitation dynamics

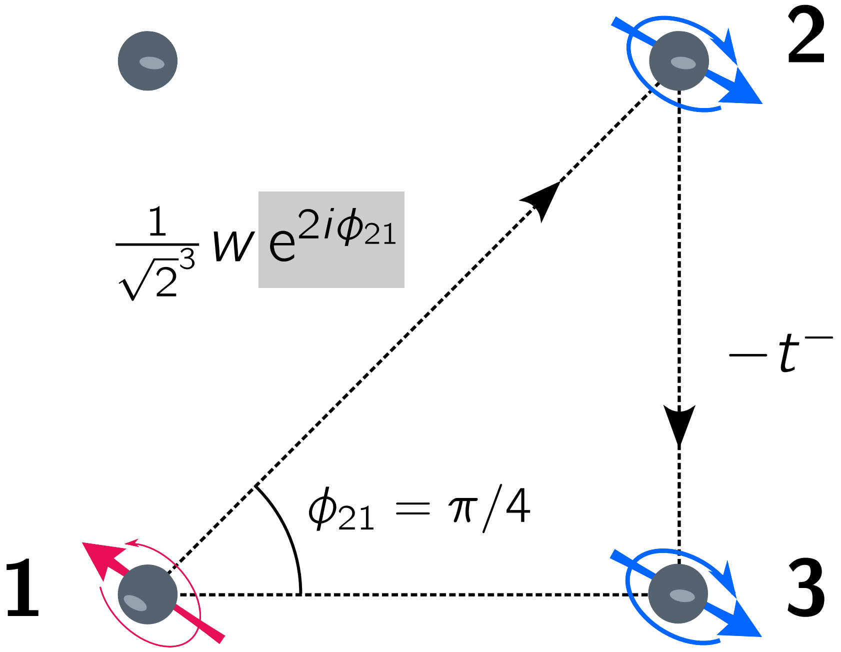

2
$\phi_{21} = \pi/4$
1
3

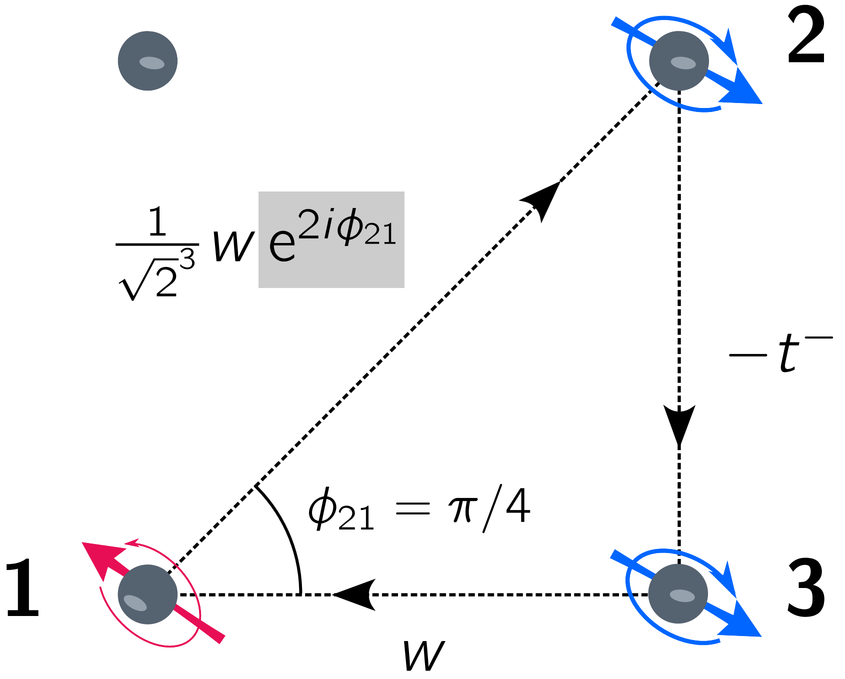
tunneling phases



$$\frac{1}{\sqrt{2}^3} \, w \, e^{2i\phi_{21}}$$

$$\phi_{21} = \pi/4$$


2
$\frac{1}{\sqrt{2}^3} w \, e^{2i\phi_{21}}$
$-t^-$
$\phi_{21} = \pi/4$
1
3



$$\varphi_{\text{flux}} = \arg\left(\prod_{\triangle} t_{ij}\right) = -\pi/2$$

Describe excitations as effective particles (hardcore bosons)

$$b^\dagger_{i,+} = |+\rangle_i\langle 0|_i$$
$$b^\dagger_{i,-} = |-\rangle_i\langle 0|_i$$

$$\psi^\dagger_i = \begin{pmatrix} b^\dagger_{i,+} \\ b^\dagger_{i,-} \end{pmatrix}$$

Tunneling Hamiltonian

$$H = \sum_{i\neq j} \underbrace{\frac{a^3}{|\boldsymbol{R}_{ij}|^3}}_{\text{long-range tunneling}} \psi^\dagger_i \begin{pmatrix} -t^+ & w\,\mathrm{e}^{-2i\phi_{ij}} \\ \underbrace{w\,\mathrm{e}^{2i\phi_{ij}}}_{\text{spin-orbit coupling}} & -t^- \end{pmatrix} \psi_j$$

$\mathcal{T}$-symmetry: $t^+ = t^-$

bosonic model

Describe excitations as effective particles (hardcore bosons)

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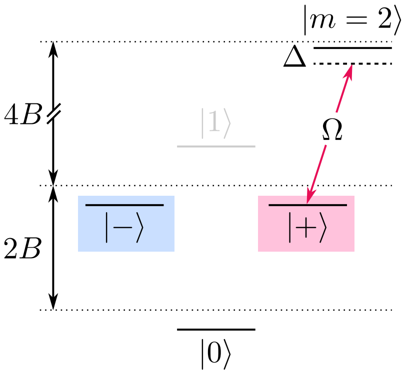

Tunneling Hamiltonian

$$H = \sum_{i \neq j} \frac{a^3}{|\boldsymbol{R}_{ij}|^3} \psi^\dagger_i \begin{pmatrix} -t^+ & w\,\mathrm{e}^{-2i\phi_{ij}} \\ w\,\mathrm{e}^{2i\phi_{ij}} & -t^- \end{pmatrix} \psi_j$$

$$+ \sum_i \psi^\dagger_i \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix} \psi_i$$

~~$\mathcal{T}$-symmetry: $t^+ = t^-$~~

momentum space

$$H = \sum_{\boldsymbol{k}} \psi_{\boldsymbol{k}}^{\dagger} \begin{pmatrix} -t^{+}\,\epsilon_{\boldsymbol{k}}^{0} + \mu & w\,\epsilon_{\boldsymbol{k}}^{-2} \\ w\,\epsilon_{\boldsymbol{k}}^{2} & -t^{-}\,\epsilon_{\boldsymbol{k}}^{0} - \mu \end{pmatrix} \psi_{\boldsymbol{k}}$$

$$\psi_{\boldsymbol{k}} = \frac{1}{\sqrt{N_s}} \sum_{j} \psi_j \, \mathrm{e}^{i\boldsymbol{k}\boldsymbol{R}_j}$$

$$\epsilon_{\boldsymbol{k}}^{m} = \sum_{j\neq 0} \frac{a^3}{|\boldsymbol{R}_j|^3} \, \mathrm{e}^{i\boldsymbol{k}\boldsymbol{R}_j + im\phi_j}$$

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$$= \sum_{\boldsymbol{k}} \psi_{\boldsymbol{k}}^{\dagger} \left(n_{\boldsymbol{k}}^{0}\,\mathbb{1} + \boldsymbol{n}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}\right) \psi_{\boldsymbol{k}}$$

$$n_{\boldsymbol{k}}^{0} = -\bar{t}\,\epsilon_{\boldsymbol{k}}^{0}$$

spin-orbit coupling with: $\boldsymbol{n}_{\boldsymbol{k}} = \begin{pmatrix} w\,\mathrm{Re}\,\epsilon_{\boldsymbol{k}}^{2} \\ w\,\mathrm{Im}\,\epsilon_{\boldsymbol{k}}^{2} \\ \mu + t\,\epsilon_{\boldsymbol{k}}^{0} \end{pmatrix}$

$$\bar{t} = \left(t^{-} + t^{+}\right)/2$$

$$t = \left(t^{-} - t^{+}\right)/2$$

$$H = \sum_{\boldsymbol{k}} \psi^\dagger_{\boldsymbol{k}} \begin{pmatrix} -t^+\,\epsilon^0_{\boldsymbol{k}} + \mu & w\,\epsilon^{-2}_{\boldsymbol{k}} \\ w\,\epsilon^2_{\boldsymbol{k}} & -t^-\,\epsilon^0_{\boldsymbol{k}} - \mu \end{pmatrix} \psi_{\boldsymbol{k}}$$

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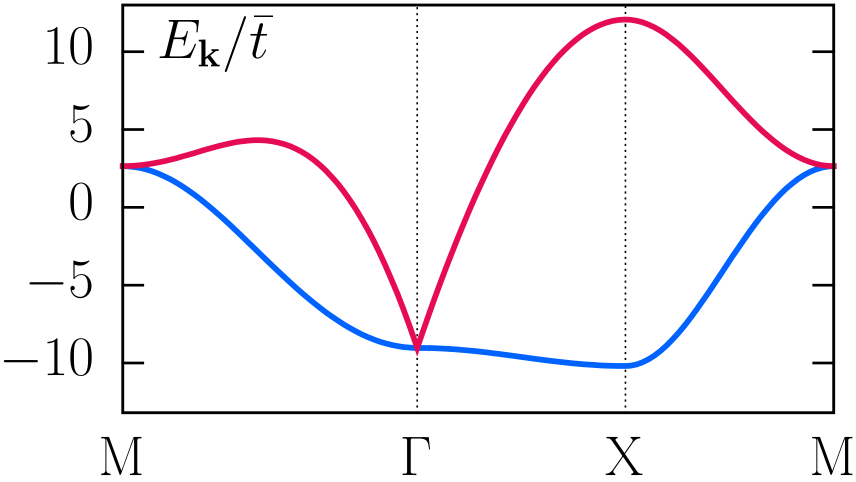
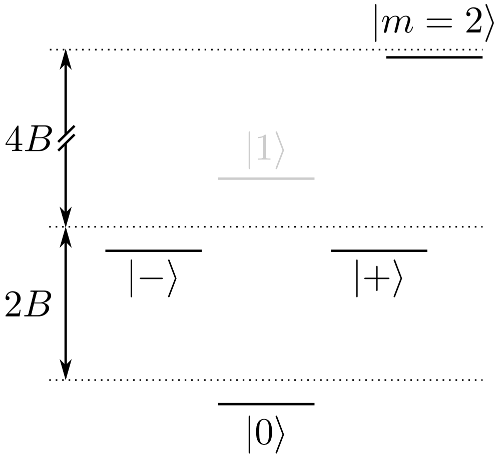
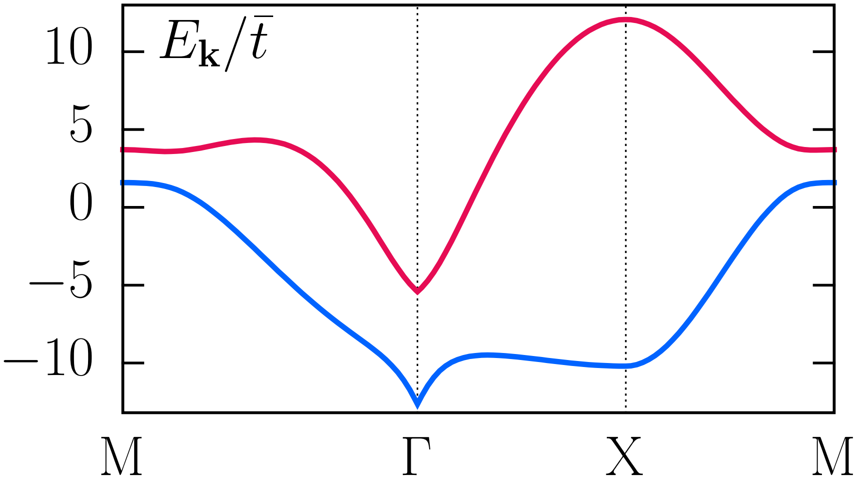
$$t = \left(t^- - t^+\right)/2$$

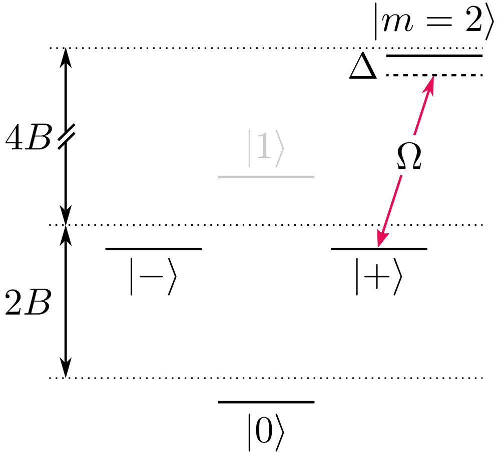
$$E_\pm(\boldsymbol{k}) = n^0_{\boldsymbol{k}} \pm |\boldsymbol{n}_{\boldsymbol{k}}| = -\bar{t}\,\epsilon^0_{\boldsymbol{k}} \pm \sqrt{w^2\left|\epsilon^2_{\boldsymbol{k}}\right|^2 + \left(\mu + t\,\epsilon^0_{\boldsymbol{k}}\right)^2}$$

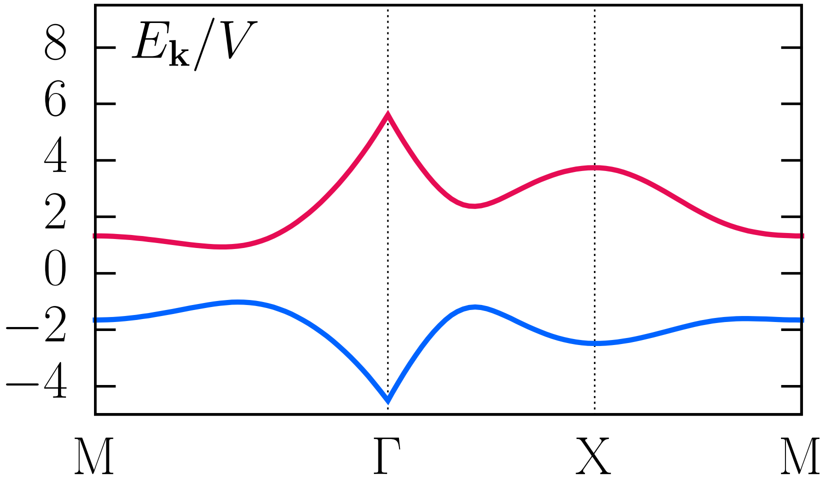
band structure






$E_{\mathbf{k}}/\bar{t}$
10
5
0
-5
-10
M
Γ
X
M


$|m=2\rangle$
Δ
4B
Ω
$|1\rangle$
$|-\rangle$
$|+\rangle$
2B
$|0\rangle$

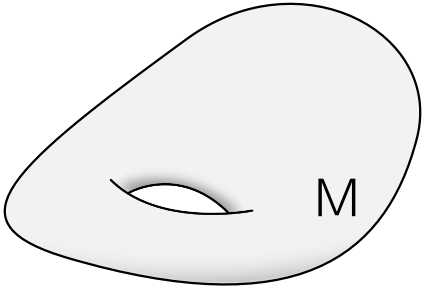


← after tuning parameters

are these bands topological?

Gauss-Bonnet-Theorem

For a two-dimensional closed manifold M:

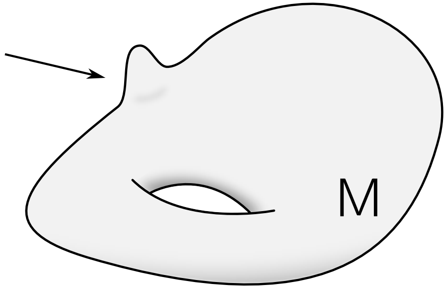


$$\int_{\mathrm{M}} \mathrm{d}S\, K = 4\pi(1 - g)$$

locally defined quantity
(Gaussian curvature)

topological index
(genus $g \in \mathbb{N}_0$)

Review on topological insulators:
M. Hasan and C. Kane, Rev. Mod. Phys. **82**, 3045 (2010)



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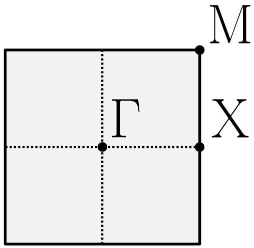


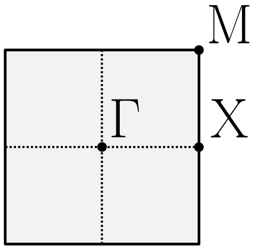
Chern number

$$\frac{1}{2\pi}\int_{\mathrm{BZ}} \mathrm{d}^2\mathbf{k}\,\mathcal{B}_\nu(\boldsymbol{k}) = C_\nu$$

locally defined quantity
(Berry curvature)

topological index
(Chern number $C_\nu \in \mathbb{Z}$)

Chern number



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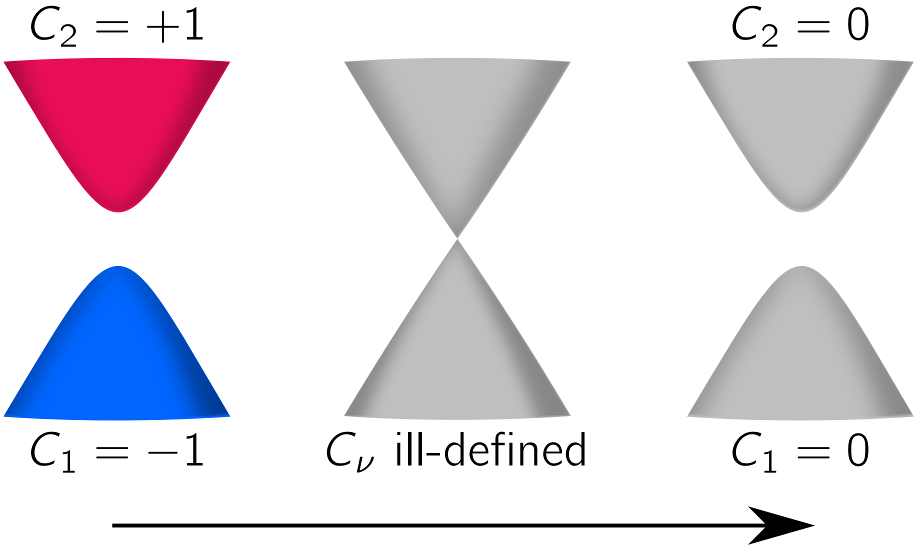
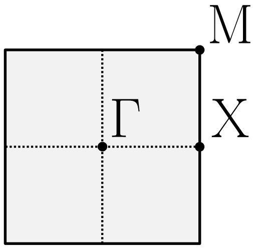




Chern number

$$\frac{1}{2\pi}\int_{\mathrm{BZ}} \mathrm{d}^2\mathbf{k}\,\mathcal{B}_\nu(\boldsymbol{k}) = C_\nu$$

$$\mathcal{B}_\nu(\boldsymbol{k}) = \partial_{k_x}\mathcal{A}^y_\nu(\boldsymbol{k}) - \partial_{k_y}\mathcal{A}^x_\nu(\boldsymbol{k})$$

$$\mathcal{A}^j_\nu(\boldsymbol{k}) = i\langle u_\nu(\boldsymbol{k})|\partial_{k_j}|u_\nu(\boldsymbol{k})\rangle$$

Berry vector potential
(gauge dependent)

Bloch functions
of the ν-th band

Chern number as a winding number

Gapped system: $\boldsymbol{n}_{\boldsymbol{k}} = \begin{pmatrix} w \,\mathrm{Re}\, \epsilon^2_{\boldsymbol{k}} \\ w \,\mathrm{Im}\, \epsilon^2_{\boldsymbol{k}} \\ \mu + t\, \epsilon^0_{\boldsymbol{k}} \end{pmatrix} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$

$E_\pm(\boldsymbol{k}) = n^0_{\boldsymbol{k}} \pm |\boldsymbol{n}_{\boldsymbol{k}}|$

Chern number as a winding number

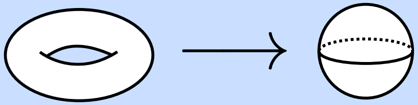
Gapped system: $\boldsymbol{n}_{\boldsymbol{k}} = \begin{pmatrix} w \, \mathrm{Re} \, \epsilon^2_{\boldsymbol{k}} \\ w \, \mathrm{Im} \, \epsilon^2_{\boldsymbol{k}} \\ \mu + t \, \epsilon^0_{\boldsymbol{k}} \end{pmatrix} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$

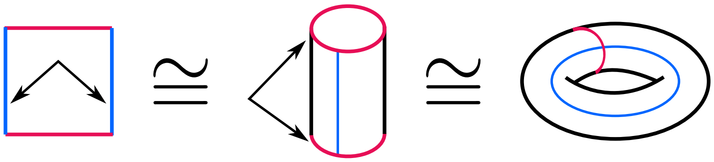
$$E_\pm(\boldsymbol{k}) = n^0_{\boldsymbol{k}} \pm |\boldsymbol{n}_{\boldsymbol{k}}|$$

normalized vector!

$$\hat{\boldsymbol{n}} : T^2 \longrightarrow S^2$$

$\hat{\boldsymbol{n}}$:





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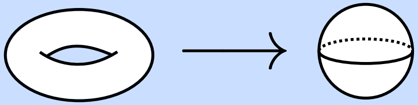
normalized vector!

$$\hat{\boldsymbol{n}} : T^2 \longrightarrow S^2$$

$\hat{\boldsymbol{n}} :$



How can we classify such mappings $\hat{\boldsymbol{n}}$?

- winding number:
  how many times does $\hat{\boldsymbol{n}}$ wrap the sphere (as we integrate over the BZ)?

Chern number as a winding number

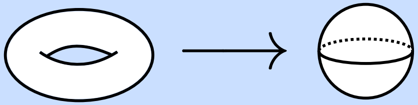
Gapped system: $\boldsymbol{n}_{\boldsymbol{k}} = \begin{pmatrix} w\,\mathrm{Re}\,\epsilon^2_{\boldsymbol{k}} \\ w\,\mathrm{Im}\,\epsilon^2_{\boldsymbol{k}} \\ \mu + t\,\epsilon^0_{\boldsymbol{k}} \end{pmatrix} \in \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$

$$E_\pm(\boldsymbol{k}) = n^0_{\boldsymbol{k}} \pm |\boldsymbol{n}_{\boldsymbol{k}}|$$

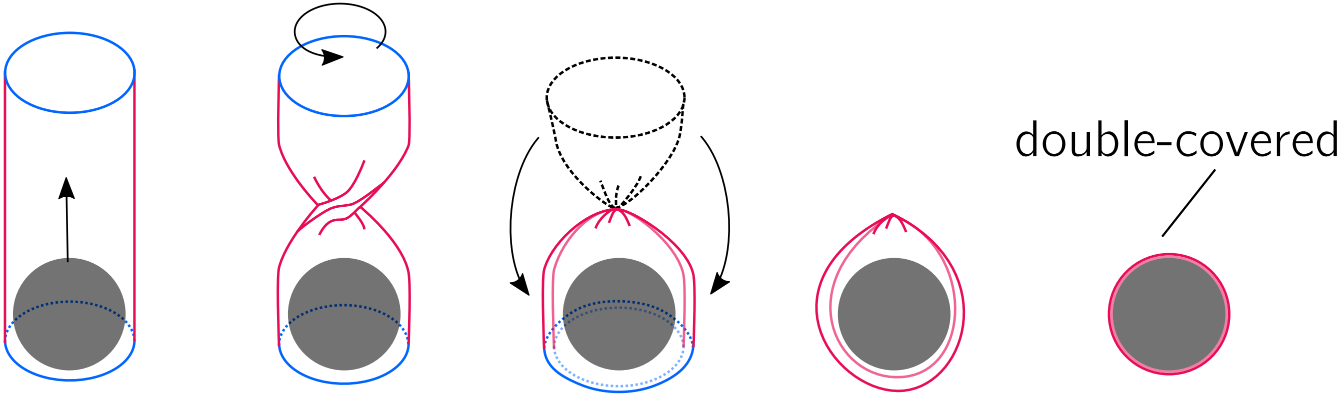
normalized vector!

$\hat{\boldsymbol{n}} : T^2 \longrightarrow S^2$

$\hat{\boldsymbol{n}}$:



How does a nontrivial mapping with $C = 2$ look like?



Chern number as a winding number

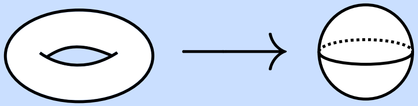
Gapped system: $\boldsymbol{n}_{\boldsymbol{k}} = \begin{pmatrix} w\,\mathrm{Re}\,\epsilon^2_{\boldsymbol{k}} \\ w\,\mathrm{Im}\,\epsilon^2_{\boldsymbol{k}} \\ \mu + t\,\epsilon^0_{\boldsymbol{k}} \end{pmatrix} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$

$$E_{\pm}(\boldsymbol{k}) = n^0_{\boldsymbol{k}} \pm |\boldsymbol{n}_{\boldsymbol{k}}|$$

normalized vector!

$$\hat{\boldsymbol{n}} : T^2 \longrightarrow S^2$$

$\hat{\boldsymbol{n}}$:



- second homotopy group of two-sphere:

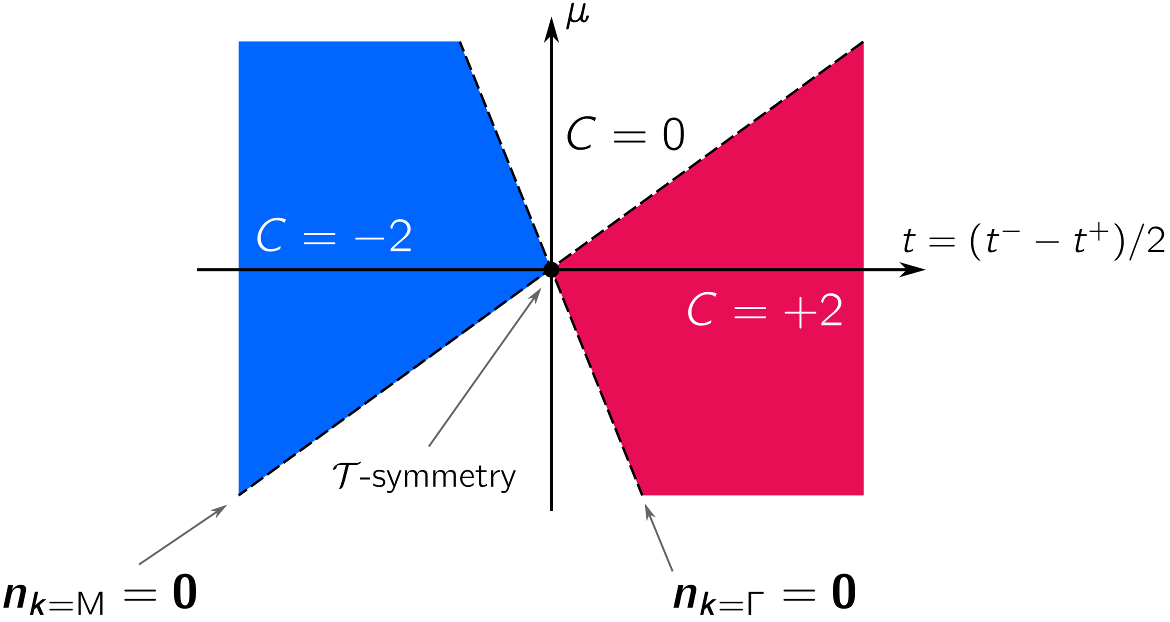
$$\pi_2(S^2) = \mathbb{Z}$$

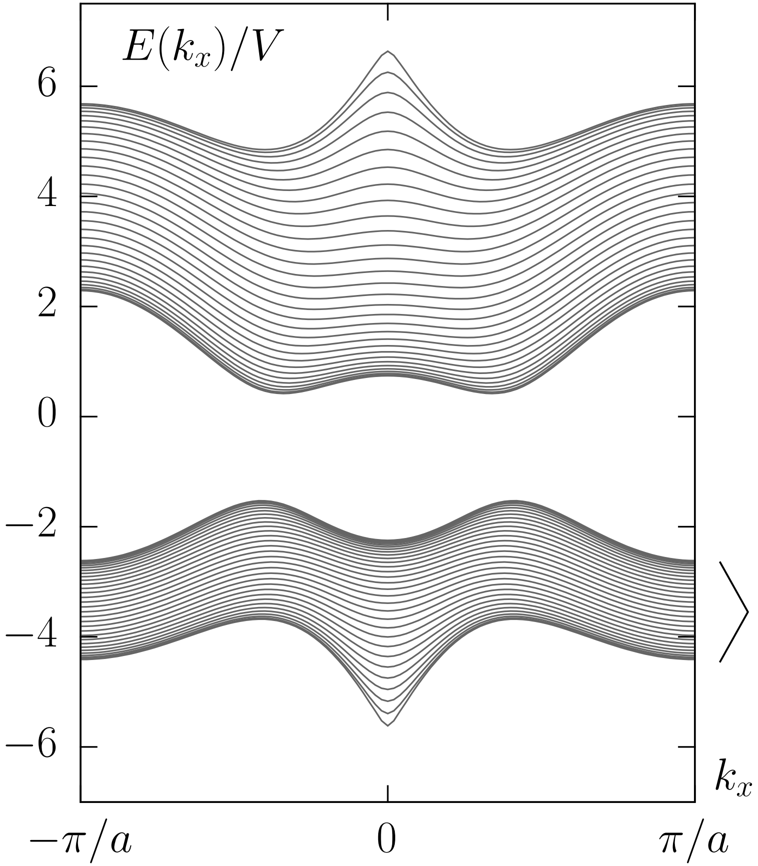
The Chern number or
the winding number of the vector $\hat{\boldsymbol{n}}$

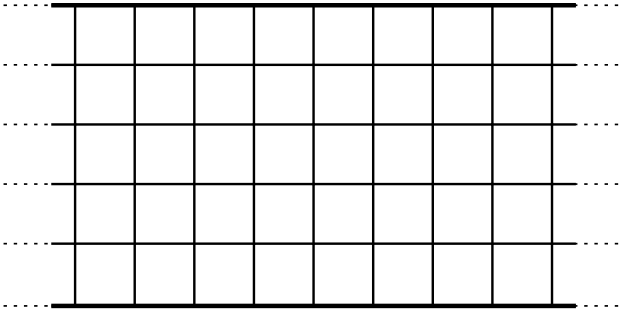
topological phase diagram

C as winding number

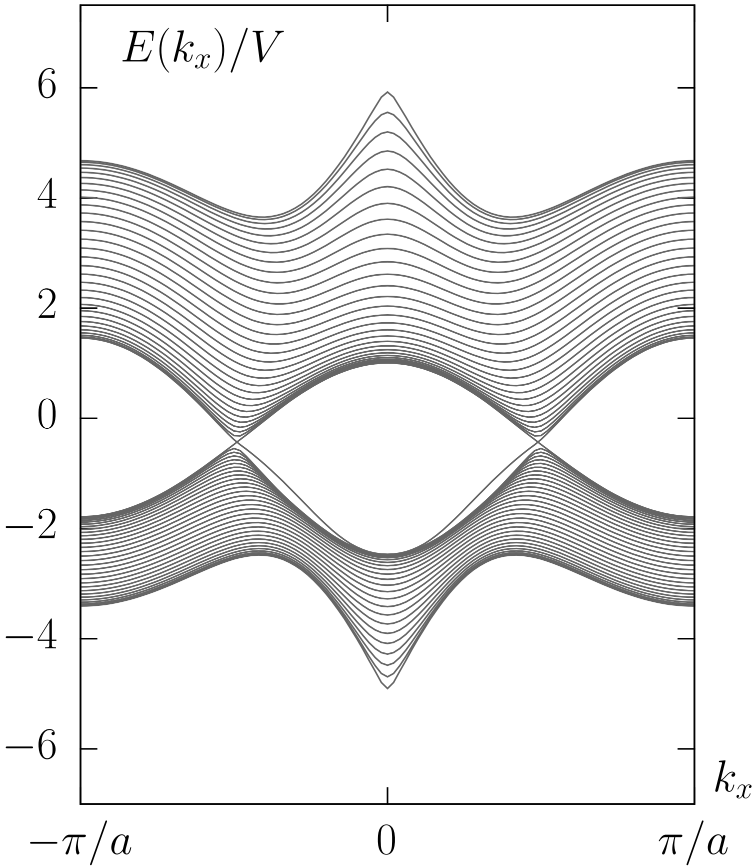
$$C = \frac{1}{4\pi} \int_{\mathrm{BZ}} \mathrm{d}^2\boldsymbol{k}\, (\partial_{k_x} \hat{\boldsymbol{n}}_{\boldsymbol{k}} \times \partial_{k_y} \hat{\boldsymbol{n}}_{\boldsymbol{k}}) \cdot \hat{\boldsymbol{n}}_{\boldsymbol{k}}$$

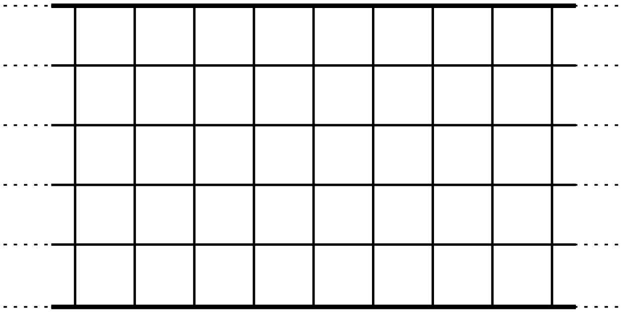




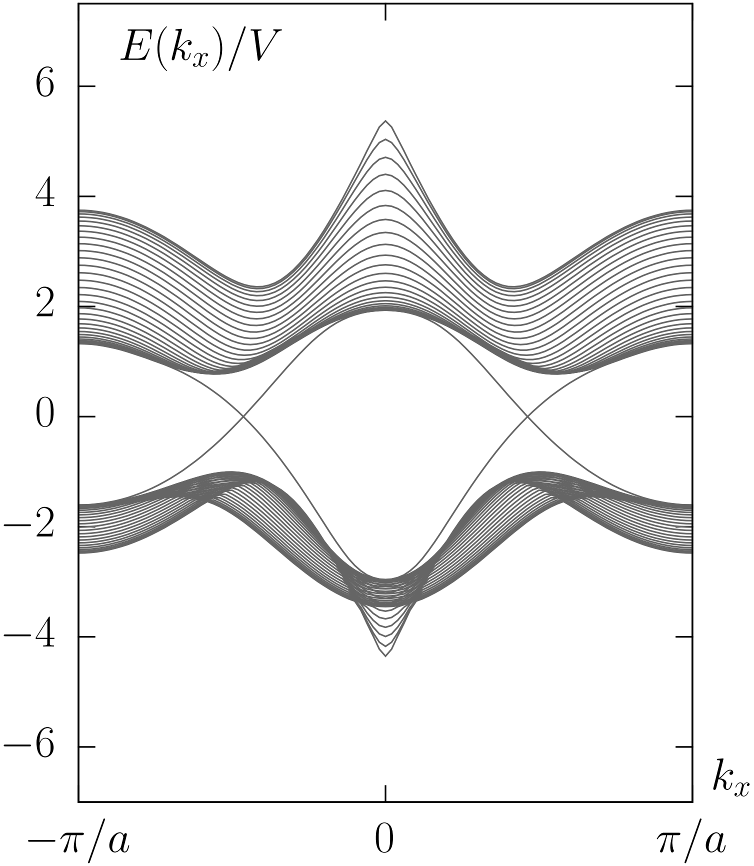


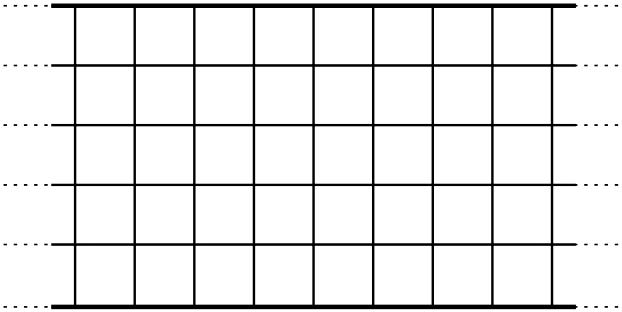
number of bands = N_y

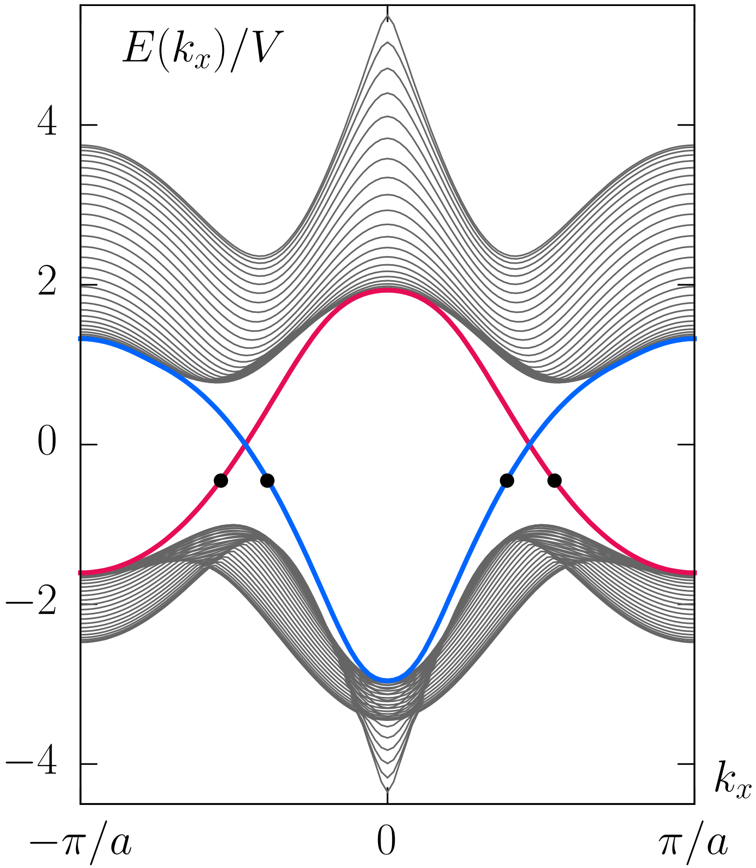

$E(k_x)/V$
6
4
2
0
-2
-4
-6
k_x
$-\pi/a$
0
π/a

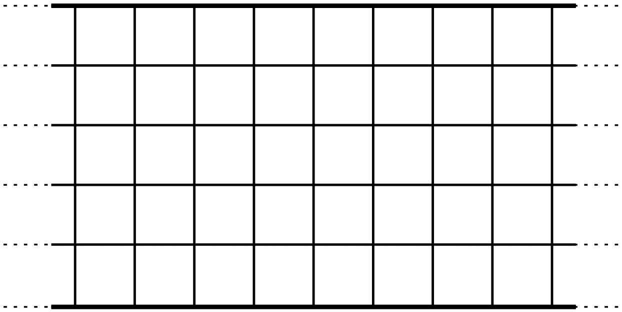


edge states









bulk–edge correspondence

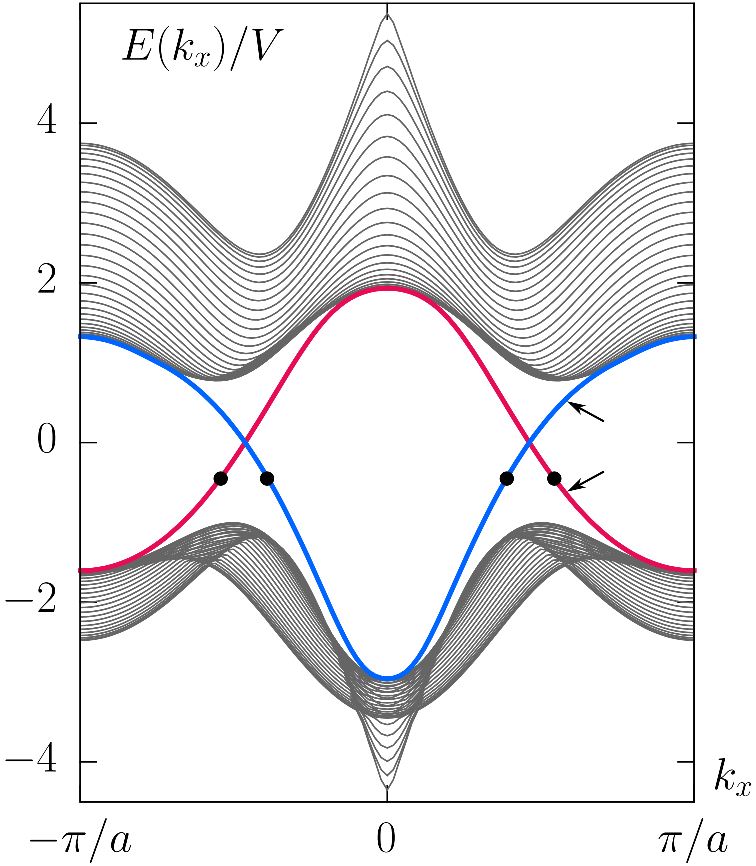
Chern number

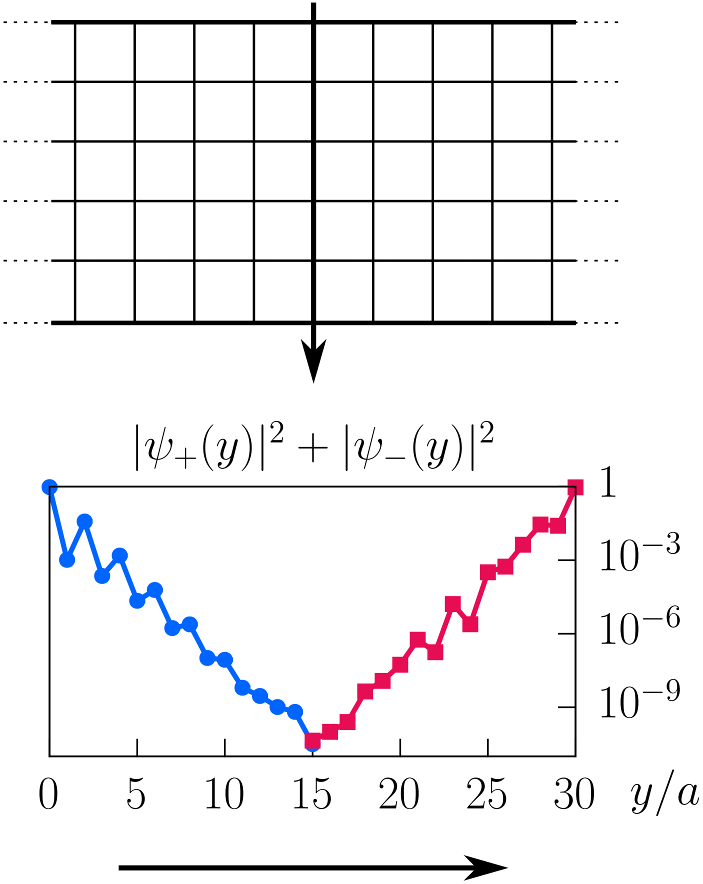
$\leftrightarrow$

number of edge modes

Hatsugai, Phys. Rev. Lett. **71**, 3697 (1993)

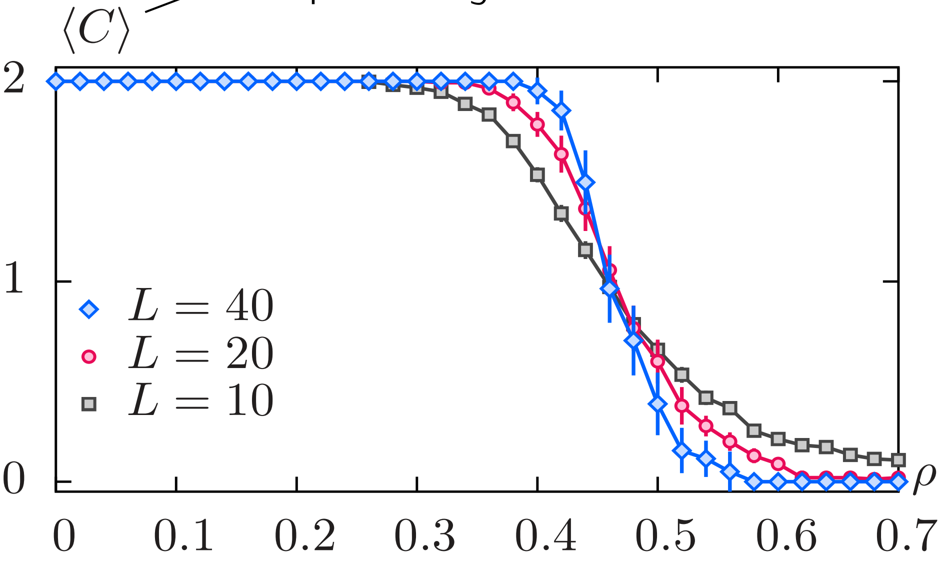
edge states


$E(k_x)/V$
4
2
0
-2
-4
k_x
$-\pi/a$
0
π/a


$|\psi_+(y)|^2 + |\psi_-(y)|^2$
1
10^{-3}
10^{-6}
10^{-9}
0
5
10
15
20
25
30
y/a

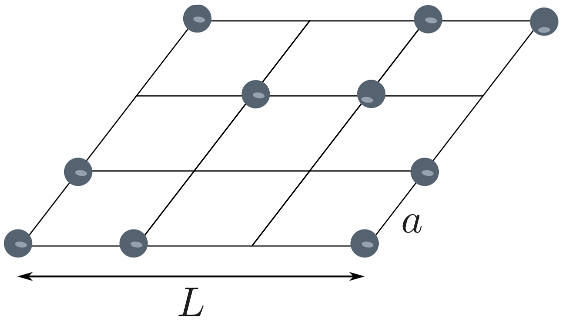
Chern number in the disordered system

What if the lattice is not perfect?



sample-averaged Chern number

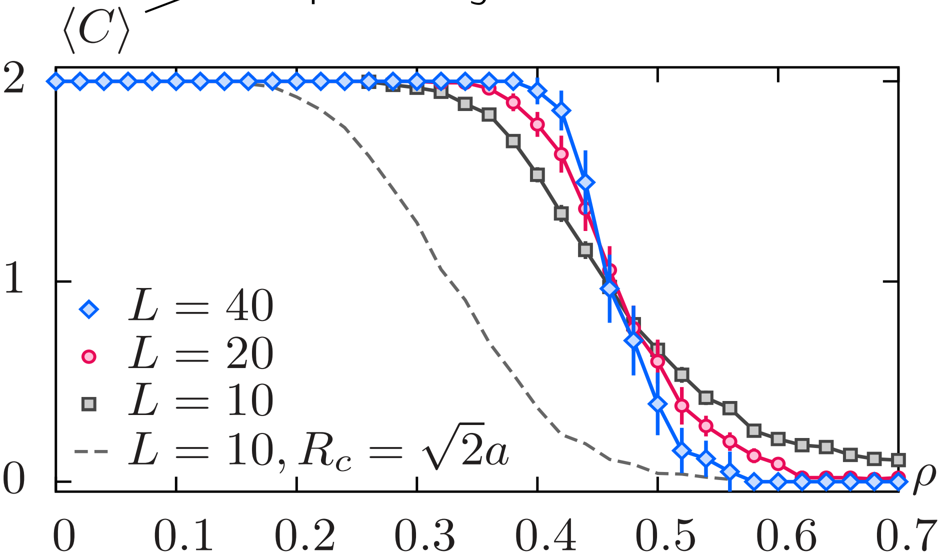
density of defects



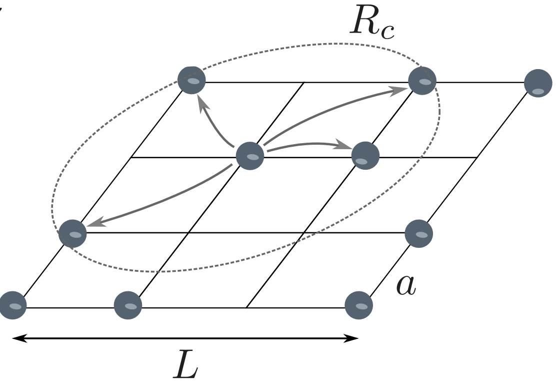
Chern number in the disordered system

What if the lattice is not perfect?

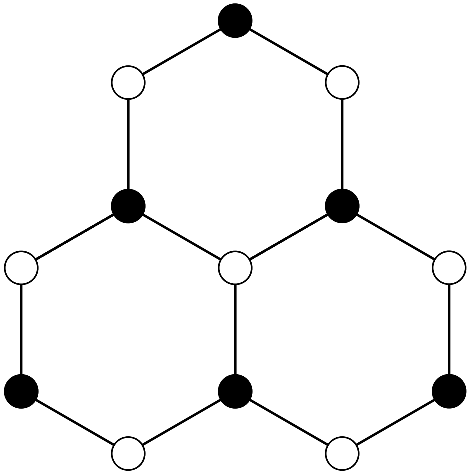
sample-averaged Chern number



density of defects

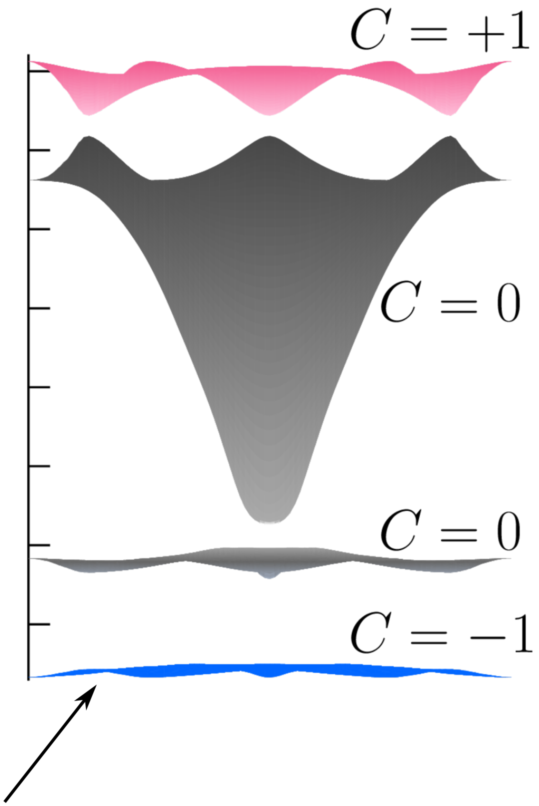


honeycomb lattice: flat bands



2 sites per unit cell × 2 orbitals

▷ 4 bands



flatness = bandwidth/bandgap ≈ 6.4

- ▷ dipolar exchange interactions & broken time-reversal symmetry naturally lead to topological bands
- ▷ Chern number depends on the underlying lattice
  square lattice: $C = 2$
  honeycomb: $C = 1 - 4$
- ▷ Robust against missing dipoles (lattice sites)
- ▷ Numerical candidate for the interacting C = 2 system[1,2]:

  bosonic fractional Chern insulator state at $\nu = {}^{2}\!/_{3}$: Halperin (221) state?

Thank you!

arXiv:1410.5667

[1] Y. Wang, Phys. Rev. B **86**, 201101 (2012)
[2] G. Möller, Phys. Rev. Lett. **103**, 105303 (2009)

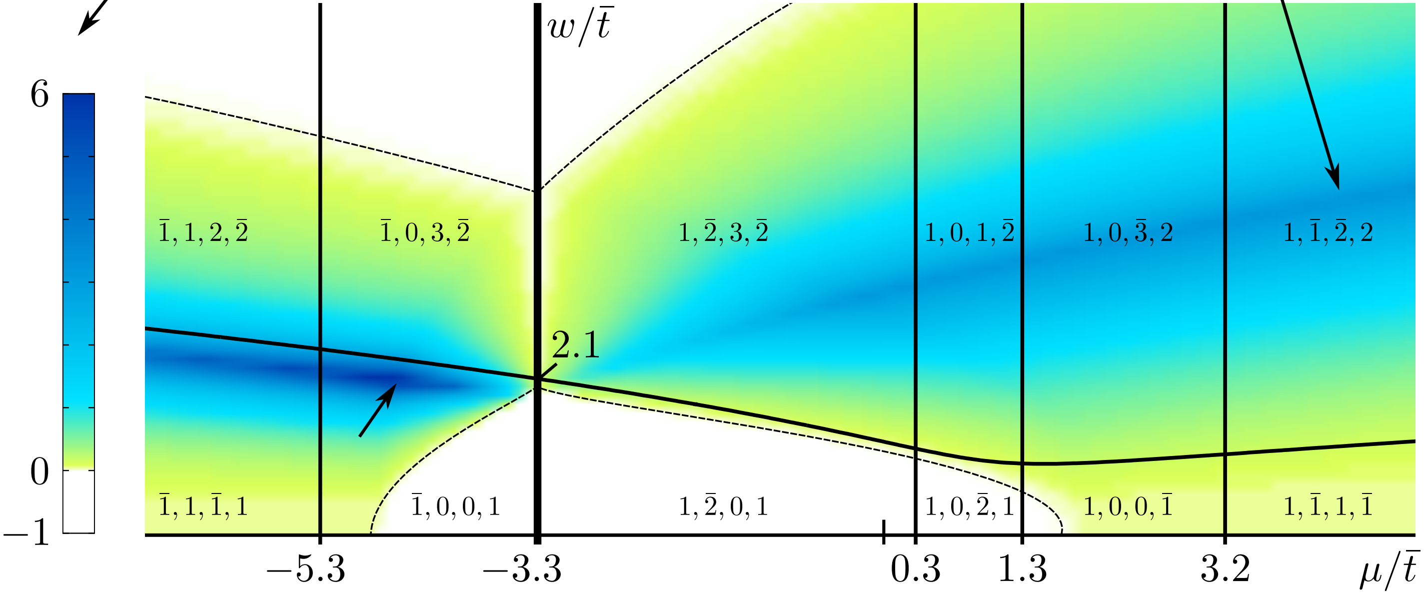
Halperin state

$$\Psi_{(l;m;n)} = \prod_{i \neq j} \left(z_i^{\downarrow} - z_j^{\downarrow}\right)^{l} \prod_{i \neq j} \left(z_i^{\uparrow} - z_j^{\uparrow}\right)^{m} \prod_{i,j} \left(z_i^{\uparrow} - z_j^{\downarrow}\right)^{n} \mathrm{e}^{-\frac{1}{4}\sum_{j,\alpha}\left|z_j^{\alpha}\right|^2}$$

honeycomb lattice: topological phase diagram

flatness

Chern numbers $(C_1, \ldots, C_4)$



How to calculate the Chern number in the disordered system?

impose twisted boundary conditions

$$\psi(x+L,y) = \mathrm{e}^{i\theta_x}\,\psi(x,y)$$

$$\psi(x,y+L) = \mathrm{e}^{i\theta_y}\,\psi(x,y)$$

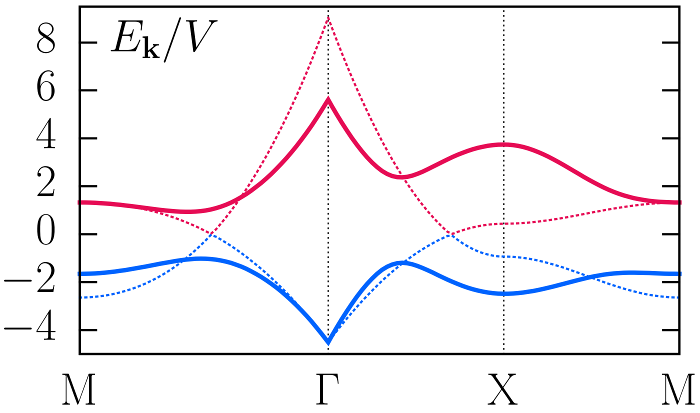
$$C = \frac{1}{2\pi}\iint \mathrm{d}\theta_x \mathrm{d}\theta_y\, F(\theta_x,\theta_y)$$

many-body Berry curvature

Slater-determinant of lower "band" $\Psi(\theta_x,\theta_y)$

$$F(\theta_x,\theta_y) = \mathrm{Im}\Big(\Big\langle \frac{\partial\Psi}{\partial\theta_y}\Big|\frac{\partial\Psi}{\partial\theta_x}\Big\rangle - \Big\langle \frac{\partial\Psi}{\partial\theta_x}\Big|\frac{\partial\Psi}{\partial\theta_y}\Big\rangle\Big)$$

dispersion relation x/1 model



Landau levels

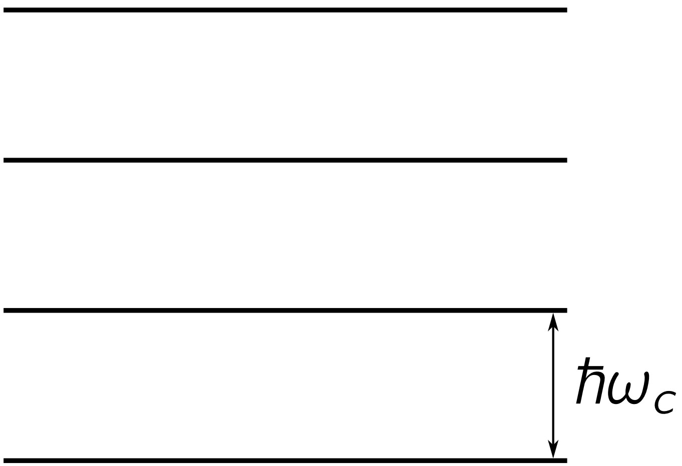
Analogy with Landau levels:

$$H = \hbar\omega_c\left(n + \tfrac{1}{2}\right)$$

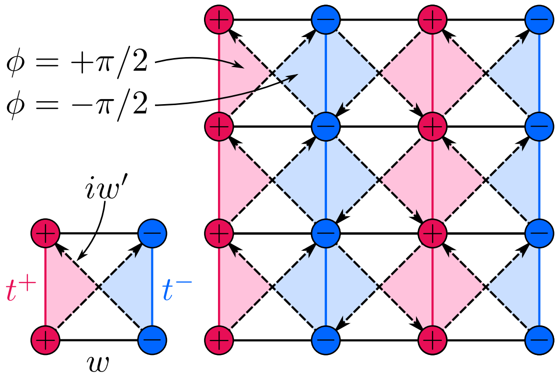
with $\omega_c = eB/m$

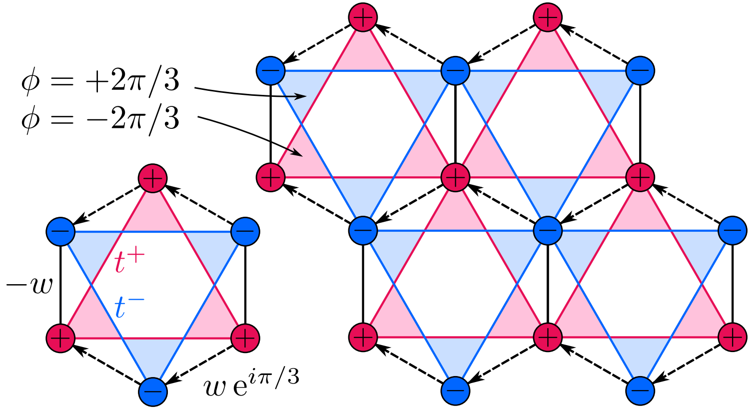
▷ perfectly flat band(s) with Chern number $C = 1$

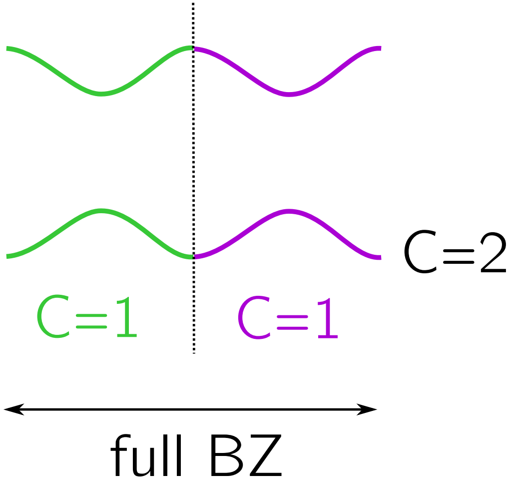
Hall conductivity: $\sigma_{xy} = \frac{e^2}{h} \times C$

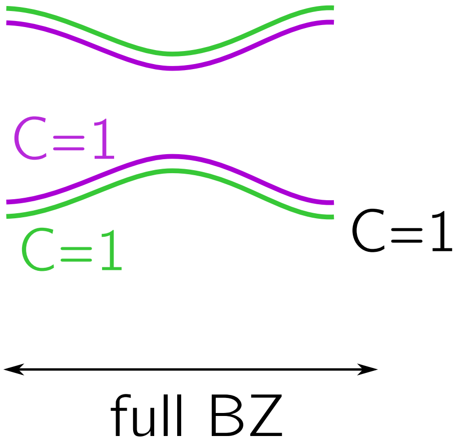


double-layer picture









1 + 1 = 2

1 + 1 = 2 x 1

outline

Setup:
▷ Polar molecules
▷ Dipolar exchange interactions

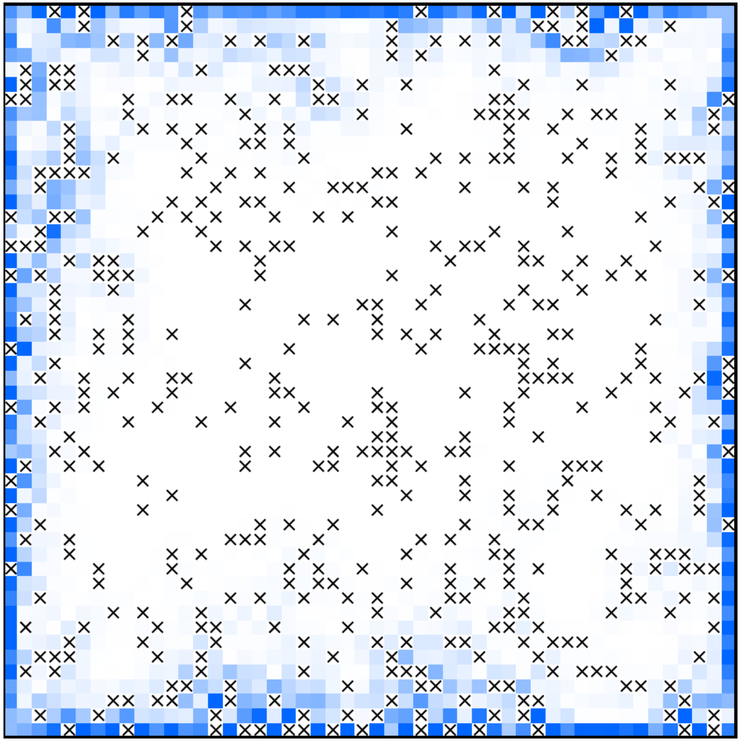
Two-band model
▷ Mapping to bosons
▷ Introduction to topological bands

Square lattice:
▷ Edge states
▷ Robustness against disorder

Honeycomb lattice:
▷ flat bands
▷ phase diagram

What if the lattice is not perfect?

edge states are robust against disorder:



20% of the molecules randomly removed