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You can find detailed information about the lecture and the problem sets on the website

`https://www.itp3.uni-stuttgart.de/teaching/electrodynamics/index.html`.

Problems are divided into two groups. *Written* problems have to be handed in and are graded by the tutors. *Oral* problems have to be prepared for the tutorials and presented by one of the students. In order to be admitted to the exam, we require (i) 80% of the written problems to be solved or treated adequately, (ii) 66% of the oral problems to be prepared, and (iii) two problems to be presented at the black board. The third requirement will be lifted if only virtual tutorials can be hold.

This first problem set repeats concepts important for the lecture. Solutions must be handed in by 22th of April. As a preparation, you should familiarize yourself with basic math, algebra, and calculus from the lecture "Mathematische Methoden der Physik".

Problem 1: Vector Calculus (Written)

Learning objective

Here, we recall some standard identities of vector calculus which we will use throughout the lecture.

Definitions and conventions: We write the vectorial differentiation operators grad, div, rot using the vector $\boldsymbol{\nabla}$ of partial derivatives $\nabla_i := \partial/\partial x_i$ as

$$\operatorname{grad} F := \boldsymbol{\nabla} F\,, \qquad \operatorname{div} \mathbf{A} := \boldsymbol{\nabla}\cdot\mathbf{A}\,, \qquad \operatorname{rot} \mathbf{A} := \boldsymbol{\nabla}\times\mathbf{A}\,. \tag{1}$$

The components of a three-dimensional vector product $\mathbf{a}\times\mathbf{b}$ are given by

$$(\mathbf{a}\times\mathbf{b})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}\, a_j b_k\,, \tag{2}$$

here ε_{ijk} is the totally anti-symmetric tensor in $\mathbb{R}^3$ with $\varepsilon_{123} = +1$.

a) Show that

$$\sum_{i=1}^{3} \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \qquad \text{and} \qquad \frac{1}{2}\sum_{i,j=1}^{3} \varepsilon_{ijk}\,\varepsilon_{ijl} = \delta_{kl}\,. \tag{3}$$

b) Now show the following identities for vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\,, \tag{4a}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\,, \tag{4b}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\,. \tag{4c}$$

c) Prove the following identities for the scalar fields F and vector fields $\mathbf{A}$, $\mathbf{B}$:

$$\nabla \times (\nabla F) = 0\,, \tag{5a}$$
$$\nabla\cdot(\nabla \times \mathbf{A}) = 0\,, \tag{5b}$$
$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A}\,, \tag{5c}$$
$$\nabla\cdot(F\mathbf{A}) = (\nabla F)\cdot\mathbf{A} + F\nabla\cdot\mathbf{A}\,, \tag{5d}$$
$$\nabla \times (F\mathbf{A}) = (\nabla F) \times \mathbf{A} + F\nabla \times \mathbf{A}\,, \tag{5e}$$
$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})\,, \tag{5f}$$
$$\nabla\cdot(\mathbf{A} \times \mathbf{B}) = \mathbf{B}\cdot(\nabla \times \mathbf{A}) - \mathbf{A}\cdot(\nabla \times \mathbf{B})\,. \tag{5g}$$

Problem 2: Gauß's Theorem (Written)

Learning objective

This problem recapitulates Gauß's theorem which will be useful for solving problems in electrostatics. We calculate the surface integrals of different vector fields over a closed surface (i.e. the flux through the surface) and show explicitely that they are equal to the volume integrals of the divergence of the fields over the region inside the surface.

Consider the following vector fields $\mathbf{A}_i$ in two dimensions

$$\mathbf{A}_1 = \big(3xy(y-x), x^2(3y-x)\big)\ , \tag{6a}$$
$$\mathbf{A}_2 = \big(x^2(3y-x), 3xy(x-y)\big)\ , \tag{6b}$$
$$\mathbf{A}_3 = \big(x/(x^2+y^2), y/(x^2+y^2)\big) = \mathbf{x}/|\mathbf{x}|^2\,. \tag{6c}$$

a) Compute the flux of $\mathbf{A}_i$ through the boundary of the square Q with corners $\mathbf{x} = (\pm 1, \pm' 1)$

$$I_i = \oint_{\partial Q} dx\, \mathbf{n}\cdot\mathbf{A}_i\,. \tag{7}$$

b) Calculate the divergence of $\mathbf{A}_i$ and its integral over the area of this square Q

$$I_i' = \int_Q d^2x\, \nabla\cdot\mathbf{A}_i\,. \tag{8}$$

Problem 3: Stokes' Theorem (Written)

Learning objective

This problem recapitulates Stokes' theorem which has important applications in electromagnetism. We calculate the line integral of a vector field around the boundary of a surface and show explicitely that it is equal to the integral of the curl of the field over the surface.

Consider the vector field

$$\mathbf{A} = \left(x^2 y, x^3 + 2xy^2, xyz\right) . \tag{9}$$

a) Compute the integral along the circle S around the origin in the xy-plane with radius R

$$I = \oint_S d\mathbf{x}\cdot\mathbf{A} \, . \tag{10}$$

b) Calculate the curl $\mathbf{B}$ of the vector field $\mathbf{A}$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \, . \tag{11}$$

c) Determine the flux of the curl $\mathbf{B}$ through the disk D whose boundary is S, $\partial D = S$

$$I' = \int_D d^2x \, \mathbf{n}\cdot\mathbf{B} \, . \tag{12}$$