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Problem 1: Cross section of two scattering particles (Written, 4 points)

Learning objective

In this problem, we will study the cross section of two scattering particles within the ϕ^4-theory. It serves as an example of the use of Feynman diagrams to calculate scattering cross sections.

From the lecture, you know the relation between S-matrix elements and cross sections which is given by

$$d\sigma = \frac{1}{4|E_{\text{in}}p^z_{\text{in}} - E'_{\text{in}}p'^z_{\text{in}}|}\left(\prod \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}(p_{\text{in}}, p'_{\text{in}} \to \{p_f\})|^2(2\pi)^4\delta^{(4)}(p_{\text{in}}+p'_{\text{in}}-\sum_f p_f)\,, \quad (1)$$

where we chose the reference frame such that the particles collide along the z-axis.

Specialize now to the case of two particles, with the same mass m, interacting with the Hamiltonian

$$H_I = \frac{\lambda}{4!}\int d^4x\,\phi_I^4(x)\,. \quad (2)$$

a) Transform into the center-of-mass frame and integrate over the final momenta. Show that to lowest order the differential cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\lambda^2}{64\pi^2E^2_{\text{cm}}}\,, \quad (3)$$

where E_{cm} is the total initial energy.

Hint: In general, the invariant matrix element $\mathcal{M}$ is defined by

$$\langle \mathbf{p}_1\mathbf{p}_2\cdots|\,iT\,|\mathbf{p}_{\text{in}}\mathbf{p}'_{\text{in}}\rangle = (2\pi)^4\delta^{(4)}(p_{\text{in}}+p'_{\text{in}}-\sum_f p_f)i\mathcal{M}(p_{\text{in}}, p'_{\text{in}} \to \{p_f\}) \quad (4)$$

and

$$\langle \mathbf{p}_1\mathbf{p}_2\cdots|\,iT\,|\mathbf{p}_{\text{in}}\mathbf{p}'_{\text{in}}\rangle = \lim_{T\to\infty(1-i\varepsilon)}\left({}_0\langle \mathbf{p}_1\mathbf{p}_2\cdots|\,\mathcal{T}\exp\left(-i\int_{-T}^{T}dt\,H_I(t)\right)|\mathbf{p}_{\text{in}}\mathbf{p}'_{\text{in}}\rangle_0\right)_{\substack{\text{connected,}\\ \text{amputated}}}\,, \quad (5)$$

with the asymptotic incoming and outgoing states $|\mathbf{p}_{\text{in}}\mathbf{p}'_{\text{in}}\rangle_0$ and $|\mathbf{p}_1\mathbf{p}_2\cdots\rangle_0$, respectively.

b) Calculate the total cross section of the scattering process.

Problem 2: Rutherford scattering (Oral)

Learning objective

Here, you will derive the famous Rutherford formula for the differential cross section for the elastic scattering of non-relativistic charged particles interacting via the Coulomb interaction.

The cross section for the scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = \int d^3x \, e\bar{\psi}\gamma^\mu\psi A_\mu, \tag{6}$$

where $\psi(x)$ is the usual quantized Dirac field.

a) Show that the T-matrix element for electron scattering off a localized classical potential is, to lowest order,

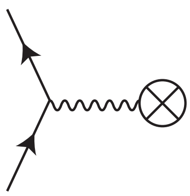
$$\langle p'| \, iT \, |p\rangle = -ie\bar{u}(p')\gamma^\mu u(p)\widetilde{A}_\mu(p'-p), \tag{7}$$

where $\widetilde{A}_\mu(q) = \int d^4x \, e^{iqx} A_\mu(x)$.

b) If $A_\mu(x)$ is time independent, its Fourier transform contains a δ-function of energy. It is then natural to define

$$\langle p'| \, iT \, |p\rangle \equiv -i\mathcal{M} \cdot (2\pi)\delta(E_f - E_i), \tag{8}$$

where E_i and E_f are the initial and final energies of the particle, and to adopt a new Feynman rule for computing $\mathcal{M}$:

 $= -ie\gamma^\mu \widetilde{A}_\mu(\mathbf{q})$

where $\widetilde{A}(\mathbf{q})$ is the three-dimensional Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross section for scattering off a time-dependent localized potential is

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\mathcal{M}(p_i - p_f)|^2(2\pi)\delta(E_f - E_i), \tag{9}$$

where v_i is the particle's initial velocity. Integrate over $|\mathbf{p}_f|$ to find a simple expression for $d\sigma/d\Omega$.

Hints:

- Start by considering a wave packet

$$|\phi\rangle = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} e^{-i\mathbf{b}\cdot\mathbf{p}}\phi(\mathbf{p}) \, |\mathbf{p}\rangle \, , \tag{10}$$

where $\mathbf{b}$ is the impact parameter accounting for the transverse displacement of the incoming wave packet and assume $\phi(\mathbf{p})$ to be narrowly peaked around $\mathbf{p} = (0, 0, p)$.

- Calculate the probability to scatter the incoming state into a final state whose momentum lies in a small region d^3p'. Take care of the proper normalization.
- In order to calculate the differential cross section, integrate this probability over the impact parameter **b**.

c) Specialize to the case of electron scattering from a Coulomb potential $(A^0 = Ze/4\pi r)$. Working in the non-relativistic limit, derive the *Rutherford formula*,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}\,, \tag{11}$$

where θ is the angle between initial and final momentum.