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Problem 1: Dimensional regularization (Written, 5 points)

> **Learning objective**
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> In this exercise we will work on the technical details of *dimensional regularization* (due to 't Hooft and Veltman). Dimensional regularization preserves the symmetries of QED and a broader class of more general theories. The idea of dimensional regularization is to extend the definition of d-dimensional volume integrals to arbitrary $d \in \mathbb{R}$. If the divergences of integrals from Feynman diagrams vanish for $d < 4$, they can be regularized if the limit $d \to 4$ is taken after evaluating physical quantities.

Let us consider spacetime to have one time dimension and $(d-1)$ space dimensions ($d = 2, 3, 4, \dots$). We are interested in solving integrals of the form

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int d\ell \, \frac{\ell^{d-1}}{(\ell^2 + \Delta)^2} \tag{1}$$

where we have Wick-rotated the time dimension so that $d^d\ell_E$ is the volume element of d-dimensional *Euclidean* space; $d\Omega_d$ denotes the angular part of the integral in d-dimensional spherical coordinates.

a) The first factor in Eq. (1) contains the area of a unit sphere in d dimensions. Show that

$$\int d\Omega_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}\,. \tag{2}$$

Use $\int dx\, e^{-x^2} = \sqrt{\pi}$ and the definition of the Gamma function $\Gamma(t) := \int_0^\infty dx\, x^{t-1} e^{-x}$.

b) With the result from a), show that Eq. (1) evaluates to

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\,. \tag{3}$$

To this end, use the substitution $x = \Delta/(\ell^2 + \Delta)$ and the definition of the beta function

$$B(\alpha, \beta) := \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,. \tag{4}$$

The expression Eq. (3) can now be used to *define* the left-hand side for $d \in \mathbb{R}$.

Where are the poles of this generalized integral in d "dimensions"?

c) Define $\epsilon = 4 - d$ and use the infinite product representation

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \tag{5}$$

(γ is the Euler-Mascheroni constant) to expand $\Gamma(2 - \frac{d}{2})$ to first order in ϵ.

d) Show that the integral (3) takes the asymptotic form

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} \xrightarrow{d\to 4} \frac{1}{(4\pi)^2}\left[\frac{2}{\epsilon}+\log\frac{4\pi}{\Delta}-\gamma+\mathcal{O}(\epsilon)\right]. \tag{6}$$

This expression extracts the diverging part of the integral for $d \to 4$ and allows for the controlled treatment of such integrals.

e) Following the previous steps, verify the more general expressions

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}, \tag{7a}$$

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}. \tag{7b}$$

These integrals are useful for the renormalization of the electric charge (see lecture).

Problem 2: Thomas-Fermi screening (Written, 3+1 points)

Learning objective

As already demonstrated in previous tasks, the machinery of quantum field theory is not restricted to high-energy physics and fundamental theories like QED; its application to condensed matter physics provides one of the most powerful tools to study strongly correlated quantum matter. In this exercise, we will study the so called *Thomas-Fermi screening* of electrons in a degenerate electron gas of density n at zero temperature.

a) Similar to the lecture, define $\Pi(q)$ to be the sum of all *one-particle-irreducible* diagrams contributing to the photon self-energy. Show by diagrammatically expanding the *full* photon propagator $D_{\text{ph}}(q)$ that

$$D_{\text{ph}}(q) = \frac{D^0_{\text{ph}}(q)}{1-D^0_{\text{ph}}(q)\Pi(q)}, \tag{8}$$

where $D^0_{\text{ph}}(q)$ is the bare photon propagator.

This approach is related to the so called *Lindhard theory* in condensed matter theory used for calculating the effects of electric field screening by electrons.

b) In condensed matter theory, the bare photon propagator in momentum space is simply given by the Fourier transform $U(\mathbf{q})$ of the (time-independent) interaction potential. Then, the denominator in (8) can be seen as a dielectric function given (in the static limit) as

$$\varepsilon(\mathbf{q}) = 1 - U(\mathbf{q})\Pi(\mathbf{q}). \tag{9}$$

Show that the bare Coulomb interaction in momentum space, $U(\mathbf{q}) = e^2/q^2$, is now modified to an effective interaction due to the screening of the electron gas in the long wavelength limit (i.e., Π is evaluated at $\mathbf{q} = \mathbf{0}$):

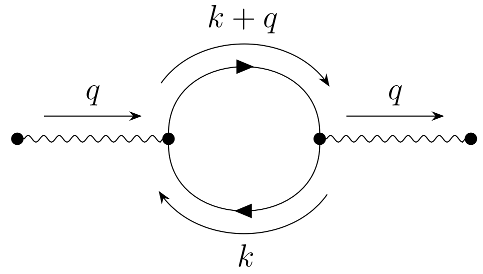
$$U_{\text{eff}}(\mathbf{q}) = \frac{e^2}{q^2+\lambda_{\text{TF}}^{-2}}, \tag{10}$$

where λ_{TF}^{-1} is the *Thomas-Fermi wave vector*.

c) Calculate the Fourier transform $U_{\text{eff}}(\mathbf{x})$ of the effective potential (10) and discuss your result.

d) **Optional:**

Calculate the Thomas-Fermi wave vector in the long wavelength limit ($\mathbf{q} \to \mathbf{0}$) and in the so-called *random-phase approximation*, where $\Pi(q)$ consists only of the particle-hole(=antiparticle) loop (neglecting the in- and outgoing lines):



According to the Feynman rules in condensed matter theory, $\Pi(q)$ is given by

$$\Pi(q) = -2i \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} G^0(\omega, \mathbf{k}) G^0(\omega, \mathbf{k} + \mathbf{q}) \,, \tag{11}$$

where the propagator/Green's function reads

$$G^0(\omega, \mathbf{k}) = \frac{1}{\omega - \xi(\mathbf{k}) + i\delta \, \text{sgn}(\xi(\mathbf{k}))} \tag{12}$$

with $\xi(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - E_F$ and E_F the Fermi energy. δ is to be taken positive but small (i.e. $\delta \to 0^+$) and $\text{sgn}(x)$ refers to the signum function, which gives the sign of x and $\text{sgn}(0) = 0$.

Hint: In 3D, the Fermi energy is given by $E_F = (3\pi^2 n)^{2/3}/(2m)$ with electron density n and mass m.