Problem 1: Propagator in the path integral formalism (Oral, 3 points)

Learning objective

In this problem, you use path integrals to construct the propagator (or two-point correlation function) of a generic quadratic field theory. You show that the propagator is given by the inverse of the quadratic form associated with the field theory.

Consider a generic quadratic theory in momentum space with action

\[ S = \frac{1}{2} \sum_k \phi_{i,k}^* M_{ij} \phi_{j,-k} \]  

(1)

with a symmetric matrix \( M \), which also satisfies \( M_k = M_{-k} \). Since the fields \( \phi_i \) are assumed to be real-valued, their Fourier components fulfill the relation \( \phi_{i,-k} = \phi_{i,k}^* \) and we can rewrite the action as

\[ S = \sum_{k^0 > 0} \phi_{i,k} M_{ij} \phi_{j,k}^* . \]  

(2)

Note that the sum over the momenta now only runs over one half of the four-momentum space, that is \( k^0 > 0 \).

In order to calculate the two-point correlation function, it proves useful to introduce the generating functional

\[ Z[J, J^*] = \int \mathcal{D}\phi \exp \left( i \sum_{k^0 > 0} \left[ \phi_{i,k} M_{ij} \phi_{j,k}^* + (J_i^*)^* \phi_{i,k} + J_i^* \phi_{j,k} \right] \right) . \]  

(3)

a) Calculate the path integral in (3) and show that the generating functional is given by

\[ Z[J, J^*] = \prod_{k^0 > 0} \left( \frac{i\pi}{\det M_k} \right) \exp \left( -iJ_{i,k} (M_k^{-1})^{ij} J_{j,k}^* \right) . \]  

(4)

Hints:

- Since the matrix \( M \) is symmetric, it can be diagonalized by a unitary transformation, that is \( M = U^\dagger DU \), where \( D \) is diagonal and \( U \) is unitary.
- After diagonalizing the matrix and decoupling the fields, split each field into real and imaginary part, \( \phi_{i,k} = \phi_{i,k}^{re} + i\phi_{i,k}^{im} \).
- The path integral measure is given by \( \mathcal{D}\phi = \prod_{j,k^0 > 0} d\phi_{i,j}^{re} d\phi_{i,j}^{im} \).
- The remaining Gaussian integrals can be calculated by completing the square.
b) Use (3) to relate the two-point correlation function in Fourier space to the generating functional 
\[ Z[J, J^*] \] and show that
\[
\tilde{D}^ij_F(k) = \left. \frac{1}{Z[0,0]} \left( -i \frac{\partial}{\partial J^*_{i,k}} \right) \left( -i \frac{\partial}{\partial J_{j,k}} \right) Z[J, J^*] \right|_{J, J^* = 0}.
\] (5)

c) Finally, use (4) to show that \[ \tilde{D}^ij_F(k) = i \left( M^{-1} \right)^{ij}. \]

Problem 2: Path integral and Weyl order (Written, 5 points)

**Learning objective**

This problem deals with the connection between the transition amplitude of a quantum system and the path integral formalism. In doing so, the peculiarity of non-commuting operators arises which can be resolved by employing a special ordering of operators in the Hamiltonian called Weyl order. As an example, you will calculate the transition amplitude for a non-relativistic particle in one dimension.

Consider a general quantum system described by a set of coordinates \( q^i \), conjugate momenta \( p^i \) and Hamiltonian \( H(q, p) \). In the lecture, it was shown that the transition amplitude \( U(q_a, q_b; T) = \langle q_b | e^{-iHT} | q_a \rangle \) can be computed by breaking the time interval into \( N \) short slices of length \( \epsilon \) and inserting a complete set of intermediate states between each slice such that

\[
U(q_a, q_b; T) = \prod_i \prod_{k,l} \int dq_k^i dq_{k+1}^i e^{-i\epsilon H} |q_k^i\rangle, \] (6)

where \( k = 0, \ldots, N - 1 \), \( l = 1, \ldots, N - 1 \) and \( q_a = q_0 \) and \( q_b = q_N \). Since \( \epsilon \to 0 \), we may expand the exponential as \( e^{-i\epsilon H} = 1 - i\epsilon H + \cdots \). In a first step, consider a Hamiltonian which is a pure function of either \( q \) or \( p \), that is \( H(q, p) = f(q) \) or \( H(q, p) = f(p) \).

a) Show that if \( H \) is only a function of the coordinates, the matrix element can be written as
\[
\langle q_{k+1}^i | f(q) | q_k^i \rangle = f \left( \frac{q_{k+1}^i + q_k^i}{2} \right) \left( \prod_i \int \frac{dp_k^i}{2\pi} \right) \exp \left( i \sum_i p_k^i (q_{k+1}^i - q_k^i) \right). \] (7)

b) Now consider the case when the Hamiltonian only depends on the momenta. Show that
\[
\langle q_{k+1}^i | f(p) | q_k^i \rangle = \left( \prod_i \int \frac{dp_k^i}{2\pi} \right) f(p_k) \exp \left( i \sum_i p_k^i (q_{k+1}^i - q_k^i) \right). \] (8)

Show also that if \( H \) contains only terms of the form \( f(q) \) and \( f(p) \), its matrix elements can be written
\[
\langle q_{k+1}^i | H(q, p) | q_k^i \rangle = \left( \prod_i \int \frac{dp_k^i}{2\pi} \right) H \left( \frac{q_{k+1}^i + q_k^i}{2}, p_k \right) \exp \left( i \sum_i p_k^i (q_{k+1}^i - q_k^i) \right). \] (9)
c) In general, (9) is false when there are products of $q$’s and $p$’s in the Hamiltonian as on the left-hand side the order of the (non-commuting) operators matters while on the right-hand side we only deal with numbers. Show this explicitly for $H = p^2 q^2$ and show that putting the Hamiltonian into Weyl order, that is

$$H(q, p) = p^2 q^2 \mapsto H_W(q, p) = \frac{1}{4}(q^2 p^2 + 2qp^2 q + p^2 q^2),$$

resolves this issue. Note that any Hamiltonian can be put into Weyl order by commuting $p$’s and $q$’s on the cost of some extra terms appearing on the right-hand side of (9).

d) Show that for a Weyl-ordered Hamiltonian, the propagator (6) is given by

$$U(q_N, q_0; T) = \left( \prod_{i,k,l} \int dq_i \int dp_i \frac{dp_i}{2\pi} \right) \exp \left[ i \sum_k \left( \sum_i p_i^k(q_{k+1} - q_k) - \epsilon H \left( \frac{q_{k+1} + q_k}{2}, p_k \right) \right) \right].$$

(11)

This expression is the discretized form of

$$U(q_a, q_b; T) = \int Dq(t) Dp(t) \exp \left[ i \int_0^T dt \left( \sum_i p^i q^i - H(q, p) \right) \right]$$

(12)

and defines what we understand as a path integral.

e) As a special case, consider the Hamiltonian $H = p^2/2m + V(q)$ of a single, non-relativistic particle in one dimension. Show that the transition amplitude reads

$$U(q_a, q_b; T) = \left( \frac{1}{C(\epsilon)} \prod_{k,l} \int dq_l \frac{dq_l}{C(\epsilon)} \right) \exp \left[ i \sum_k \left( \frac{m}{2} \left( \frac{q_{k+1} - q_k}{\epsilon} \right)^2 - \epsilon V \left( \frac{q_{k+1} + q_k}{2} \right) \right) \right]$$

(13)

and determine $C(\epsilon)$. 