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You can find detailed information about the lecture and the exercises on the website <https://www.itp3.uni-stuttgart.de/teaching/quantumfieldtheory/index.html>. Problems are divided into two different groups. *Written* problems have to be handed in and will be graded by the tutors. *Oral* problems have to be prepared for the exercise session and will be presented by one of the students. In order to be admitted to the exam, we require (a) 80% of the written problems to be solved or treated adequately, (b) 66% of the oral problems to be prepared and (c) two problems to be presented at the black board.

The first problem set serves as a repetition for some important concepts from classical field theory.

### Problem 1: Functional derivative (Oral)

For a given manifold  $\mathcal{M}$  of functions  $\phi$  and a functional  $F$  with  $F : \mathcal{M} \rightarrow \mathbb{R}$  or  $\mathbb{C}$ , the functional derivative  $\frac{\delta F[\phi]}{\delta \phi}$  is defined as

$$\int dx' \frac{\delta F[\phi]}{\delta \phi(x')} f(x') = \lim_{\varepsilon \rightarrow 0} \frac{F[\phi(x) + \varepsilon f(x)] - F[\phi(x)]}{\varepsilon} = \left. \frac{d}{d\varepsilon} F[\phi + \varepsilon f] \right|_{\varepsilon=0} \quad (1)$$

for all test functions  $f \in \mathcal{M}$ .

a) Show that for two functionals  $F$  and  $G$

$$\frac{\delta(F + \lambda G)[\phi]}{\delta \phi(x)} = \frac{\delta F[\phi]}{\delta \phi(x)} + \lambda \frac{\delta G[\phi]}{\delta \phi(x)} \quad \text{for } \lambda \in \mathbb{R} \quad (2)$$

$$\frac{\delta(FG)[\phi]}{\delta \phi(x)} = \frac{\delta F[\phi]}{\delta \phi(x)} G[\phi] + F[\phi] \frac{\delta G[\phi]}{\delta \phi(x)}, \quad \text{with } (FG)[\phi] = F[\phi]G[\phi]. \quad (3)$$

If  $G[\phi]$  is a function-like functional, i.e. can be treated as a function itself

$$\frac{\delta F[G[\phi]]}{\delta \phi(y)} = \int dx \frac{\delta F[G]}{\delta G(x)} \frac{\delta G[\phi](x)}{\delta \phi(y)}. \quad (4)$$

b) Calculate the functional derivative  $\frac{\delta F[\phi]}{\delta \phi(x)}$  for the following functionals:

$$F[\phi] = \int dx' K(y, x') \phi(x'), \quad \text{where } K(y, x') \text{ is a so called } \textit{integral kernel} \quad (5)$$

$$F[\phi] = \phi(y) \quad (6)$$

$$F[\phi] = \phi'(y) \quad (7)$$

$$F[\phi] = \int dy f(\phi(y), \phi'(y)) \quad \text{for a differentiable function } f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \quad (8)$$

**Problem 2: Lorentz covariance (Oral)**

This exercise should serve as a **brief** revision of Lorentz covariance and the covariant formulation of classical electromagnetism. In the following, we will work in units where  $c = 1$ . Further, we will make use of *Einstein notation* where summation over indices appearing twice is assumed.

We first introduce the four-vector

$$x^\mu = (t, \mathbf{r}), \quad \mu = 0, 1, 2, 3, \quad (9)$$

which we will call a **contravariant** vector or tensor of first order. The vector  $x_\mu$  is called **covariant** vector. In special relativity, the metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (10)$$

and the relationship between co- and contravariant vectors is given by

$$x_\mu = g_{\mu\nu} x^\nu. \quad (11)$$

A **Lorentz vector** is an object that under a Lorentz transformation  $\Lambda^\mu{}_\nu$  transforms like

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu. \quad (12)$$

In tensors of higher order, each index transforms as a Lorentz vector, e.g.

$$\tilde{A}^{\alpha\beta\gamma}{}_{\delta\varepsilon} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \Lambda^\gamma{}_\xi \Lambda^\delta{}_\rho \Lambda^\varepsilon{}_\sigma A^{\mu\nu\xi}{}_{\rho\sigma}. \quad (13)$$

A **Lorentz scalar** is a quantity that is invariant under Lorentz transformations.

a) Show that  $x^\mu x_\mu$  is a Lorentz scalar, i.e. show that  $x^\mu x_\mu = \tilde{x}^\sigma \tilde{x}_\sigma$ .

Another important object is the four-gradient

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = (\partial_t, \nabla). \quad (14)$$

b) Compute the d'Alembert operator  $\partial^\mu \partial_\mu$ . Is this quantity a Lorentz scalar?

In a covariant formulation of electromagnetism, the electric and magnetic field  $\mathbf{E}$  and  $\mathbf{B}$ , respectively, are given by the anti-symmetric field tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (15)$$

The fields can also be described by the four-potential  $A_\mu = (\Phi, -\mathbf{A})$ , where  $\Phi$  is a scalar potential and  $\mathbf{A}$  is a vector potential.

- c) Show that  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  reproduces the fields  $\mathbf{E}$  and  $\mathbf{B}$ .
- d) Show that  $F_{\mu\nu}$  is invariant under the gauge transformation  $A_\mu \rightarrow A_\mu - \partial_\mu f$ , where  $f$  is an arbitrary function.
- e) Show that in Lorenz gauge,  $\partial_\nu A^\nu = 0$ , and for no external sources, the Maxwell equations  $\partial_\mu F^{\mu\nu} = 0$  reduce to  $\partial^\mu \partial_\mu A^\nu = 0$ .

### Problem 3: Maxwell equations (Written, 4 points)

The Maxwell equations for classical electromagnetism in vacuum can be derived from the action

$$S = \int d^4x \mathcal{L} = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (16)$$

with  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ .

- a) Derive the Maxwell equations from the action (16). Use the Euler-Lagrange equations and treat the components  $A_\mu(x)$  as the dynamic variables. Write the two 'inhomogeneous' Maxwell equations in their standard form by using  $E^i = -F^{0i}$  and  $\varepsilon^{ijk} B^k = -F^{ij}$ ,  $i = x, y, z$ . What happens with the two homogeneous equations?
- b) Calculate the energy-momentum tensor  $T^{\mu\nu}$  for the electromagnetic field

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L}. \quad (17)$$

- c) This tensor, however, is not symmetric which can be fixed by adding a term of the form  $\partial_\lambda K^{\lambda\mu\nu}$ , where  $K^{\lambda\mu\nu}$  is antisymmetric in its first two indices. Calculate the symmetric energy-momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \quad (18)$$

with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu. \quad (19)$$

- d) Show that the symmetrized tensor yields the standard form of the electromagnetic energy density and the momentum density (Poynting vector)

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2) \quad \text{and} \quad \mathbf{S} = \mathbf{E} \times \mathbf{B}. \quad (20)$$