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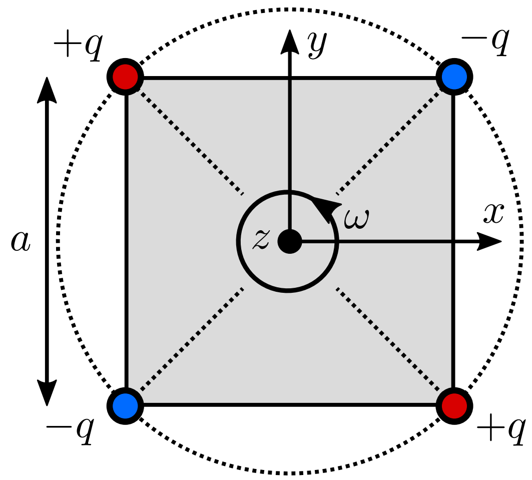
Problem 1: Rotating quadrupole (Written)

Learning objective

In this exercise we will learn about the time dependent quadrupole tensor and electromagnetic waves radiated by a rotating quadrupole. Then we will derive the angular power distribution of a rotating quadrupole and compare it with the angular power distribution of an oscillating dipole.

Consider the quadrupole setup depicted below, with two pairs of opposing charges $\pm q$ fixed at the corners of a square of size a. The square lies in the xy-plane and rotates with frequency $\boldsymbol{\omega} = \omega \mathbf{e}_z$ around its center:



We will focus on the electromagnetic waves radiated by this setup.

a) Write down the time-dependent charge distribution $\rho(\mathbf{x}, t)$ and calculate the quadrupole tensor

$$Q_{ij}(t) = \int_{\mathbb{R}^3} \mathrm{d}^3x\, \rho(\mathbf{x}, t) \left(3x_i x_j - |\mathbf{x}|^2 \delta_{ij}\right) \,. \tag{1}$$

For this, consider the initial condition where the $+q$ charges are located on the x-axis at time $t = 0$.

Result:

$$\begin{aligned} Q_{3i} &= Q_{i3} = 0,\ i = 1,2,3 \\ Q_{11} &= -Q_{22} = 3qa^2\ \mathrm{Re}\, e^{-2i\omega t} \\ Q_{21} &= Q_{12} = 3qa^2\ \mathrm{Re}\, ie^{-2i\omega t} \end{aligned}$$

b) Show that the general expression for the angular power distribution in the far-field reads

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mu_0}{4\pi}\frac{c^3k^6}{288\pi}\,|\hat{\mathbf{r}} \times Q_0\hat{\mathbf{r}}|^2 \tag{2}$$

with $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. Here, Q_0 is the quadrupole tensor, a 3×3 amplitude matrix defined by components Q_{ij} without the oscillating factor.

Hint: The fields in the far-field approximation are given by

$$\mathbf{B} = -\frac{\mu_0}{4\pi}\frac{ik^3c}{6}\frac{e^{ikr}}{r}\,\hat{\mathbf{r}} \times Q_0\hat{\mathbf{r}} \quad \text{and} \quad \mathbf{E} = c\,\mathbf{B} \times \hat{\mathbf{r}}\,, \tag{3}$$

and the angular power distribution reads

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{r^2}{2\mu_0}\,\hat{\mathbf{r}}\cdot(\mathbf{E} \times \mathbf{B}^*)\ . \tag{4}$$

c) Evaluate Eq. (2) with $k = 2\omega/c$ in spherical coordinates for the given setup of rotating quadrpole in the Fig. Justify why the frequency is 2ω? Compare the result of rotating quadrupole with the angular power distribution of an oscillating *dipole*.
Hint: Use the result from task a).

Problem 2: Spherical Bessel functions (Written)

Learning objective

The spherical Bessel and Hankel functions j_l and $h_l^{(1)} \equiv h_l^+$ play a crucial role for the expansion of the vector potential. In this exercise we will derive explicit expressions for these functions.

The spherical Bessel equation describes the radial part $R_l(r)$ of the solution $\Phi(r,\theta,\varphi) = R_l(r)Y_{lm}(\theta,\varphi)$ of the Helmoltz equation $[\Delta + k^2]\Phi = 0$ and reads

$$\left[\partial_x^2 + \frac{2}{x}\partial_x + \left(1 - \frac{l(l+1)}{x^2}\right)\right]R_l(x) = 0 \qquad \text{for} \qquad l \in \mathbb{N}_0 \tag{5}$$

with $x = kr$.

a) As a warm-up, show that for half integer $\nu = l + \frac{1}{2}$ the substitution $R_l(x) = \frac{u_l(x)}{\sqrt{x}}$ converts the spherical Bessel equation to the ordinary Bessel equation

$$\left[\partial_x^2 + \frac{1}{x}\partial_x + \left(1 - \frac{\nu^2}{x^2}\right)\right]u_l(x) = 0. \tag{6}$$

Provide solutions of Eq. (5) in terms of the Bessel and Neumann functions $J_\nu(x)$ and $N_\nu(x)$ which have been introduced in the lecture during the discussion of electrostatics.

b) The solutions derived from $J_\nu(x)$ and $N_\nu(x)$ are denoted as $j_l(x)$ and $n_l(x)$ and referred to as *spherical* Bessel and Neumann functions, respectively (up to normalizing factors). In the remainder of this exercise, you will derive explicit expressions for these functions.

To this end, prove that the *spherical Hankel functions*

$$h_l^{\pm}(x) = \mp\frac{(x/2)^l}{l!}\int_{\pm 1}^{i\infty} \mathrm{d}t\, e^{ixt}(1-t^2)^l \tag{7}$$

are solutions of Eq. (5) for $x > 0$ and $l \in \mathbb{N}_0$.

Hint: Use $x^{-1}\partial_x^2 x = 2x^{-1}\partial_x + \partial_x^2$ and write the integrand as a total derivative with respect to t.

c) Now show that $h_l^{\pm}$ satisfy the recursion relation

$$\frac{\mathrm{d}h_l^{\pm}(x)}{\mathrm{d}x} = \frac{l}{x}h_l^{\pm}(x) - h_{l+1}^{\pm}(x)\,. \tag{8}$$

d) The spherical Hankel functions are a basis of the two-dimensional solution space for every l. Another common basis is given by the linear combinations

$$j_l(x) = \frac{1}{2}[h_l^{+}(x) + h_l^{-}(x)] \qquad \text{and} \qquad n_l(x) = \frac{1}{2i}[h_l^{+}(x) - h_l^{-}(x)] \tag{9}$$

which are the *spherical* Bessel and Neumann functions as introduced in task a).

Use the recursion from task c) to prove the explicit expressions

$$j_l(x) = (-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\sin(x)}{x} \tag{10a}$$

$$n_l(x) = -(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\frac{\cos(x)}{x} \tag{10b}$$

These are known as *Rayleigh's formulas.*

Hint: Mathematical induction.

e) Use the above results to write down $j_l(x), n_l(x)$ and $h_l^{+}(x), h_l^{-}(x)$ for $l = 0, 1$ and sketch the graphs of $j_l(x), n_l(x)$.

Problem 3: Lifetime of "classical" atoms (Written)

Learning objective

In this problem we will calculate the lifetime of a classical atom in presence of an oscillating dipole. We will learn about the stability of the atom caused by the dipole radiation in a classical and a quantum picture.

In the lecture it was shown that an oscillating dipole $\mathbf{p}(t) = \mathbf{p}_0 e^{i\omega t}$ radiates the power

$$P_{\text{Dipole}} = \frac{1}{4\pi\varepsilon_0}\frac{ck^4}{3}|\mathbf{p}_0|^2 \tag{11}$$

with vacuum dispersion $ck = \omega$.

Pretend you forgot everything you have learned in your quantum mechanics course last semester and think of the hydrogen atom as a positive central charge $+e$ with mass m_p (the proton) orbited classically by a charge with opposite sign $-e$ and mass m_e (the electron). Due to $m_p \gg m_e$ you may consider the proton stationary, $\mathbf{r}_p(t) \equiv \mathbf{0}$, and the electron's position parameterized by $\mathbf{r}_e(t) = a_B \cos(\omega_e t)\mathbf{e}_x + a_B \sin(\omega_e t)\mathbf{e}_y$ where $a_B = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$ is the Bohr radius.

Starting from (11), derive an expression for the radiated power of this "classical" atom in terms of ω_e and a_B. Get an estimate for ω_e using Coulomb's law and plug in the numbers to derive the timescale that governs the lifetime τ of this system. Are these results consistent with your experience? How does quantum mechanics mend this matters?