Prof. Dr. Hans Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

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Problem 1: Infrared divergence of the electron vertex function (Written, 4 points)

Learning objective

The calculation of the one-loop correction of the electron vertex function is riddled with both an ultraviolet and an infrared divergence—caused by the momentum integration of the loop. While the ultraviolet divergence is controlled by Pauli-Villars regularization, the infrared divergence can be parametrized by introducing a small, artificial photon mass $\mu > 0$. It is important to extract the asymptotic behaviour of this divergence for $\mu \to 0$ to prepare its cancellation with a similar term found for soft bremsstrahlung. Here you work out the details of this asymptotic behaviour.

As shown in the lecture, the regularized form factor F_1 of the electron vertex in QED up to one-loop order reads

$$F_1(q^2) = 1 + \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right)\right. \tag{1}$$

$$\left. + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \mu^2 z} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \mu^2 z}\right] + \mathcal{O}(\alpha^2)\,.$$

Here, x, y, z are Feynman parameters, m is the electron mass, $q = p' - p$ the momentum transfer and μ the artificial photon mass to regularize the integral; α is the fine structure constant.

We are interested in the (physical) limit of vanishing photon mass ($\mu \to 0$) where Eq. (1) diverges.

a) Show that the dominant terms of Eq. (1) in this limit read

$$F_1^{(1)}(q^2) := \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \tag{2}$$

$$\times\left[\frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \mu^2 z} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \mu^2 z}\right]\,.$$

Hint: Show that the virtual photon is spacelike, i.e., show that $q^2 < 0$; then show that the argument of the logarithm is bounded in the relevant region.

b) Using the previous result, show that the asymptotic behaviour of F_1 is captured by the simpler expression

$$F_1^{(2)}(q^2) := \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z} dy\left[\frac{-2m^2 + q^2}{m^2(1-z)^2 - q^2(1-z-y)y + \mu^2} - \frac{-2m^2}{m^2(1-z)^2 + \mu^2}\right]\,. \tag{3}$$

c) Use the substitution $y = (1-z)\xi$ and $w = 1-z$ to show that

$$F_1^{(3)}(q^2) := \frac{\alpha}{4\pi} \int_0^1 d\xi \left[\frac{-2m^2 + q^2}{m^2 - q^2\xi(1-\xi)} \log\left(\frac{m^2 - q^2\xi(1-\xi)}{\mu^2} \right) + 2\log\left(\frac{m^2}{\mu^2} \right) \right] . \quad (4)$$

d) Finally, show that the asymptotics of F_1 is given by $F_1(q^2) \approx 1 + F_1^{(4)}(q^2) + \mathcal{O}(\alpha^2)$ with

$$F_1^{(4)}(q^2) := -\frac{\alpha}{2\pi} f_{\mathrm{IR}}(q^2) \log\left(\frac{A}{\mu^2} \right) \quad (5)$$

where the function $f_{\mathrm{IR}}(q^2)$ has to be determined and both choices $A \in \{-q^2, m^2\}$ give rise to valid expressions.

Hint: Use that adding constants (with respect to μ) to $F_1^{(3)}$ does not change its asymptotic behaviour for $\mu \to 0$.

What is the sign of $f_{\mathrm{IR}}(q^2)$?

This expression can now be used to cancel the infrared divergence of the electron vertex function with the corresponding divergence found for soft *bremsstrahlung* to obtain a finite result independent of μ (see lecture).

Problem 2: Dimensional regularization (Oral)

Learning objective

In this exercise we will work on the technical details of *dimensional regularization* (due to 't Hooft and Veltman). Dimensional regularization preserves the symmetries of QED and a broader class of more general theories. The idea of dimensional regularization is to extend the definition of d-dimensional volume integrals to arbitrary $d \in \mathbb{R}$. If the divergences of integrals from Feynman diagrams vanish for $d < 4$, they can be regularized if the limit $d \to 4$ is taken after evaluating physical quantities.

Let us consider spacetime to have one time dimension and $(d-1)$ space dimensions ($d = 2, 3, 4, \ldots$). We are interested in solving integrals of the form

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int d\ell \, \frac{\ell^{d-1}}{(\ell^2 + \Delta)^2} \tag{6}$$

where we have Wick-rotated the time dimension so that $d^d\ell_E$ is the volume element of d-dimensional *Euclidean* space; $d\Omega_d$ denotes the angular part of the integral in d-dimensional spherical coordinates.

a) The first factor in Eq. (6) contains the area of a unit sphere in d dimensions. Show that

$$\int d\Omega_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)} \,. \tag{7}$$

Use $\int dx\, e^{-x^2} = \sqrt{\pi}$ and the definition of the Gamma function $\Gamma(t) := \int_0^\infty dx\, x^{t-1} e^{-x}$.

b) With the result from a), show that Eq. (6) evaluates to

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \,. \tag{8}$$

To this end, use the substitution $x = \Delta/(\ell^2 + \Delta)$ and the definition of the beta function

$$B(\alpha, \beta) := \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \,. \tag{9}$$

The expression Eq. (8) can now be used to *define* the left-hand side for $d \in \mathbb{R}$.

Where are the poles of this generalized integral in d "dimensions"?

c) Define $\epsilon = 4 - d$ and use the infinite product representation

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \tag{10}$$

(γ is the Euler-Mascheroni constant) to expand $\Gamma(2 - \frac{d}{2})$ to first order in ϵ.

d) Show that the integral (8) takes the asymptotic form

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} \xrightarrow{d \to 4} \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} + \log\frac{4\pi}{\Delta} - \gamma + \mathcal{O}(\epsilon)\right] \,. \tag{11}$$

This expression extracts the diverging part of the integral for $d \to 4$ and allows for the controlled treatment of such integrals.

e) Following the previous steps, verify the more general expressions

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}, \tag{12a}$$

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}. \tag{12b}$$

These integrals are useful for the renormalization of the electric charge (see lecture).