Prof. Dr. Hans Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

June 24th, 2020
SS 2020

Problem 1: Ideal wave guide, part 1 (Written)

> **Learning objective**
>
> The problem set deals with wave guides, i.e. structures which confine the propagation of electromagnetic waves to one direction. We start with analyzing the ideal wave guide. Here, in the first part of the problem, we calculate modes and prove that no transverse electromagnetic waves exist in this wave guide.

We consider electromagnetic waves which are confined to an ideal, cylindrical wave guide and propagate in z-direction. The term "ideal" means that the boundary surface of the wave guide is a perfect conductor. The medium inside the cylinder is assumed to be homogeneous with permeability $\mu = \mu_0\mu_r$ and permittivity $\varepsilon = \varepsilon_0\varepsilon_r$. For this geometry, we can separate off the propagation in z-direction:

$$\mathbf{E}(x, y, z, t) = \mathbf{E}_0(x, y)e^{i(kz-\omega t)} \tag{1}$$

$$\mathbf{H}(x, y, z, t) = \mathbf{H}_0(x, y)e^{i(kz-\omega t)}. \tag{2}$$

The boundary conditions at the interior boundary $\partial\mathcal{V}$ of the cylinder are $\mathbf{E}^{\parallel} = 0$ and $\mathbf{H}^{\perp} = 0$.

We distinguish between different type of electromagnetic waves: We define transverse magnetic (TM) modes and transverse electric (TE) modes via

$$E_z|_{\partial\mathcal{V}} = 0\,, \quad H_z = 0 \qquad \text{(TM modes)}\,, \tag{3a}$$

$$\partial_n H_z|_{\partial\mathcal{V}} = 0\,, \quad E_z = 0 \qquad \text{(TE modes)}\,. \tag{3b}$$

If both $E_z = 0$ and $H_z = 0$, we have transverse electromagnetic (TEM) modes.

a) Start with Maxwell equations and show that the z-components of the fields must fulfill the eigenvalue equations

$$\left(\nabla_t^2 + \gamma_\lambda^2\right) E_z = 0\,, \tag{4a}$$

$$\left(\nabla_t^2 + \gamma_\lambda^2\right) H_z = 0\,, \tag{4b}$$

with $\gamma_\lambda^2 = \mu_r\varepsilon_r\omega^2/c^2 - k_\lambda^2$ (the index λ labels the eigenvalues) and $\nabla_t = \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y$. Show that the boundary conditions are $E_z|_{\partial\mathcal{V}} = 0$ and $\partial_n H_z|_{\partial\mathcal{V}} = 0$.

b) Show that the solutions for the transverse field components $\mathbf{E}_t = E_x\mathbf{e}_x + E_y\mathbf{e}_y$ and $\mathbf{H}_t = H_x\mathbf{e}_x + H_y\mathbf{e}_y$ are determined by the solutions for the z-components of the fields via

$$\mathbf{E}_t = \frac{ik_\lambda}{\gamma_\lambda^2}\nabla_t E_z, \quad \mathbf{H}_t = \frac{\varepsilon_r\omega}{ck_\lambda}\mathbf{e}_z \times \mathbf{E}_t \qquad \text{(TM modes)}\,, \tag{5a}$$

$$\mathbf{H}_t = \frac{ik_\lambda}{\gamma_\lambda^2}\nabla_t H_z, \quad \mathbf{E}_t = -\frac{\mu_r\omega}{ck_\lambda}\mathbf{e}_z \times \mathbf{H}_t \qquad \text{(TE modes)}\,. \tag{5b}$$

c) Solve the equations for the TE modes by taking into account that the wave guide is a cylinder. Show that the solutions are given by Bessel functions.

d) Show that in an ideal wave guide, no TEM mode exist.

Hint: Use Gauss' and Faraday's law as well as the boundary condition for $\mathbf{E}^{\parallel}$ to show that there are no TEM waves in a single, hollow, cylindrical conductor.

Problem 2: Ideal wave guide, part 2 (Written)

Learning objective

Here, in the second part of the problem, we calculate the energy flow in the ideal wave guide.

Introducing the critical frequency $\omega_\lambda = \frac{c}{\sqrt{\mu_r \varepsilon_r}}\gamma_\lambda$ allows us to write $k_\lambda^2 = \frac{\mu_r \varepsilon_r}{c^2}(\omega^2 - \omega_\lambda^2)$ for the wave number that describes the propagation along the z-axis of the waveguide.

The flow of energy is given by the *complex* pointing vector

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* \tag{6}$$

where $*$ denotes complex conjugation.

a) Employ the solutions from the first part of the exercise to show that the pointing vector takes the form

$$\mathbf{S} = \frac{\omega k_\lambda}{2c\gamma_\lambda^4} \begin{cases} \varepsilon_r[|\nabla_t E_z|^2 \mathbf{e}_z + i\frac{\gamma_\lambda^2}{k_\lambda} E_z \nabla_t E_z^*]\,, & \text{(TM modes)} \\ \mu_r[|\nabla_t H_z|^2 \mathbf{e}_z - i\frac{\gamma_\lambda^2}{k_\lambda} H_z^* \nabla_t H_z]\,. & \text{(TE modes)} \end{cases} \tag{7}$$

b) Which contribution in Eq. (7) determines the energy flow in z-direction? Integrate this part over the cross section S of the waveguide for both TE and TM modes and show that the propagating power is given by

$$\begin{Bmatrix} P_{\text{TM}} \\ P_{\text{TE}} \end{Bmatrix} = \frac{1}{2\sqrt{\mu_r \varepsilon_r}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} \begin{Bmatrix} \varepsilon_r \\ \mu_r \end{Bmatrix} \int_S \mathrm{d}A \begin{Bmatrix} |E_z|^2 \\ |H_z|^2 \end{Bmatrix}\,. \tag{8}$$

Hint: Use Green's first identity for two scalar fields Ψ and Φ

$$\int_U \mathrm{d}V\, \left[\Phi \nabla^2 \Psi + \nabla\Phi \cdot \nabla\Psi\right] = \oint_{\partial U} \mathrm{d}A\, \Phi \frac{\partial \Psi}{\partial n} \tag{9}$$

and the boundary conditions given in (3). Here, $U \subset \mathbb{R}^n$ is some n-dimensional subset, ∂U its boundary, and $\partial_n \Psi = \mathbf{n} \cdot \nabla\Psi$ is the normal derivative with respect to ∂U. Eq. (4) may be useful as well.

c) Along the same lines, calculate the energy $U_{\text{TM/TE}}$ *per unit length* of the waveguide and show that

$$\begin{Bmatrix} U_{\text{TM}} \\ U_{\text{TE}} \end{Bmatrix} = \frac{1}{2} \left(\frac{\omega}{\omega_\lambda}\right)^2 \begin{Bmatrix} \varepsilon_r \\ \mu_r \end{Bmatrix} \int_S \mathrm{d}A \begin{Bmatrix} |E_z|^2 \\ |H_z|^2 \end{Bmatrix}\,. \tag{10}$$

Hint: The time-averaged energy u *per volume* (energy density) is given by

$$u = \frac{1}{4}\left(\varepsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2\right) \ . \tag{11}$$

d) Finally, combine the results (8) and (10) to derive an expression for the velocity of the energy flux and compare your result with the group velocity $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k_\lambda}$.

Problem 3: Coaxial cable (Written)

Learning objective

In this problem, we analyze the coaxial cable as another example of a wave guide. We show that a current flowing through the cable gives rise to transverse electromagnetic waves.

A coaxial cable is a thin conducting wire (with radius a) surrounded by a conducting cylindrical shield (with radius $b > a$). Write down solutions for TEM waves in the coaxial cable for a current I flowing through the thin wire.

Hint: The problem can be reduced to equations of magnetostatics and electrostatics in two dimensions.