

**Problem 1: Propagator in the path integral formalism (Written, 3 points)**

**Learning objective**

In this problem, you use path integrals to construct the propagator (or two-point correlation function) of a generic quadratic field theory. You show that the propagator is given by the inverse of the quadratic form associated with the field theory.

Consider a generic quadratic theory in momentum space with action

$$S = \frac{1}{2} \sum_k \phi_{i,k} M_k^{ij} \phi_{j,-k} \tag{1}$$

with a symmetric matrix  $M$ , which also satisfies  $M_k = M_{-k}$ . Since the fields  $\phi_i$  are assumed to be real-valued, their Fourier components fulfill the relation  $\phi_{i,-k} = \phi_{i,k}^*$  and we can rewrite the action as

$$S = \sum_{k^0 > 0} \phi_{i,k} M_k^{ij} \phi_{j,k}^* \tag{2}$$

Note that the sum over the momenta now only runs over one half of the four-momentum space, that is  $k^0 > 0$ .

In order to calculate the two-point correlation function, it proves useful to introduce the *generating functional*

$$Z[J, J^*] = \int \mathcal{D}\phi \exp \left( i \sum_{k^0 > 0} [\phi_{i,k} M^{ij} \phi_{j,k}^* + (J_k^i)^* \phi_{i,k} + J_k^j \phi_{j,k}^*] \right) \tag{3}$$

a) Calculate the path integral in (3) and show that the generating functional is given by

$$Z[J, J^*] = \prod_{k^0 > 0} \left( \frac{i\pi}{\det M_k} \right) \exp \left( -i J_{i,k} (M_k^{-1})^{ij} J_{j,k}^* \right) \tag{4}$$

**Hints:**

- Since the matrix  $M$  is symmetric, it can be diagonalized by a unitary transformation, that is  $M = U^\dagger D U$ , where  $D$  is diagonal and  $U$  is unitary.
- After diagonalizing the matrix and decoupling the fields, split each field into real and imaginary part,  $\phi_{i,k} = \phi_{i,k}^{\text{re}} + i\phi_{i,k}^{\text{im}}$ .
- The path integral measure is given by  $\mathcal{D}\phi = \prod_{j,k^0 > 0} d\phi_{j,k}^{\text{re}} d\phi_{j,k}^{\text{im}}$ .
- The remaining Gaussian integrals can be calculated by completing the square.

- b) Use (3) to relate the two-point correlation function in Fourier space to the generating functional  $Z[J, J^*]$  and show that

$$\tilde{D}_F^{ij}(k) = \langle \phi_k^i \phi_k^{j*} \rangle = \frac{1}{Z[0, 0]} \left( -i \frac{\partial}{\partial J_{i,k}^*} \right) \left( -i \frac{\partial}{\partial J_{j,k}} \right) Z[J, J^*] \Big|_{J, J^* = 0}. \quad (5)$$

- c) Finally, use (4) to show that  $\tilde{D}_F^{ij}(k) = i (M_k^{-1})^{ij}$ .

**Problem 2: Path integral and Weyl order (Written, 5 points)**

**Learning objective**

This problem deals with the connection between the transition amplitude of a quantum system and the path integral formalism. In doing so, the peculiarity of non-commuting operators arises which can be resolved by employing a special ordering of operators in the Hamiltonian called *Weyl order*. As an example, you will calculate the transition amplitude for a non-relativistic particle in one dimension.

Consider a general quantum system described by a set of coordinates  $q^i$ , conjugate momenta  $p^i$  and Hamiltonian  $H(q, p)$ . In the lecture, it was shown that the transition amplitude  $U(q_a, q_b; T) = \langle q_b | e^{-iHT} | q_a \rangle$  can be computed by breaking the time interval into  $N$  short slices of length  $\epsilon$  and inserting a complete set of intermediate states between each slice such that

$$U(q_a, q_b; T) = \prod_i \prod_{k,l} \int dq_l^i \langle q_{k+1} | e^{-i\epsilon H} | q_k \rangle, \quad (6)$$

where  $k = 0, \dots, N - 1, l = 1, \dots, N - 1$  and  $q_a = q_0$  and  $q_b = q_N$ . Since  $\epsilon \rightarrow 0$ , we may expand the exponential as  $e^{-i\epsilon H} = 1 - i\epsilon H + \dots$ . In a first step, consider a Hamiltonian which is a pure function of either  $q$  or  $p$ , that is  $H(q, p) = f(q)$  or  $H(q, p) = f(p)$ .

- a) Show that if  $H$  is only a function of the coordinates, the matrix element can be written as

$$\langle q_{k+1} | f(q) | q_k \rangle = f\left(\frac{q_{k+1} + q_k}{2}\right) \left( \prod_i \int \frac{dp_k^i}{2\pi} \right) \exp\left(i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right). \quad (7)$$

- b) Now consider the case when the Hamiltonian only depends on the momenta. Show that

$$\langle q_{k+1} | f(p) | q_k \rangle = \left( \prod_i \int \frac{dp_k^i}{2\pi} \right) f(p_k) \exp\left(i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right). \quad (8)$$

Show also that if  $H$  contains only terms of the form  $f(q)$  and  $f(p)$ , its matrix elements can be written

$$\langle q_{k+1} | H(q, p) | q_k \rangle = \left( \prod_i \int \frac{dp_k^i}{2\pi} \right) H\left(\frac{q_{k+1} + q_k}{2}, p_k\right) \exp\left(i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right). \quad (9)$$

- c) In general, (9) is false when there are products of  $q$ 's and  $p$ 's in the Hamiltonian as on the left-hand side the order of the (non-commuting) operators matters while on the right-hand side we only deal with numbers. Show this explicitly for  $H = p^2 q^2$  and show that putting the Hamiltonian into *Weyl order*, that is

$$H(q, p) = p^2 q^2 \mapsto H_W(q, p) = \frac{1}{4}(q^2 p^2 + 2qp^2q + p^2 q^2), \quad (10)$$

resolves this issue. Note that any Hamiltonian can be put into Weyl order by commuting  $p$ 's and  $q$ 's on the cost of some extra terms appearing on the right-hand side of (9).

- d) Show that for a Weyl-ordered Hamiltonian, the propagator (6) is given by

$$U(q_N, q_0; T) = \left( \prod_{i,k} \int dq_k^i \int \frac{dp_k^i}{2\pi} \right) \exp \left[ i \sum_k \left( \sum_i p_k^i (q_{k+1}^i - q_k^i) - \epsilon H \left( \frac{q_{k+1} + q_k}{2}, p_k \right) \right) \right]. \quad (11)$$

This expression is the discretized form of

$$U(q_a, q_b; T) = \int \mathcal{D}q(t) \mathcal{D}p(t) \exp \left[ i \int_0^T dt \left( \sum_i p^i \dot{q}^i - H(q, p) \right) \right] \quad (12)$$

and defines what we understand as a *path integral*.

- e) As a special case, consider the Hamiltonian  $H = p^2/2m + V(q)$  of a single, non-relativistic particle in one dimension. Show that the transition amplitude reads

$$U(q_a, q_b; T) = \left( \frac{1}{C(\epsilon)} \prod_k \int \frac{dq_k}{C(\epsilon)} \right) \exp \left[ i \sum_k \left( \frac{m}{2} \frac{(q_{k+1} - q_k)^2}{\epsilon} - \epsilon V \left( \frac{q_{k+1} + q_k}{2} \right) \right) \right] \quad (13)$$

and determine  $C(\epsilon)$ .