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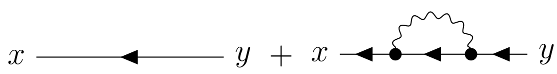
Problem 1: The electron self-energy (Written, 3 points)

Learning objective

The mass-energy equivalence inherent to any relativistic theory implies for quantum field theories that fluctuations of fields around particles with "bare" mass m_0 shift the latter to a larger, observable mass m. In QED, virtual photons that couple to the charged electron make up for its *self-energy* which, in turn, contributes to its mass m; we say that the mass is *renormalized*. Here you work out the details of the one-loop correction discussed in the lecture. As a result, we find that m_0 and m differ by an infininity.

The electron two-point function is given by the sum of diagrams

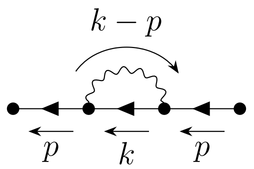
$$\langle\Omega|\,\mathcal{T}\Psi(x)\bar{\Psi}(y)\,|\Omega\rangle = \quad (1)$$



where the first diagram is just the free-field propagator,

$$= \frac{i(\not{p}+m_0)}{p^2-m_0^2+i\epsilon}\,, \quad (2)$$

and the second diagram (the *electron self-energy*) yields the expression



$$= \frac{i(\not{p}+m_0)}{p^2-m_0^2}[-i\Sigma_2(p)]\frac{i(\not{p}+m_0)}{p^2-m_0^2} \quad (3)$$

according to the Feynman rules of QED (for the sake of simplicity, we omit the term $e^{-ip(x-y)}$ and the integral $\int d^4p/(2\pi)^4$ for the external points). Here

$$-i\Sigma_2(p) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\,\gamma^\mu\,\frac{i(\not{k}+m_0)}{k^2-m_0^2+i\epsilon}\,\gamma_\mu\,\frac{-i}{(p-k)^2-\mu^2+i\epsilon} \quad (4)$$

contains the two loop propators with their two vertices. m_0 is the bare mass of the electron and $\mu > 0$ is a small photon mass to regulate the infrared divergence of the integral.

a) Using Feynman parameters, show that the second-order self-energy $-i\Sigma_2(p)$ takes the form

$$-i\Sigma_2(p) = -e^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{-2x\not{p}+4m_0}{[\ell^2-\Delta_\mu+i\epsilon]^2}\,, \quad (5)$$

where Δ_μ has to be determined.

b) To control the ultraviolet divergence of the integral (5), use the *Pauli-Villars regularization*

$$\frac{1}{(p-k)^2-\mu^2+i\epsilon} \to \frac{1}{(p-k)^2-\mu^2+i\epsilon} - \frac{1}{(p-k)^2-\Lambda^2+i\epsilon} \tag{6}$$

for $\Lambda \to \infty$ and show that

$$\Sigma_2(p) = \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0 - x\not{p})\,\log\left[\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right] \tag{7}$$

in this limit.

c) Using the expression for the second-order self-energy obtained in b), calculate the mass shift

$$\delta m = m - m_0 = \Sigma_2(\not{p} = m) \approx \Sigma_2(\not{p} = m_0) \tag{8}$$

in first order of α.

Show that the bare mass m_0 and the measurable mass m differ by a diverging quantity.

Problem 2: Thomas-Fermi screening (Oral)

Learning objective

As already demonstrated in previous tasks, the machinery of quantum field theory is not restricted to high-energy physics and fundamental theories like QED; its application to condensed matter physics provides one of the most powerful tools to study strongly correlated quantum matter. In this exercise, we will study the so called *Thomas-Fermi screening* of electrons in a degenerate electron gas of density n at zero temperature.

a) Similar to the lecture, define $\Pi(q)$ to be the sum of all *one-particle-irreducible* diagrams contributing to the photon self-energy. Show by diagrammatically expanding the *full* photon propagator $D_{\text{ph}}(q)$ that

$$D_{\text{ph}}(q) = \frac{D^0_{\text{ph}}(q)}{1 - D^0_{\text{ph}}(q)\Pi(q)}\,, \tag{9}$$

where $D^0_{\text{ph}}(q)$ is the bare photon propagator.

This approach is related to the so called *Lindhard theory* in condensed matter theory used for calculating the effects of electric field screening by electrons.

b) In condensed matter theory, the bare photon propagator in momentum space is simply given by the Fourier transform $U(\mathbf{q})$ of the (time-independent) interaction potential. Then, the denominator in (9) can be seen as a dielectric function given (in the static limit) as

$$\varepsilon(\mathbf{q}) = 1 - U(\mathbf{q})\Pi(\mathbf{q})\,. \tag{10}$$

Show that the bare Coulomb interaction in momentum space, $U(\mathbf{q}) = e^2/q^2$, is now modified to an effective interaction due to the screening of the electron gas in the long wavelength limit (i.e., Π is evaluated at $\mathbf{q} = \mathbf{0}$):

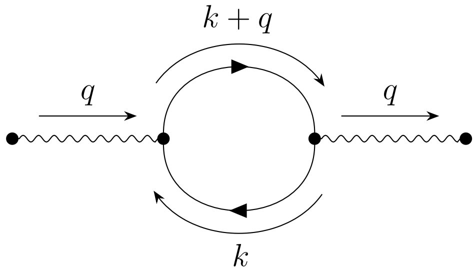
$$U_{\text{eff}}(\mathbf{q}) = \frac{e^2}{q^2 + \lambda_{\text{TF}}^{-2}}\,, \tag{11}$$

where λ_{TF}^{-1} is the *Thomas-Fermi wave vector.*

c) Calculate the Fourier transform $U_{\text{eff}}(\mathbf{x})$ of the effective potential (11) and discuss your result.

d) **Optional (+1 point):**

Calculate the Thomas-Fermi wave vector in the long wavelength limit ($\mathbf{q} \to \mathbf{0}$) and in the so-called *random-phase approximation*, where $\Pi(q)$ consists only of the particle-hole(=antiparticle) loop (neglecting the in- and outgoing lines):



According to the Feynman rules in condensed matter theory, $\Pi(q)$ is given by

$$\Pi(q) = -2i \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} G^0(\omega, \mathbf{k}) G^0(\omega, \mathbf{k} + \mathbf{q}) \,, \tag{12}$$

where the propagator/Green's function reads

$$G^0(\omega, \mathbf{k}) = \frac{1}{\omega - \xi(\mathbf{k}) + i\delta \,\mathrm{sgn}(\xi(\mathbf{k}))} \tag{13}$$

with $\xi(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - E_F$ and E_F the Fermi energy. δ is to be taken positive but small (i.e. $\delta \to 0^+$) and $\mathrm{sgn}(x)$ refers to the signum function, which gives the sign of x and $\mathrm{sgn}(0) = 0$.

Hint: In 3D, the Fermi energy is given by $E_F = (3\pi^2 n)^{2/3}/(2m)$ with electron density n and mass m.