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Problem 1: Important relations for gamma matrices (Written, 4 points)

> **Learning objective**
>
> The evaluation of scattering amplitudes in quantum electrodynamics becomes much easier with an appropriate compendium of relations for gamma matrices. Here you derive the most important ones from their algebraic properties.

Consider a set of four matrices γ^μ satisfying the *Dirac algebra*

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \,. \tag{1}$$

In addition, we introduce a fifth matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \,. \tag{2}$$

If not stated otherwise, all results below should be derived *algebraically*, i.e., without reference to a specific representation of the Dirac algebra.

a) As warm-up, show that

$$\left(\gamma^5\right)^2 = 1 \,, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0 \,. \tag{3}$$

b) Show that the *Weyl (chiral) representation* satisfies the additional *Hermiticity condition*

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \qquad \text{and that this implies} \qquad \left(\gamma^5\right)^\dagger = \gamma^5 \,. \tag{4}$$

Is this true for arbitrary representations?

c) Prove the following *trace identities*:

$$\operatorname{tr}\left[\gamma^\mu\right] = 0 \tag{5a}$$
$$\operatorname{tr}\left[\gamma^\mu\gamma^\nu\right] = 4g^{\mu\nu} \tag{5b}$$
$$\operatorname{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \tag{5c}$$
$$\operatorname{tr}\left[\gamma^5\right] = 0 \tag{5d}$$
$$\operatorname{tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] = 0 \tag{5e}$$
$$\operatorname{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right] = -4i\varepsilon^{\mu\nu\rho\sigma} \tag{5f}$$
$$\operatorname{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\dots\right] = \operatorname{tr}\left[\dots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right] \tag{5g}$$

Hint: Only for the last relation (5g) the Hermiticity condition $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ is required.

d) Finally, prove the following *contraction identities*:

$$\gamma^\mu\gamma_\mu = 4 \tag{6a}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{6b}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} \tag{6c}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu \tag{6d}$$

Problem 2: Scattering cross section for $e^+e^- \to \mu^+\mu^-$ in QED (Oral)

Learning objective

The scattering of an electron e^- and a positron e^+ into a muon/antimuon pair $\mu^+\mu^-$ is the simplest and one of the most important scattering processes described by QED. In the lecture you already evaluated the cross section for vanishing electron mass. Here you will retrace this calculation step by step with non-vanishing electron mass to practice the evaluation of cross sections.

We want to calculate the *unpolarized* scattering cross section of the elementary QED process

$$e^+e^- \to \mu^+\mu^- \tag{7}$$

to lowest order. Here "unpolarized" means that the spins of incoming particles are distributed uniformly and the spins of the scattering products cannot be resolved by the experiment.

a) Draw and label the lowest-order Feynman diagram for this process and calculate the amplitude

$$i\mathcal{M}[e^-(p,s)e^+(p',s') \to \mu^-(k,r)\mu^+(k',r')] \tag{8}$$

with momenta p, p', k, k' and spins s, s', r, r'.

b) Since we want to calculate the *unpolarized* scattering cross section, we have to average uniformly over all incoming spins and sum over the outgoing ones. I.e., calculate

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{2}\sum_{s}\frac{1}{2}\sum_{s'}\sum_{r,r'}|\mathcal{M}|^2 \tag{9}$$

and evaluate the traces using the identities from Problem 1.

Hint: You should end up with

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{(p+p')^4}\left[(p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k) + (p\cdot p')m_\mu^2 + (k\cdot k')m_e^2 + 2m_\mu^2 m_e^2\right],$$

where m_μ and m_e are the masses of the muon and electron, respectively.

c) In the center-of-mass frame, calculate the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ and the total cross section σ in terms of the energy E of the incoming particles and the scattering angle θ of the outgoing particles.

Hint: As there are only two particles in the final state, you can use the simplified expression for the differential scattering cross section

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CM}} = \frac{1}{2E_A 2E_B|v_A - v_B|}\,\frac{|\mathbf{k}|}{(2\pi)^2\,4E_{\text{cm}}}\,\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \tag{10}$$

where $E_A = E_B = E$, $E_{\text{cm}} = 2E$ and $v_A = \sqrt{E^2 - m_e^2}/E = -v_B$ since the two incoming particles have the same mass.

d) In reality, the muon is much heavier than the electron: $m_e/m_\mu \approx 1/200$. Show that in the limit of small electron mass, $m_\mu \gg m_e$, the differential and total cross section take the form derived in the lecture.

e) Starting from your result for finite m_e, calculate the differential and total cross section in the high-energy limit $E \gg m_\mu, m_e$ to the lowest order where the results depend on both masses m_e and m_μ.