Prof. Dr. Hans Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

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Problem 1: Elliptically polarized waves (Written)

> **Learning objective**
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> In this exercise we discuss the concept of polarization for electromagnetic plane waves based on the example of elliptically polarized light.

A wave $\mathbf{E}(\mathbf{x}, t)$ with wave vector $\mathbf{k} = k\mathbf{e}_z$ is given by

$$E_x(\mathbf{x}, t) = A\cos(kz - \omega t) \tag{1}$$
$$E_y(\mathbf{x}, t) = B\cos(kz - \omega t + \phi). \tag{2}$$

a) Show that the trajectory of the vector $\mathbf{E}(\mathbf{0}, t)$, which describes the polarization of the wave, is an ellipse. For which values of A, B and ϕ is this trajectory a circle ?
Hint: Use the trigonometric addition theorem

$$\cos(\omega t - \phi) = \cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi) \tag{3}$$

in order to transform the equations into the form of a conic section:

$$ax^2 + 2bxy + cy^2 + f = 0 \tag{4}$$

Which conditions does one need to impose on a, b, c ?

b) Show that for general A and B the wave can be written as the superposition of two oppositely circularly polarized waves

$$\mathbf{E}(\mathbf{x}, t) = \mathrm{Re}(\mathbf{E}_+(z, t) + \mathbf{E}_-(z, t)), \tag{5}$$

where $\mathbf{E}_\pm(z, t) = A_\pm \epsilon_\pm e^{i(kz-\omega t)}$. Here $A_\pm$ are constants and $\epsilon_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_x \pm i\mathbf{e}_y)$. Determine $A_\pm$ as a function of A, B and ϕ.
Hint: Write the wave as the real part of a complex vector and solve $\epsilon_\pm$ for $\mathbf{e}_x$ and $\mathbf{e}_y$.

Problem 2: Reflection and transmission (Written)

> **Learning objective**
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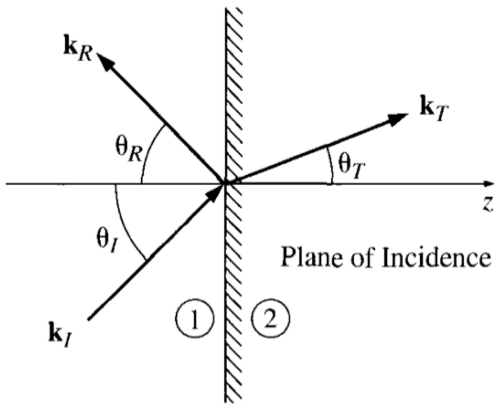
> In this exercise the application of the general theory of reflection and transmission at the interface of two media is discussed.

The space is filled with two different non-conductive media. Consider incident, reflected and transmitted monochromatic plane waves as in the figure below.



a) The frequency of each plane wave is fixed. Why and how are the wave numbers $\mathbf{k}_i$ related to each other in terms of the angles θ_i, where $i \in \{I, T, R\}$?

b) In general, how are the electric and the magnetic fields related to each other at the interface? Using the results from task a) simplify these relations.

c) Show for s-polarized light (E-field perpendicular to the incidence plane, also referred to as Transverse Electric (TE) waves) that $1 + r = t$ where $r = E_R/E_I$ and $t = E_T/E_I$. Next, derive a relation of the form $1 - |r|^2 = c|t|^2$. Determine the constant c. Then, find the corresponding equations for p-polarized light (B-field perpendicular to the plane of incidence, E-field parallel to the page, also called Transverse Magnetic (TM) waves).

d) Consider a plane wave that passes from the vacuum to a medium with $n' \in \mathbb{R}$. The interface between the vacuum and the medium is perpendicular to the propagation direction. Find the reflected and the transmitted power.

Problem 3: Propagation of wave packets in media (Written)

Learning objective

In media, the propagation of wave packets exhibits a dispersion, and the propagation speed of the wave packet is no longer described by the light velocity, but rather the group velocity. In this exercise, these concepts are studied in a simple example.

We start with a general, scalar field Ψ the time evolution of which is given as a superposition of plain waves,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}k\, \hat{\Psi}_0(k) \exp\left(i[kx - \omega(k)t]\right) \tag{6}$$

where $\hat{\Psi}_0(k) \equiv \mathcal{F}[\Psi_0](k)$ is the Fourier transform of the initial wave packet $\Psi_0 = \Psi(x, t = 0)$. The function $\omega = \omega(k)$ is called *dispersion relation* and determined by the differential equation that governs the dynamics of Ψ.

a) Give two paradigmatic examples of differential equations ("wave equations") with general solutions given by (6) and compare their corresponding dispersion relations $\omega = \omega(k)$.

b) Assume that $\hat{\Psi}_0(k)$ is sharply peaked around k_0. Then it is reasonable to expand $\omega(k)$ at k_0 for small $k - k_0$ up to first order (Why?). Use this expansion with Eq. (6) to show that $\Psi(x,t)$ can be written in the form

$$\Psi(x,t) = e^{i\phi(x-v_p t)} \cdot \psi(x - v_g t) \tag{7}$$

where $\phi(x)$ is a real function and $\psi(x)$ an arbitrary scalar field. Give expressions for v_p and v_g in terms of $\omega(k)$. v_p and v_g are called *phase*- and *group* velocity, respectively.

In the following we focus on a special case, namely a Gaussian wave packet at $t = 0$:

$$\Psi_0(x) \equiv \Psi(x, t=0) = \psi_0 \exp\left(-\frac{x^2}{2\sigma_x^2}\right) , \tag{8}$$

the propagation of which is still governed by an arbitrary dispersion relation $\omega = \omega(k)$. Here, σ_x^2 is the variance that describes the width of the wave packet.

c) Show that the initial wave packet in Fourier representation $\hat{\Psi}_0(k)$ is Gaussian as well, i.e.,

$$\hat{\Psi}_0(k) = \hat{\psi}_0 \exp\left(-\frac{k^2}{2\sigma_k^2}\right) . \tag{9}$$

What is the relation of σ_x and σ_k and how can one interpret this result?

Hint:

$$\int_{\mathbb{R}} \mathrm{d}x \, e^{-\frac{x^2}{2\sigma^2}} = \sqrt{2\pi}\sigma \quad \& \quad \text{completing the square} \tag{10}$$

d) Assume that $\hat{\Psi}_0(k)$ is peaked around $k_0 =?$ so that an expansion of $\omega(k)$ up to second order in $k - k_0$ is a valid approximation (What is the requirement on σ_x for $\hat{\Psi}_0(k)$ to be sharply "peaked"?):

$$\omega(k) \approx \omega_0 + v_g(k - k_0) + \frac{1}{2} w_g (k - k_0)^2 \tag{11}$$

w_g is called *group velocity dispersion*. How does it relate to v_g?
Use Eq. (6) and your result from (c) to calculate $\Psi(x,t)$ explicitely.