Quantum Field Theory

Lecture Notes • Summer Term 2020

Nicolai Lang*

Institute for Theoretical Physics III
University of Stuttgart

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*nicolai@itp3.uni-stuttgart.de

Contents

0 Preliminaries

Requirements

For this course, we assume that students are familiar with the following concepts:

- Non-relativistic quantum mechanics and second quantization
- The Lagrangian and Hamiltonian formalism of classical mechanics
- Special theory of relativity and tensor calculus
- Complex analysis (contour integrals, residue theorem, ...)

Literature

- Weinberg: *The Quantum Theory of Fields (Volume 1)*
  `ISBN 978-0-521-67053-1`
  Standard reference, very rigorous & mathematical, #formulas/#text = high
- Itzykson & Zuber: *Quantum Field Theory*
  `ISBN 978-0-486-44568-7`
  Standard reference, #formulas/#text = high
- **Peskin & Schroeder: *An Introduction to Quantum Field Theory***
  `ISBN 978-0-201-50397-5`
  Standard reference for courses on QFT, #formulas/#text = medium
- Zee: *Quantum Field Theory in a Nutshell*
  `ISBN 978-0-691-14034-6`
  Compact and pedagogical introduction to the field, #formulas/#text = low

Suggestion: For a first introduction to QFT, read *Peskin & Schroeder* (which we will use in this course). Then, if you are hooked and want to understand QFT *really* (in particular its mathematical foundations) read *Weinberg* afterwards.

Goals

The goal of this course is to gain a thorough understanding of relativistic quantum field theory, the concepts of Feynman diagrams, renormalization for quantum electrodynamics, and to extend this knowledge to non-abelian gauge theories. In particular (⁂ optional):

- Relativistic quantum mechanics (Klein-Gordon and Dirac field)
- Quantization of free fields
- Perturbative analysis of interacting fields
- Feynman rules and diagrams

↑ Notes

- Elementary processes and first corrections of quantum electrodynamics
- Path integral formalism
- Renormalization
- Non-abelian gauge fields ⁂
- The standard model ⁂

This course follows and partially covers Part I (field quantization, perturbation theory, Feynman rules) and Part II (path integrals, renormalization) of "An Introduction to Quantum Field Theory" by Peskin & Schroeder. If there is time, we close with a brief perspective on Part III (non-abelian gauge theories, standard model).

Note

This is **not** an extension of the material covered in the lectures but the script that I use to prepare them. Please have a look at Peskin & Schroeder for more comprehensive coverage; the corresponding pages are noted in the headers (→ P&S • pp. xx-yy).

Key

- The content of this script is color-coded as follows:
  - Text in black is written to the blackboard.
  - Notes in red should be mentioned in the lecture to prevent misconceptions.
  - Notes in blue can be mentioned/noted in the lecture if there is enough time.
  - Notes in green are hints for the lecturer.
- One page of the script corresponds roughly to one covered panel of the blackboard.
- Only equations that are later refered to are numbered.
- Enumerated lists are used for more or less rigorous chains of thought:
  1 leads to 2 leads to 3 ...
- "$\sphericalangle$ " reads "consider"
- "$\rightarrow$ " reads "therefore"
- "➔ " reads "see"
- "$\overset{\circ}{=}$" denotes a non-trivial equality that requires lengthy, but straightforward calculations
- "$\overset{*}{=}$" denotes a non-trivial equality that cannot be derived without additional input
- "$\overset{\circ}{\rightarrow}$ " reads "it is easy to show" (not the Russian style, really easy ...)
- "$\overset{*}{\rightarrow}$ " reads "it is *not* easy to show"
- "$\Rightarrow$" denotes a logical implication
- "$\wedge$" denotes a logical conjunction

↑ Notes

- "$\vee$" denotes a logical disjunction
- "$\square$" denotes a repeated expression

↑ Notes

1 Elements of Classical Field Theory

Problem Set 1

(due **17.04.2020**)

1. Functional derivatives, rules and applications
2. Lorentz covariance and application of tensor calculus to Maxwell theory
3. Maxwell equations from action principle and symmetric energy-momentum tensor

1.1 Lagrangian and Hamiltonian Formalism

Recap: Classical mechanics of "points"

With "points" we mean a discrete set of degrees of freedom.

1. Degrees of freedom q_i labeled by $i = 1, \dots, N$
2. Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t) = T - V$
   We write q for $\{q_i\} = \{q_1, \dots, q_N\}$.
   T is the kinetic, V the potential energy.
3. Action $S[q] = \int dt\ L(q(t), \dot{q}(t), t)\ \in \mathbb{R}$
   This is a functional of trajectories $q = q(t)$.
4. Hamilton's principle of least action:
   $$\frac{\delta S[q]}{\delta q} \overset{!}{=} 0 \quad \Leftrightarrow \quad \delta S = \int dt\, \delta L \overset{!}{=} 0$$
   δ denotes functional derivatives/variations (➔ Problemset 1).
5. Euler-Lagrange equations ($i = 1, \dots, N$):
   $$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0$$

Analogous: Lagrangian Field Theory

Now we consider a continuous set of degrees of freedom.

1. One or more fields $\phi(x)$ on spacetime $x \in \mathbb{R}^{1,3}/\mathbb{R}^4$
   with derivatives $\partial_\mu \phi(x)$ where $\partial_0 = \partial_t$ and $\partial_{i=1,2,3} = \partial_{x,y,z}$
   $\mathbb{R}^{1,3}$ for Minkowski metric, $\mathbb{R}^4$ for Euclidean metric.

2. Lagrangian *density* $\mathcal{L}(\phi, \partial\phi, x)$
   Most general form: $\mathcal{L}(\{\phi_k\}, \{\partial_\mu \phi_k\}, \{x^\mu\})$. No explicit x^μ-dependence in the following!
   $\rightarrow$ Lagrangian $L = \int d^3x\ \mathcal{L}(\phi, \partial\phi)$
   We omit the "density" in the following.

3. Action:

$$S[\phi] = \int dt\ L = \int dt\ d^3x\ \mathcal{L}(\phi, \partial\phi) = \int d^4x\ \mathcal{L}(\phi, \partial\phi)$$

   $S[\phi]$ is a functional of "field trajectories" in $\mathbb{R}^{1,3}$.

4. Action principle:

$$\begin{aligned}
0 \stackrel{!}{=} \delta S[\phi] &= \int d^4x\, \delta\mathcal{L} \\
&= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) \right\}
\end{aligned}$$

   Add zero and use $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) \right\}$$

   Gauss theorem

$$= \int_\partial dA\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\delta\phi}_{=0} + \int d^4x \underbrace{\left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) \right\}}_{=0} \delta\phi$$

   Note that ϕ is fixed on the boundary ∂ and therefore $\delta\phi = 0$.
   The second term vanishes because the integral must vanish for arbitrary variations $\delta\phi$.

5. Euler-Lagrange equations (one for each field ϕ):

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0}$$

   Note the Einstein summation! This expression is manifestly Lorentz invariant.

Recap: Hamiltonian Mechanics

$$\begin{array}{ccc}
\text{Lagrangian} & \xrightarrow{\text{Legendre transformation}} & \text{Hamiltonian} \\
L(q, \dot q, t) & \text{Conjugate momentum} & H(q,p,t) = p\dot q - L(q,\dot q,t) \\
 & p \equiv \frac{\partial L}{\partial \dot q} \Leftrightarrow \dot q = \dot q(p) &
\end{array}$$

↑ Notes

Analogous: Hamiltonian Field Theory

1. Let $x = x_i \mathrel{\widehat{=}} i$ be discrete *spatial* coordinates:

$$\frac{\partial L}{\partial \dot{q}_i} = p_i \mathrel{\widehat{=}} p(x) = \frac{\partial L}{\partial \dot\phi(x)} = \frac{\partial}{\partial \dot\phi(x)} \int d^3y\, \mathcal{L}(\phi(y), \dot\phi(y))$$

$$\sim \sum_y d^3y \underbrace{\frac{\partial}{\partial \dot\phi(x)} \mathcal{L}(\phi(y), \dot\phi(y))}_{\delta_{x,y} \frac{\partial \mathcal{L}}{\partial \dot\phi}\big|_{y=x}} = \underbrace{\frac{\partial \mathcal{L}}{\partial \dot\phi(x)}}_{\equiv \pi(x)} d^3x$$

In $\mathcal{L}(\phi(y), \dot\phi(y))$ we omit the time dependece!
→ Momentum *density* conjugate to ϕ is $\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi}$

2. Hamiltonian:

$$H = \sum_x \overbrace{p(x)}^{\pi(x)\, d^3x}\ \dot\phi(x) - \overbrace{L}^{\sum_x \mathcal{L}(\phi(x),\dot\phi(x))\, d^3x}$$

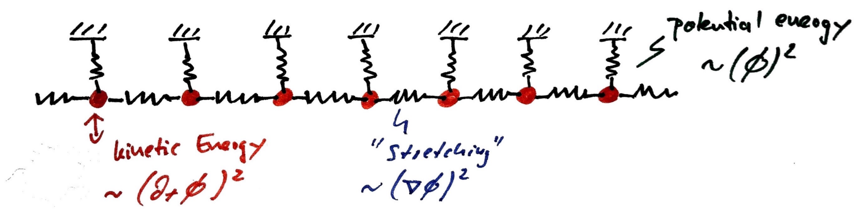
Therefore

$$H = \int d^3x \underbrace{\{\pi(x)\dot\phi(x) - \mathcal{L}(\phi, \dot\phi)\}}_{\text{Hamiltonian density } \mathcal{H}(\phi,\pi)}$$

Note that $\dot\phi = \dot\phi(\pi)$.

Example 1.1: Free scalar field

1. Real field $\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ with $(\vec{x}, t) \mapsto \phi(\vec{x}, t) = \phi(x)$
2. Lagrangian (density): $\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$
   It is $(\partial_\mu\phi)^2 \equiv \partial_\mu\phi\partial^\mu\phi = (\partial_t\phi)^2 - (\partial_x\phi)^2 - (\partial_y\phi)^2 - (\partial_z\phi)^2$ with signature $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$. Note that then $\partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$.
3. Interpretation:



In $\mathcal{L}$, m is refered to as *mass*. This is not the inertial mass of the pendula but the stiffness of the harmonic potential!
Continuum of spring-coupled pendula for $m = 0 \Leftrightarrow$ 1D rubber band

↑ Notes

4. Equation of motion ("field equation"):

$$-m^2\phi - \partial_\mu(\partial^\mu\phi) = 0 \quad \Leftrightarrow \quad (\partial_\mu\partial^\mu + m^2)\phi = 0$$

This is the classical (!) Klein-Gordon equation.

5. Conjugate momentum field: $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$

6. Hamiltonian (density):

$$\begin{aligned}\mathcal{H} &= \pi\dot{\phi} - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\end{aligned}$$

The Hamiltonian is $H = \int d^3x\, \mathcal{H}(\phi,\pi)$.

1.2 Symmetries and Conservation Laws

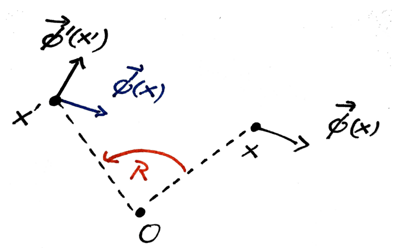
What follows is based on Sénéchal *"Conformal Field Theory"* (pp. 36–42,45–46)

1. Consider transformations on coordinates and fields:

$$x \to x' = x'(x) \quad \text{and} \quad \phi(x) \to \phi'(x') = \mathcal{F}(\phi(x))$$

Two effects: coordinates *and* fields transformed
These are *active transformations* that change physics!

Example 1.2: Rotation of a vector field $\vec{\phi}$



a) $\lhd$ 3-component field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and $R \in SO(3)$ rotation

b) $\vec{x}' = R\vec{x}$ and $\vec{\phi}'(x') = R\vec{\phi}(x) = R\vec{\phi}(R^{-1}x')$
This defines a *vector field*.

2. Change of the action under $\phi \mapsto \phi'$:

$$S' \equiv S[\phi'] = \int d^d x \, \mathcal{L}(\phi'(x), \partial_\mu \phi'(x))$$

Rename integration variables $x \to x'$

$$= \int d^d x' \, \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'))$$

Definition

$$= \int d^d x' \, \mathcal{L}(\mathcal{F}(\phi(x)), \partial'_\mu \mathcal{F}(\phi(x)))$$

Substitution

$$= \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}\left(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\phi(x)) \right) \tag{1.1}$$

Skip first step, use colors for primes.

Example 1.3: Translations

1. $x' := x + a$ and $\phi'(x') := \phi(x) = \phi(x' - a)$
   This defines a *scalar* field!
2. $\mathcal{F}$ trivial, $\phi'(x') = \mathcal{F}(\phi(x)) = \phi(x(x'))$, and $\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu$
3. Action:

$$S[\phi'] = \int d^d x \, \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) = \int d^d x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = S[\phi]$$

   The action is translation invariant: $S = S'$!
   This follows generally from the missing x-dependence of $\mathcal{L}$ for scalar fields.

↑ Notes

Example 1.4: Scale transformations

1. $x' := \lambda x$ and $\phi'(x') := \lambda^{-\Delta}\phi(x) = \lambda^{-\Delta}\phi(\lambda^{-1}x')$
   Δ is the *scaling dimension* of the field ϕ

2. $\mathcal{F}(\phi) = \lambda^{-\Delta}\phi$ and $\frac{\partial x^\nu}{\partial x'^\mu} = \lambda^{-1}\delta^\nu_\mu$ and $\left|\frac{\partial x'}{\partial x}\right| = \lambda^d$

3. Action:

$$S[\phi'] = \lambda^d \int d^dx\, \mathcal{L}(\lambda^{-\Delta}\phi(x), \lambda^{-1-\Delta}\partial_\mu\phi(x))$$

Massless scalar field $S[\phi] = \frac{1}{2}\int d^dx\,(\partial_\mu\phi)^2$

$$= \lambda^{d-2-2\Delta}\int d^dx\,\mathcal{L}(\phi(x),\partial_\mu\phi(x)) = \lambda^{d-2-2\Delta}S[\phi]$$

$\rightarrow S' = S$ iff $\Delta = \frac{d}{2} - 1$
This is an example of a *conformal field theory (CFT)*.

Example 1.5: Phase rotation

1. $x' := x$ and $\phi'(x') := e^{i\theta}\phi(x)$
   $\rightarrow$ There are symmetries that only transform the fields but not the coordinates.

2. $\mathcal{F}(\phi) = e^{i\theta}\phi$ and $\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu$ and $\left|\frac{\partial x'}{\partial x}\right| = 1$

Infinitesimal Transformations

We are interested in *continuous* symmetries (Lie groups).

1. Infinitesimal transformations (IT):

$$x'^\mu = x^\mu + w_a\frac{\delta x^\mu}{\delta w_a}(x) \quad \text{and} \quad \phi'(x') = \phi(x) + w_a\frac{\delta\mathcal{F}}{\delta w_a}(x) \tag{1.2}$$

   Here, w_a denotes infinitesimal parameters of the transformation (sum over a implied!). They may vary from point to point: $w_a = w_a(x)$ (see below).

2. *Generator* of IT:

$$\delta_w\phi(x) := \phi'(x) - \phi(x) \equiv -i\,w_a\,G_a\phi(x)$$

   With

$$\begin{aligned}\phi'(x') &= \phi(x) + w_a\frac{\delta\mathcal{F}}{\delta w_a}(x)\\ &= \phi(x') - w_a\frac{\delta x^\mu}{\delta w_a}\partial_\mu\phi(x') + w_a\frac{\delta\mathcal{F}}{\delta w_a}(x') + \mathcal{O}(w^2)\end{aligned}$$

↑ Notes

Omit first line and refer to previous equation!
it follows

$$iG_a\phi = \frac{\delta x^\mu}{\delta w_a}\partial_\mu\phi - \frac{\delta\mathcal{F}}{\delta w_a}$$

"Change of field at the same point."

Example 1.6: Translations

1. $x'^\mu = x^\mu + w^\mu = x^\mu + w^\nu \frac{\delta x^\mu}{\delta w^\nu}$ with $\frac{\delta x^\mu}{\delta w^\nu} = \delta^\mu_\nu$
2. $\frac{\delta\mathcal{F}}{\delta w^\nu} = 0$
   For a scalar or a vector field.
3. $iG_\mu\phi = \delta^\nu_\mu\partial_\nu\phi - 0$ and therefore

$$G_\mu = -i\partial_\mu \equiv P_\mu$$

→ The "momentum operator" generates translations.

Example 1.7: Scale Transformations

$G = -ix^\mu\partial_\mu \equiv D$
Generates "dilations" in spacetime. This simple form is valid for a scalar field with scaling dimension $\Delta = 0$.

Example 1.8: Spatial Rotations

$G_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + S_{\mu\nu}$ for $\mu,\nu = 1,2,3$
First term generates coordinate rotations (orbital angular momentum operator).
$S_{\mu\nu}$ are spin matrices that generate field transformations.
Question: What generates $G_{\mu\nu}$ if either $\mu = 0$ or $\nu = 0$? Answer: Boosts.

Noether's Theorem

1. Transformation (1.2) is symmetry of the action
   $:\Leftrightarrow S[\phi] = S[\phi']$ for w_a independent of x (*rigid transformation*)
2. *Assume* that (1.2) ist *not* rigid: $w_a = w_a(x)$
3. Jacobian: $\frac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu + \partial_\mu\left(w_a\frac{\delta x^\nu}{\delta w_a}\right) \to \left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu\left(w_a\frac{\delta x^\mu}{\delta w_a}\right)$
   Use $\det(1+A) = 1 + \mathrm{Tr}[A] + \mathcal{O}(A^2)$.
4. Inverse Jacobian matrix: $\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \partial_\mu\left(w_a\frac{\delta x^\nu}{\delta w_a}\right)$

↑ Notes

5. Use (1.1):

$$S' = \int d^dx \left[1 + \partial_\mu\left(w_a \frac{\delta x^\mu}{\delta w_a}\right)\right]$$
$$\times \mathcal{L}\left(\phi + w_a \frac{\delta \mathcal{F}}{\delta w_a}, \left[\delta^\nu_\mu - \partial_\mu\left(w_a \frac{\delta x^\nu}{\delta w_a}\right)\right] \times \left[\partial_\nu \phi + \partial_\nu\left(w_a \frac{\delta \mathcal{F}}{\delta w_a}\right)\right]\right)$$

6. Expand in 1st order of w_a and $\frac{\partial w_a}{\partial x^\mu}$
   We assume that w_a is not only small but also varies smoothly so that $\frac{\partial w_a}{\partial x^\mu} \ll 1$.

7. $\sphericalangle\ \delta S \equiv S' - S \to$ Only terms $\propto \frac{\partial w_a}{\partial x^\mu}$ remain
   Because the transformation is a symmetry of the action by assumption, i.e., for $w_a =$ const (a rigid transformation) it is $S' = S$!
   This is equivalent to the *definition* of a symmetry (of the action).

8. For generic, non-rigid transformation we find

$$\delta S = -\int d^dx\, j^\mu_a \partial_\mu w_a$$

   with the *current*

$$j^\mu_a = \left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right\}\frac{\delta x^\nu}{\delta w_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\delta \mathcal{F}}{\delta w_a} \tag{1.3}$$

   associated to the IT $\frac{\delta x^\nu}{\delta w_a}$ and $\frac{\delta \mathcal{F}}{\delta w_a}$.
   This is only true for transformations that are symmetries of the action!

9. Integration by parts: $\delta S = \int d^dx\, w_a\, \partial_\mu j^\mu_a$

10. Let ϕ obey the equations of motion $\to \delta S = 0$ for *arbitrary* variations $\phi' = \phi + \delta\phi$
    In particular, for arbitrary non-rigid transformations $w_a(x)$!

    It follows *Noether's (first) theorem:*

$$\partial_\mu j^\mu_a = 0 \qquad \forall_{x,a}$$

    This is a *conservation law* with *conserved current* j^μ_a.

11. *Conserved charge:*

$$Q_a := \int_{\text{Space}} d^{d-1}x\, j^0_a$$

    Indeed

$$\frac{\mathrm{d}Q_a}{\mathrm{d}t} = \int_{\text{Space}} d^{d-1}x\, \partial_0 j^0_a \overset{\text{Noether}}{=} -\int_{\text{Space}} d^{d-1}x\, \partial_k j^k_a \overset{\text{Gauss}}{=} -\int_{\text{Surface}} d\sigma_k\, j^k_a = 0$$

    Here we assume that $j^k_a \equiv 0$ at spatial infinity, i.e., the universe is closed.
    $k = 1, 2, 3$ denotes the spatial coordinates.

↑ Notes

Note 1.1

The current (1.3) is called *canonical current* as there is an ambiguity:

$$\tilde{j}_a^\mu := j_a^\mu + \partial_\nu B_a^{\mu\nu} \quad \text{with} \quad B_a^{\mu\nu} = -B_a^{\nu\mu} \quad \text{arbitrary} \quad \Rightarrow \quad \partial_\mu \tilde{j}_a^\mu = 0$$

Note 1.2

$$\underbrace{\text{Symmetric Lagrangian} \quad \Rightarrow \quad \text{Symmetric action}}_{\to \text{Conserved currents}} \quad \Rightarrow \quad \text{Symmetric EOMs}$$

Continuous symmetries of the EOMs do *not* imply conserved currents!

Application: The Energy-Momentum-Tensor (EMT)

Special relativity: global spacetime symmetries (Lorentz transformations).
General relativity: local spacetime symmetries (diffeomorphisms → gauge symmetries).

1. $\sphericalangle$ infinitesimal spacetime translations: $x'^\mu = x^\mu + \varepsilon^\mu \to \frac{\delta x^\mu}{\delta \varepsilon^\nu} = \delta^\mu_\nu$, $\frac{\delta \mathcal{F}}{\delta \varepsilon^\nu} = 0$
2. $\sphericalangle$ translation invariant action: $S' = S$
   This includes translations in time!
3. Conserved currents:

$$T^\mu{}_\nu = \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi - \delta^\mu_\rho \mathcal{L} \right\} \underbrace{\frac{\delta x^\rho}{\delta \varepsilon^\nu}}_{\delta^\rho_\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

$$T^{\mu\nu} = g^{\nu\rho} T^\mu{}_\rho = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \qquad \text{(Energy-Momentum Tensor)}$$

with $\partial_\mu T^{\mu\nu} = 0$ and four conserved charges

$$P^\nu = \int d^3x\, T^{0\nu}$$

4. *Energy* ($\nu = 0$):

$$P^0 = \int d^3x\, T^{00} = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial \dot\phi} \dot\phi - \mathcal{L} \right\} = \int d^3x\, \mathcal{H}(\phi, \pi) = H$$

Skip first step.
This is only true if the EOMs are satisfied.
The Hamiltonian is the component of a tensor and not Lorentz invariant!
By contrast, the Lagrangian *is* Lorentz invariant.

↑ Notes

5. *Kinetic momentum* ($\nu = i$):

$$P^i = \int d^3x\, T^{0i} = \int d^3x\, \frac{\partial \mathcal{L}}{\partial \dot\phi}(-\partial_i \phi) = -\int d^3x\, \pi\, \partial_i \phi$$

π is the *canonical* momentum.

Note 1.3

In general $T^{\mu\nu} \neq T^{\nu\mu}$ for the canonical EMT. But:

$$\tilde{T}^{\mu\nu} := T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} \quad \text{with} \quad K^{\rho\mu\nu} = -K^{\mu\rho\nu}$$

Choose $K^{\rho\mu\nu}$ such that $\tilde{T}^{\mu\nu} = \tilde{T}^{\nu\mu}$ (→ *Belinfante EMT*)

Example 1.9: Electromagnetism (EM) in the vacuum

1. Four-component field: $A^\mu = (\phi, A^1, A^2, A^3)$
2. EM field tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
   Contains E- and B-field components.
3. Lagrangian: $\mathcal{L}_{\text{em}}(A^\nu, \partial_\mu A^\nu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$
4. Action: $S_{\text{em}} = \int d^4x\, \mathcal{L}_{\text{em}}$
5. Euler-Lagrange equations: $\partial_\mu F^{\mu\nu} = 0$ (inhomogeneous Maxwell equations)
6. S_{em} is Lorentz invariant and translation invariant (= Poincaré invariant)
   → EMT = conserved currents
   Ask students why this is obvious.
7. Canonical EMT: $T^{\mu\nu}_{\text{em}} = \frac{\partial \mathcal{L}_{\text{em}}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L}_{\text{em}}$
8. Symmetric EMT via $K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$:

$$\boxed{\tilde{T}^{\mu\nu}_{\text{em}} = \frac{1}{4} g^{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - F^{\mu\rho} F^{\nu}{}_{\rho}}$$

- $\tilde{T}^{00} = \frac{1}{2}(E^2 + B^2)$ (energy density)
- $\tilde{T}^{0i} = (\vec{E} \times \vec{B})_i$ (Pointing vector)
- $\tilde{T}^{ij} = \sigma_{ij}$ (Maxwell stress tensor)

Details ➔ Problemset 1.

↑ Notes

2 The Klein-Gordon Field

Problem Set 2

(due **24.04.2020**)

1. The classical complex Klein-Gordon field and its Noether current
2. The quantized complex Klein-Gordon field and its conserved charge

2.1 Quantization

1. Theory:
   a) *Real* field $\phi(x)$ (➔ Problemset 2 for the complex analog)
   b) Lagrangian: $\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$ (free scalar field)
   c) EOM: $(\partial^2 + m^2)\phi = 0$ (Klein-Gordon equation)
   d) Hamiltonian: $\mathscr{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$
2. *Canonical quantization:*

$$\begin{aligned} \left[\phi(\vec{x}), \pi(\vec{y})\right] &= i\delta^{(3)}(\vec{x}-\vec{y}) \\ \left[\phi(\vec{x}), \phi(\vec{y})\right] &= 0 \\ \left[\pi(\vec{x}), \pi(\vec{y})\right] &= 0 \end{aligned} \tag{2.1}$$

   with $\phi^\dagger = \phi$ and $\pi^\dagger = \pi$ ("real" field operators).

   This is completely analog to the canonical quantization of "points" known from undergraduate courses on quantum mechanics if Kronecker deltas are replaced by delta distributions: $[q_i, p_i] = i\delta_{ij} \;\to\; [\phi(x), \pi(y)] = i\delta(x-y)$.

   For now, we are in the Schrödinger picture where the fields do not depend on time!
3. *Goal:* Spectrum of Hamiltonian?
4. *Motivation:*
   a) Fourier transform of KG equation in space:

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \tilde{\phi}(\vec{p}, t)$$

   Then

$$\left[\partial_t^2 + (|\vec{p}|^2 + m^2)\right]\tilde{\phi}(\vec{p}, t) = 0$$

   → decoupled harmonic oscillators with frequency $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$
   (and constraint $\tilde{\phi}^*(\vec{p}, t) = \tilde{\phi}(-\vec{p}, t)$ since $\phi^* = \phi$)

↑ Notes

b) $\lessdot$ Hamiltonian $H_{\text{SHO}} = \frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\omega^2\tilde{\phi}^2$ and introduce
$\tilde{\phi} = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$ and $\tilde{\pi} = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$ with $[a, a^\dagger] = 1$
$\rightarrow H_{\text{SHO}} = \omega(a^\dagger a + \frac{1}{2})$ (diagonal!)

5. This motivates the *Field operators:*

$$\begin{aligned}\phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}}\right)\\ &= \int \frac{d^3p}{(2\pi)^3}\underbrace{\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}} + a^\dagger_{-\vec{p}}\right)}_{\tilde{\phi}(\vec{p})}e^{i\vec{p}\vec{x}}\\ \pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\underbrace{(-i)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}} - a^\dagger_{-\vec{p}}\right)}_{\tilde{\pi}(\vec{p})}e^{i\vec{p}\vec{x}}\end{aligned} \tag{2.2}$$

(Use colors to skip second line.)
The $-\vec{p}$ is necessary to make the fields "real", i.e., $\phi^\dagger = \phi$.
with *momentum modes*

$$\left[a_{\vec{p}}, a^\dagger_{\vec{q}}\right] = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{q}) \tag{2.3}$$

$\overset{\circ}{\rightarrow}$ (2.3)$\wedge$(2.2)$\Rightarrow$(2.1).

6. *Hamiltonian:*

$$H \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left(a^\dagger_{\vec{p}}a_{\vec{p}} + \frac{1}{2}\underbrace{\left[a_{\vec{p}}, a^\dagger_{\vec{p}}\right]}_{\propto\delta(0)=\infty}\right)$$

Ignore the infinite term since only relative energies are physical!
This infinity accounts for the zero-point energies of all harmonic oscillator modes.
Dropping this infinity is called *normal ordering* (see later).

7. *Eigenstates & Spectrum:*

a) $\overset{\circ}{\rightarrow} [H, a^\dagger_{\vec{p}}] = \omega_{\vec{p}}a^\dagger_{\vec{p}}$

b) Vacuum $|0\rangle \rightarrow$ Eigenstates $a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}\dots|0\rangle$ (span complete Hilbert space)

c) Energy: $E_{\vec{p}} = \omega_{\vec{p}} = +\sqrt{|\vec{p}|^2 + m^2}$ (relativistic dispersion, positive energies!)

d) (Kinetic) momentum:

$$P^i = \int d^3x\,\pi(\vec{x})(-\partial_i)\phi(\vec{x}) \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3}p^i a^\dagger_{\vec{p}}a_{\vec{p}} \tag{2.4}$$

This is now an operator!

e) Statistics: $a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}|0\rangle = a^\dagger_{\vec{q}}a^\dagger_{\vec{p}}|0\rangle$

$\rightarrow$ Excitations $a^\dagger_{\vec{p}}$ commute and carry additive energy & momentum
$\rightarrow$ *Bosonic particles* (in momentum space)

↑ Notes

8. *Normalization:*

   a) $\sphericalangle\ \Lambda = \mathcal{R}' L_3(\beta)\mathcal{R} \in \mathrm{SO}^+(1,3) \to p' = (E_{\vec p'}, \vec p') = \Lambda p$ with $p = (E_{\vec p}, \vec p)$
   Recall that all Lorentz transformations can be generated from spatial rotations and a boost $L_3(\beta)$ in z-direction!

   b) Jacobian in space: $\det\left(\frac{\partial \vec p'}{\partial \vec p}\right) \stackrel{\circ}{=} \frac{\mathrm{d}p_3'}{\mathrm{d}p_3} \stackrel{\circ}{=} \frac{E_{\vec p'}}{E_{\vec p}}$
   $\to \delta^{(3)}(\vec p - \vec q) = \frac{E_{\vec p'}}{E_{\vec p}} \delta^{(3)}(\vec p' - \vec q')$
   $\to \delta^{(3)}(\vec p - \vec q)$ is not Lorentz invariant but $E_{\vec p}\,\delta^{(3)}(\vec p - \vec q)$ is!
   Use colors to shorten this!
   3D volumes are *not* invariant under boosts due to Lorentz contraction!

   c) *Single-particle eigenstates:*

$$|\vec p\rangle := \sqrt{2E_{\vec p}}\, a^\dagger_{\vec p}|0\rangle \quad \Rightarrow \quad \langle \vec p|\vec q\rangle = (2\pi)^3 \underbrace{2E_{\vec p}\,\delta^{(3)}(\vec p - \vec q)}_{\text{Lorentz invariant}}$$

   This follows directly from the commutation relations.
   The 2 is just convention.

9. *Lorentz transformations* $\Lambda \in \mathrm{SO}^+(1,3)$:
   We need a unitary *representation* of the Lorentz group $\mathrm{SO}^+(1,3)$ on the Hilbert space!

$$U(\Lambda)|\vec p\rangle = |\Lambda \vec p\rangle \quad \Leftrightarrow \quad U(\Lambda) a^\dagger_{\vec p} U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda \vec p}}{E_{\vec p}}}\, a^\dagger_{\Lambda \vec p}$$

   It is $(\Lambda \vec p)^i \equiv \Lambda^i_{\ \mu} p^\mu$ (i.e., the spatial projection).
   Note that the "boost part" of Λ is hidden in the normalization of the state!

10. *Interpretation of* $\phi(\vec x)$:

$$\phi(\vec x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-i\vec p \vec x}|\vec p\rangle$$

   For non-relativistic $|\vec p| \ll m \Rightarrow E_{\vec p} \approx \text{const}$
   $\to$ State $|\vec x\rangle$ of particle at position $\vec x$
   $\to \phi(\vec x)$ creates particle at position $\vec x$
   This interpretation is also consistent with the "position-space representation" $\langle 0|\phi(\vec x)|\vec p\rangle \stackrel{\circ}{=} e^{i\vec p \vec x}$.

Note 2.1

1. Projector on single-particle sector: $\mathbb{1}_1 = \int \frac{d^3p}{(2\pi)^3} |\vec p\rangle \frac{1}{2E_{\vec p}} \langle \vec p|$
2. $\sphericalangle\ f(p)$ Lorentz invariant $\to \int \frac{d^3p}{(2\pi)^3} \frac{f(p)}{2E_{\vec p}}$ is Lorentz invariant

↑ Notes

2.2 The Klein-Gordon Field in Space-Time

So far: Schrödinger picture
Now: Heisenberg picture

1. *Heisenberg operators:* $\phi(x) = \phi(\vec{x}, t) = e^{iHt}\phi(\vec{x})e^{-iHt}$ (similar for $\pi(x)$)
2. *Heisenberg equation:* $i\partial_t \mathcal{O} = [\mathcal{O}, H]$ for $\mathcal{O} = \phi, \pi$ yields

$$
\begin{aligned}
i\partial_t\phi(x) &= \left[\phi(x), \int d^3y \left\{\frac{1}{2}\pi^2(\vec{y},t) + \frac{1}{2}(\nabla\phi(\vec{y},t))^2 + \frac{1}{2}m^2\phi^2(\vec{y},t)\right\}\right] \\
&= \int d^3y\, i\delta^{(3)}(\vec{x}-\vec{y})\pi(\vec{y},t) \\
&= i\pi(x) \\
i\partial_t\pi(x) &\stackrel{\circ}{=} -i(-\nabla^2 + m^2)\phi(x)
\end{aligned}
$$

$$\Rightarrow \quad (\partial_t^2 - \nabla^2 + m^2)\phi(x) = 0 \quad \text{(Klein-Gordon equation)}$$

3. Time-evolution of modes:

$$
\begin{aligned}
e^{iHt}a_{\vec{p}}e^{-iHt} &= a_{\vec{p}}e^{-iE_{\vec{p}}t} \\
e^{iHt}a_{\vec{p}}^\dagger e^{-iHt} &= a_{\vec{p}}^\dagger e^{+iE_{\vec{p}}t}
\end{aligned}
$$

   Use colors to skip last row.
   This can be shown informally by using $H_{\vec{p}} = E_{\vec{p}}\, a_{\vec{p}}^\dagger a_{\vec{p}}$ and counting excitations (i.e., on the number basis).

4. *Field operators:*

$$
\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-ipx} + a_{\vec{p}}^\dagger e^{ipx}\right)\Big|_{p^0=E_{\vec{p}}} \\
\pi(x) &= \partial_t\phi(x)
\end{aligned}
$$

   Here, $px = p^\mu x_\mu = E_{\vec{p}}t - \vec{p}\vec{x}$; note that $p^0 = E_{\vec{p}}$.
   In the following, $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ always denotes the time-independent *Schrödinger operators*!

Note 2.2

1. Hamiltonian generates time translations:
   $\phi(\vec{x}, t) = e^{iHt}\underbrace{\phi(\vec{x}, 0)}_{\phi(\vec{x})}e^{-iHt}$
2. Total momentum operator generates space translations:
   $\phi(\vec{x}) = e^{-i\vec{P}\vec{x}}\phi(\vec{0})e^{i\vec{P}\vec{x}}$
3. → Four-momentum operator generates space-time translations:
   $\phi(x) = e^{iPx}\phi(0)e^{-iPx}$

↑ Notes

Here, $P^\mu = (H, \vec{P})$ where $\vec{P}$ is defined in (2.4).

Note 2.3

Note that $p^0 = E_{\vec{p}}$ is always positive:

- $e^{-ipx} \leftrightarrow$ *positive-frequency* solution of KG equation $\leftrightarrow$ destruction operator $a_{\vec{p}}$
- $e^{+ipx} \leftrightarrow$ *negative-frequency* solution of KG equation $\leftrightarrow$ creation operator $a_{\vec{p}}^\dagger$

As *single-particle wavefunctions*, solutions with positive/negative frequency correspond to solutions with positive/negative energy. Note that there are only excitations with positive energy in the quantized field theory!

↑ Notes

Causality

$\sphericalangle$ Amplitude for a particle to propagate from y to x:

$$D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{-ip(x-y)} \tag{2.5}$$

This expression is Lorentz invariant, i.e., $D(\Lambda(x-y)) = D(x-y)$ for all $\Lambda \in \mathrm{SO}^+(1,3)$.
This is *not* true for improper Lorentz transformations which flip the sign of $D(x-y)$!

1. *Time-like distance:* $x^0 - y^0 = t$ and $\vec{x}-\vec{y} = 0$

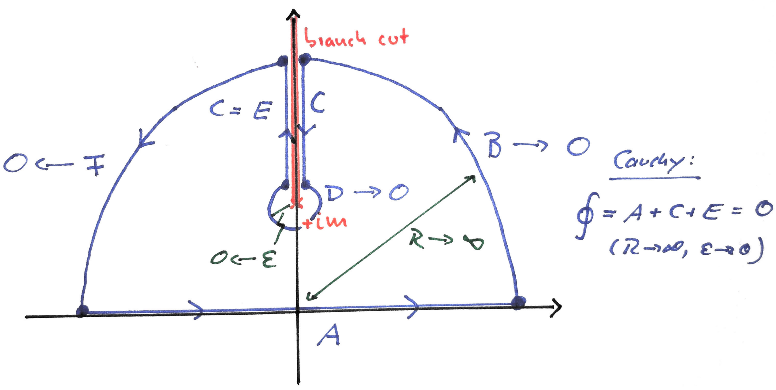
$$\begin{aligned} D(x-y) &= \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t} \\ &= \frac{1}{4\pi^2}\int_m^\infty dE\,\sqrt{E^2-m^2}\, e^{-iEt} \\ &\overset{t\to\infty}{\neq} 0 \quad \text{(actually not convergent)} \\ &\overset{t\to\infty}{\sim} e^{-imt} \quad \text{(this is very hand-wavy)} \end{aligned}$$

$\rightarrow$ does not vanish $\rightarrow$ propagation possible

2. *Space-like distance:* $x^0 - y^0 = 0$ and $\vec{x}-\vec{y} = \vec{r}$

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{i\vec{p}\vec{r}} \\ &= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp\,\frac{p^2}{2E_{\vec{p}}}\frac{e^{ipr}-e^{-ipr}}{ipr} \\ &= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^\infty dp\,\frac{p\,e^{ipr}}{\sqrt{p^2+m^2}} \end{aligned}$$

Use *Cauchy's integral theorem* with the following path:



Show that the curved sections vanish for $R \to \infty$ and $\varepsilon \to 0$, respectively!
It is $C = E$ since the minus from the opposite direction and the branch cut cancel.

Then

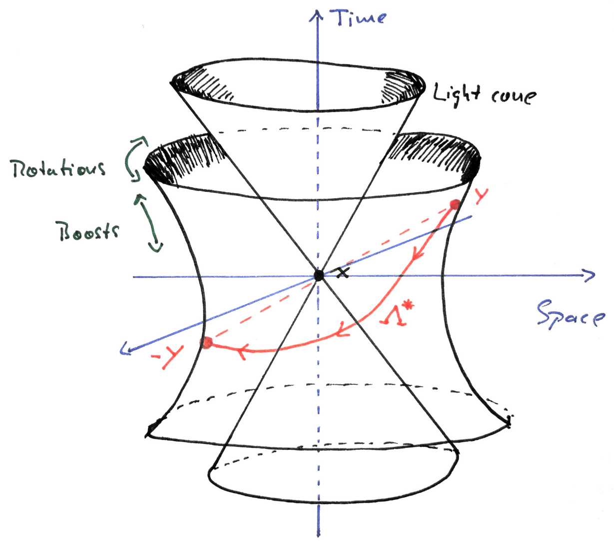
$$\begin{aligned} D(x-y) &= -C - E = -2C \\ &= \frac{-i}{(2\pi)^2 r}\int_{im}^{\infty} dp\, \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}} \\ &\overset{\rho=-ip}{=} \frac{1}{4\pi^2 r}\int_{m}^{\infty} d\rho\, \frac{\rho\, e^{-\rho r}}{\sqrt{\rho^2-m^2}} \\ &\overset{r\to\infty}{\sim} e^{-mr} \end{aligned}$$

→ vanishes exponentially (but non-zero!) → Problem?

3. $\sphericalangle$ *Measurements* A and B: can affect each other iff $[A, B] \neq 0$
Simplest choice: $A = \phi(x)$ and $B = \phi(y)$
Causality is preserved if all observables commute at space-like separations!
Since $\pi = \partial_t \phi$, it is sufficient for $[\phi(x), \phi(y)]$ to vanish for $(x-y)^2 < 0$.

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec q}}} \\ &\quad \times \left[\left(a_{\vec p}e^{-ipx} + a^\dagger_{\vec p}e^{ipx}\right), \left(a_{\vec q}e^{-iqy} + a^\dagger_{\vec q}e^{iqy}\right)\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right) \\ &= D(x-y) - D(y-x) \end{aligned}$$

Let $(x-y)^2 < 0$ space-like → $\exists \Lambda^* \in \mathrm{SO}^+(1,3)\ :\ \Lambda^*(x-y) = -(x-y)$:
The *proper orthochronous Lorentz group* $\mathrm{SO}^+(1,3)$ is a connected subgroup of the Lorentz group $\mathrm{O}(1,3)$, the elements of which connect continuously to the identity.



Continuous transformations (rotations in space and boosts) allow for $(x-y) \mapsto -(x-y)$ only if $(x-y)^2 < 0$. For time-like distances, this requires discontinuous transformations (time-reversal).

↑ Notes

Then

$$[\phi(x), \phi(y)] = D(x-y) - D(\Lambda^*(y-x)) \stackrel{(x-y)^2<0}{=} D(x-y) - D(x-y) \equiv 0 \quad \text{(Causality)}$$

For time-like separation, $(x-y)^2 > 0$, there is no such continuous transformation and the argument breaks down.

The first line follows from the Lorentz invariant integral measure in Note 2.1 and the definition of the propagator in (2.5).

The Propagator

1. Since $[\phi(x), \phi(y)] \propto \mathbb{1}$ (the commutator is a c-number), we can write ($x^0 > y^0$ for now)
"c-number" historically denotes scalar multiples of the identity, i.e. classical/commuting/complex "numbers".

$$\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip(x-y)} - e^{ip(x-y)}\right)$$

Substitute $\vec{p} \to -\vec{p}$ to obtain the second term:

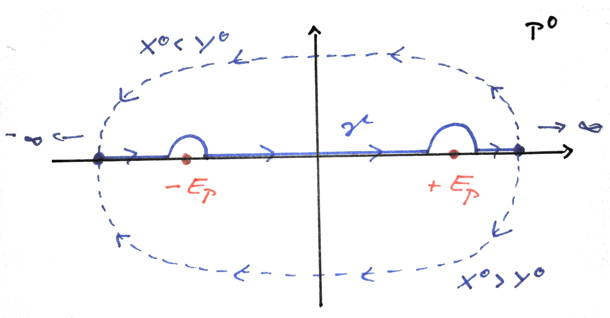
$$= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{e^{-ip(x-y)}|_{p^0=E_{\vec{p}}}}{2E_{\vec{p}}} + \frac{e^{-ip(x-y)}|_{p^0=-E_{\vec{p}}}}{-2E_{\vec{p}}} \right\}$$

Residue theorem with clockwise orientation (therefore the -1):

$$\stackrel{x^0>y^0}{=} \int \frac{d^3p}{(2\pi)^3} \int_\gamma \frac{dp^0}{2\pi i} \frac{-1}{\underbrace{p^2 - m^2}_{(p^0-E_{\vec{p}})(p^0+E_{\vec{p}})}} e^{-ip(x-y)}$$

$$= \int_\gamma \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} \tag{2.6}$$

with contours γ



The arc vanishes in the lower/upper-half pane for $x^0 > y^0$ and $x^0 < y^0$, respectively.

Therefore

$$D_R(x-y) \equiv \theta(x^0 - y^0)\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle = (2.6) + \gamma$$

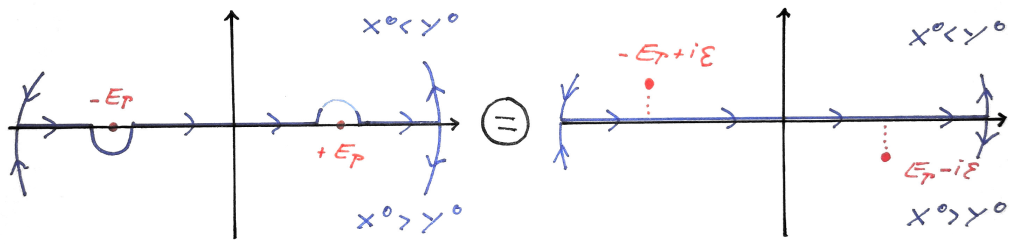
↑ Notes

2. *Interpretation:*

$$(\partial^2 + m^2) D_R(x-y) \overset{\circ}{=} -i\delta^{(4)}(x-y) \tag{2.7}$$

→ *Retarded Green's function* of Klein-Gordon operator
"R" for "retarded" since it vanishes for $x^0 < y^0$.
We could have found (2.6) directly from (2.7) by Fourier transformation.

3. Alternative contour γ^*:



- $x^0 > y^0$: close contour below
- $x^0 < y^0$: close contour above

$$D_F(x-y) \equiv (2.6) + \gamma^* = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$$

(Feynman propagator)

The infinitesimal $i\varepsilon$ shifts the poles to $p^0 \approx \pm(E_{\vec{p}} - i\varepsilon/2E_{\vec{p}}) = \pm(E_{\vec{p}} - i\epsilon)$ and yields an equivalent prescription of the Feynman propagator without the need to specify a contour. Note that $\varepsilon/2E_{\vec{p}} \equiv \epsilon$ are both infinitesimals.

We find (using (2.6) and (2.5))

$$\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & \text{for } x^0 > y^0 \\ D(y-x) & \text{for } x^0 < y^0 \end{cases} \\ &= \theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle \\ &\quad + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &\equiv \langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle \end{aligned}$$

with the *time-ordering (meta-)operator* $\mathcal{T}$
$\mathcal{T}$ orders products of operators by time with the latest to the left.
It is a *meta-operator* as it operates on *descriptions* of operators. Note that this is different from *super-operators* (such as the Lindbladian) which operate on operators.

The Feynman propagator is a Green's function of the KG equation (with different boundary conditions than the retarded/advanced Green's functions).

Later: Feynman propagator & Interactions → *Feynman rules*
However, so far we only studied the *free* KG field (→ linear field equation). Without interactions, however, there is no scattering so that there are no characteristic observations possible.

↑ Notes

3 The Dirac Field

Problem Set 3

(due **29.04.2020**)

1. Bosonic Fock states and coherent states
2. Free-particle solutions of the Dirac equation and completeness relations

3.1 The Dirac Equation

So far: Simplest relativistic field equation → Klein-Gordon equation
Now: Second simplest relativistic field equation → Dirac equation

1. *Observation:* Lorentz symmetry of the KG equation:
   We view Lorentz transformations as *active* transformations, mapping solutions to different solutions! This is equivalent to the *passive* viewpoint where the coordinate system is transformed instead.
   a) Coordinate transformation: $x' = \Lambda x$ & Field transformation: $\phi'(x') = \phi(x)$
   b) $\sphericalangle\, \phi$ with $(\partial^2 + m^2)\phi(x) = 0$ for all x
   c) $\rightarrow \phi'(x) = \phi(\Lambda^{-1}x)$ is a new solution:
   Use the chain rule in the first step twice:
   $$(g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi'(x) = [g^{\mu\nu}(\Lambda^{-1})^\sigma{}_\mu\partial_\sigma\,(\Lambda^{-1})^\rho{}_\nu\partial_\rho + m^2]\phi(\Lambda^{-1}x)$$
   Use invariance of the metric
   $$= (g^{\sigma\rho}\partial_\sigma\partial_\rho + m^2)\phi(\Lambda^{-1}x)$$
   $$= (\partial^2 + m^2)\phi(\Lambda^{-1}x) \overset{\phi \text{ solution}}{=} 0$$

2. *Observation:* Vector fields under rotations: $\vec{\phi}'(x) = R\vec{\phi}(R^{-1}x)$
   → In general, a field $\phi(x) \in \mathbb{C}^n$ can transform under Lorentz transformations as
   $$\phi'_a(x) = M_{ab}(\Lambda)\phi_b(\Lambda^{-1}x) \qquad a = 1, \dots, n$$
   where
   $$M(\Lambda')M(\Lambda)\phi(\Lambda^{-1}\Lambda'^{-1}x) \overset{!}{=} M(\Lambda'\Lambda)\phi((\Lambda'\Lambda)^{-1}x)$$
   is a n-dimensional *representation* of the Lorentz group $\mathrm{SO}^+(1,3)$

3. We want a *first-order* relativistic field equation:
   $$(\partial^\mu\partial_\mu + \text{const})\phi = 0 \quad \Rightarrow \quad (i\blacksquare^\mu\partial_\mu + \text{const})\phi = 0$$
   The i anticipates wave-like solutions for real $\blacksquare$.

↑ Notes

4. Then (combine 1 & 2)
   a) Coordinate transformation: $x' = \Lambda x$ & Field transformation: $\phi'(x') = M(\Lambda)\phi(x)$
   b) $\sphericalangle$ ϕ with $(i\blacksquare^\mu\partial_\mu + \text{const})\phi(x) = 0$ for all x
   c) When is $\phi'(x) = M(\Lambda)\phi(\Lambda^{-1}x)$ is a new solution?

$$(i\blacksquare^\mu\partial_\mu + \text{const})\phi'(x) = [i\blacksquare^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu + \text{const}]\, M(\Lambda)\phi(\Lambda^{-1}x) \stackrel{!}{=} 0$$

Multiply with $M^{-1}(\Lambda)$:

$$\Leftrightarrow \quad [i\, \underbrace{M^{-1}(\Lambda)\blacksquare^\mu M(\Lambda)(\Lambda^{-1})^\nu{}_\mu}_{\stackrel{!}{=}\blacksquare^\nu}\, \partial_\nu + \text{const}]\, \phi(\Lambda^{-1}x) \stackrel{!}{=} 0$$

$\to \blacksquare^\mu \equiv \gamma^\mu$ must be $n\times n$-matrices with

$$\boxed{M^{-1}(\Lambda)\gamma^\mu M(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu} \tag{3.1}$$

The γ-matrices "translate" the "spinor"-representation $M(\Lambda)$ into the "vector"-representation Λ and vice versa.

5. How to find γ^μ and $M(\Lambda)$? $\mathrm{SO}^+(1,3)$ is a *Lie group*:

$$\begin{aligned}\Lambda &= \exp\left[-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\right] \stackrel{\omega\ll 1}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta} \\ M(\Lambda) &= \exp\left[-\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right] \stackrel{\omega\ll 1}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\end{aligned} \tag{3.2}$$

$\omega_{\alpha\beta}$ antisymmetric tensor → 3 rotations (angles) + 3 boosts (rapidities)
It is $(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu)$.
The 4×4 matrices $\mathcal{J}^{\alpha\beta}$ generate the vector-representation Λ, $(\frac{1}{2},\frac{1}{2})$, the $n\times n$-matrices $S^{\alpha\beta}$ the spinor-representation $M(\Lambda)$, $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. The generators are antisymmetric in the spacetime indices.

- Infinitesimal form of (3.1):

$$\left[\gamma^\mu, S^{\alpha\beta}\right] \stackrel{\circ}{=} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu\gamma^\nu \stackrel{\circ}{=} i(g^{\alpha\mu}\gamma^\beta - g^{\beta\mu}\gamma^\alpha) \tag{3.3}$$

- $\mathcal{J}^{\alpha\beta} \to$ *Lie-algebra* of Lorentz group ($J = S, \mathcal{J}$)

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] \stackrel{\circ}{=} i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \tag{3.4}$$

The Lie algebra defines the structure of the Lie group by integration and is therefore the same for all representations.

6. *Solution:* Dirac's trick: $\sphericalangle$ γ^μ such that

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}_{n\times n} \quad \text{(Dirac algebra)}}$$

↑ Notes

This is the *Clifford algebra* $\mathrm{C}\ell_{1,3}(\mathbb{C})$.
Then

$$S^{\mu\nu} \equiv \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] \tag{3.5}$$

satisfies the Lorentz algebra (3.4) and (3.3)

7. *Representations:*
   - At least 4-dimensional
     (think of the γ^μ as Majorana modes and construct ladder operators → 2 modes)
   - All 4-dimensional representations are unitarily equivalent
     (actually, they constitute the *unique* irrep of the Dirac algebra which is 4-dimensional)
   - We use the *Weyl representation* (sometimes called *chiral representation*):

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

   - Henceforth: $\Lambda_{\frac{1}{2}} \equiv M(\Lambda)$
     Two "copies" of a spin-$\frac{1}{2}$ projective representation.

8. Setting const $= -m$, we find the

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \quad \text{(Dirac equation)} \tag{3.6}$$

$\Psi(x)$ is a bispinor-field with values in $\mathbb{C}^4$.

9. The components of the Dirac spinor field satisfy the KG equation:

$$0 = (-i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\Psi \mathrel{\hat{=}} (\partial^2 + m^2)\Psi$$

The Dirac differential operator is the "square root" of the Klein-Gordon differential operator.

10. *Dirac adjoint:*
    Goal: Lagrangian (which must be a Lorentz scalar).
    How to form Lorentz scalars from spinors?
    a) First try: $\Psi^\dagger\Psi$

$$\Psi'^\dagger\Psi' = \Psi^\dagger \underbrace{\Lambda_{\frac{1}{2}}^\dagger\Lambda_{\frac{1}{2}}}_{\neq\mathbb{1}} \Psi \neq \Psi^\dagger\Psi$$

$\Lambda_{\frac{1}{2}}$ is not unitary because $S^{\mu\nu}$ is not hermitian for boosts ($\mu = 0$ and $\nu = 1, 2, 3$).
This is a consequence of the non-compactness of the Lorentz group due to boosts!

↑ Notes

b) Define

$$\overline{\Psi} = \Psi^\dagger \gamma^0 \quad \text{(Dirac adjoint)}$$

$\overset{\circ}{\to} \overline{\Psi}'\Psi' = \overline{\Psi}\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\Psi = \overline{\Psi}\Psi \Rightarrow$ Lorentz scalar
Use (3.5) and (3.2) and the Dirac algebra to show this!

11. *Lagrangian:*

$$\mathcal{L}_{\text{Dirac}} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$

$\overset{\circ}{\to}$ Euler-Lagrange equations yield Dirac equation.

Note 3.1

- Let $\sigma^\mu \equiv (1, \vec{\sigma})^T$ and $\overline{\sigma}^\mu \equiv (1, -\vec{\sigma})^T$ and $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$
  $\to$ Dirac equation:

$$\begin{pmatrix} -m & i\sigma\partial \\ i\overline{\sigma}\partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

- Ψ_L and Ψ_R are called left- and right-handed *Weyl spinors*
- They do not mix under Lorentz transformations
  They form the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ projective irreps of the Lorentz group. Note that the reducibility of the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ bispinor representation is manifest in the Weyl basis:

$$S^{0i} = \frac{i}{4}\left[\gamma^0, \gamma^i\right] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \quad \text{(Boosts, antihermitian)}$$

$$S^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad \text{(Rotations, hermitian)} \tag{3.7}$$

- For $m = 0$, the Dirac equation decouples into the *Weyl equations*:

$$i\overline{\sigma}\partial\Psi_L = 0 \quad \text{and} \quad i\sigma\partial\Psi_R = 0$$

  Solutions Ψ_R and Ψ_L are eigenstates of the helicity operator $h = \hat{p}\frac{\vec{\sigma}}{2}$ with $h = +\frac{1}{2}$ called *right-handed* and $h = -\frac{1}{2}$ *left-handed*. Here, $\hat{p} = \vec{p}/E$ is the normalized 3-momentum for a massless particle.

3.2 Free-Particle Solutions of the Dirac Equation

1. (3.6) $\Rightarrow (\partial^2 + m^2)\Psi = 0$ (Klein-Gordon equation), therefore

$$\Psi^\pm(x) = \psi^\pm(p)e^{\mp ipx} \quad \text{with } p^2 = m^2 \text{ and } p^0 > 0 \tag{3.8}$$

↑ Notes

We set $p^0 > 0$ for both positive $(+)$ and negative $(-)$ frequency solutions and change the sign of p in the exponent (to simplify the discussion below).

2. (3.8) in (3.6) yields

$$(\pm\gamma^\mu p_\mu - m)\psi^\pm(p) = \begin{pmatrix} -m & \pm p\sigma \\ \pm p\bar{\sigma} & -m \end{pmatrix} \begin{pmatrix} \psi_L^\pm \\ \psi_R^\pm \end{pmatrix} = 0 \tag{3.9}$$

3. Note (➔ Problemset 3):
   - $(p\sigma)(p\bar{\sigma}) = p^2 = m^2$
   - Eigenvalues of $p\sigma$ and $p\bar{\sigma}$: $p^0 \pm |\vec{p}| \to$ for $p^0 > 0$ and $m > 0$ positive spectrum
     In particular, $p\sigma$ and $p\bar{\sigma}$ are invertible and the positive squareroots $\sqrt{p\sigma}$ and $\sqrt{p\bar{\sigma}}$ are Hermitian.

4. $\lessdot\ \psi_L^\pm = \sqrt{p\sigma}\,\xi^\pm$ with arbitrary, normalized $((\xi^\pm)^\dagger\xi^\pm = 1)$ *spinor* $\xi^\pm \in \mathbb{C}^2$

$$(3.9) \Rightarrow -m\sqrt{p\sigma}\,\xi^\pm \pm p\sigma\,\psi_R^\pm = 0 \quad \overset{\sqrt{p\sigma}\sqrt{p\bar{\sigma}}=m}{\Leftrightarrow} \quad \psi_R^\pm = \pm\frac{m}{\sqrt{p\sigma}}\xi^\pm = \pm\sqrt{p\bar{\sigma}}\xi^\pm$$

The second equation in (3.9) yields the same solution.

5. *Solutions:*
   Conventional notation: $\xi^+ \mapsto \xi$, $\xi^- \mapsto \eta$ and $\psi^+ \mapsto u$, $\psi^- \mapsto v$
   Basis states: ξ^s with $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (same for η^s)

$$\Psi^+(x) = \underbrace{\begin{pmatrix} \sqrt{p\sigma}\xi^s \\ \sqrt{p\bar{\sigma}}\xi^s \end{pmatrix}}_{u^s(p)} e^{-ipx} \quad \text{(positive frequency solution)}$$

$$\Psi^-(x) = \underbrace{\begin{pmatrix} \sqrt{p\sigma}\eta^s \\ -\sqrt{p\bar{\sigma}}\eta^s \end{pmatrix}}_{v^s(p)} e^{+ipx} \quad \text{(negative frequency solution)}$$

   with $p^2 = m^2$, $p^0 > 0$ and $s = 1, 2$

6. *Some relations* (➔ Problemset 3):
   - Orthonormality:
     Let $\bar{u}^s \equiv (u^s)^\dagger\gamma^0$ and $\bar{v}^s \equiv (v^s)^\dagger\gamma^0$, then

$$\begin{aligned} \bar{u}^r u^s &= 2m\delta^{rs} \quad \text{and} \quad (u^r)^\dagger u^s = 2E_{\vec{p}}\delta^{rs} \\ \bar{v}^r v^s &= -2m\delta^{rs} \quad \text{and} \quad (v^r)^\dagger v^s = 2E_{\vec{p}}\delta^{rs} \\ \bar{v}^r u^s &= \bar{u}^r v^s = 0 \\ u^{r\dagger}(\vec{p})v^s(-\vec{p}) &= v^{r\dagger}(-\vec{p})u^s(\vec{p}) = 0 \end{aligned} \tag{3.10}$$

     Note that $\bar{u}u$ is Lorentz invariant whereas $u^\dagger u \propto E_{\vec{p}}$ is not!
     Note that $(u^r)^\dagger v^s \neq 0$ and $(v^r)^\dagger u^s \neq 0$!
     For massless particles, the normalization condition is given by $(u^r)^\dagger u^s = 2E_{\vec{p}}\delta^{rs}$.

↑ Notes

- Spin sums:
  Let $\not{p} \equiv \gamma^\mu p_\mu$ (*Feynman slash notation*), then

$$\begin{aligned} \sum_s u^s(p)\bar{u}^s(p) &= \not{p} + m\mathbb{1} \\ \sum_s v^s(p)\bar{v}^s(p) &= \not{p} - m\mathbb{1} \end{aligned} \tag{3.11}$$

  Useful if one wants to sum over spin-polarizations of fermions.

↑ Notes

3.3 Dirac Field Bilinears

1. *Definition:*

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \overset{\text{Weyl basis}}{=} \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

with

$$(\gamma^5)^\dagger = \gamma^5, \quad (\gamma^5)^2 = \mathbb{1}, \quad \{\gamma^5, \gamma^\mu\} = 0$$

The last relation implies $[\gamma^5, S^{\mu\nu}] = 0$, i.e., the Dirac bispinor representation must be reducible according to Schur's lemma: $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

2. The following bilinears $\overline{\Psi}\Gamma\Psi$ have definite transformation properties under the Lorentz group:

$\Gamma =$	1	scalar	$\times 1$
	γ^μ	vector	$\times 4$
	$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = i\gamma^{[\mu}\gamma^{\nu]}$	tensor	$\times 6$
	$\gamma^\mu\gamma^5$	pseudo-vector	$\times 4$
	γ^5	pseudo-scalar	$\times 1$

The notation $\gamma^{[\mu} \dots \gamma^{\nu]}$ denotes the completely antisymmetrized product.
Any 4×4 matrix Γ can be decomposed into these 16 matrices with definite transformation properties under Lorentz transformations.

The prefix *pseudo-* marks quantities that transform under continuous Lorentz transformations $\Lambda \in \mathrm{SO}^+(1,3)$ as usual but pick up an additional sign under parity transformations. This is similar to the cross product $a \times b$ in three dimensions which produces a pseudo-vector from the two vectors a and b with respect to the Euclidean group (= isometries of Euclidean space). E.g., angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is not a vector but a pseudo-vector.

For example,

$$(j^\mu)' = \overline{\Psi}'\gamma^\mu\Psi' = \overline{\Psi}\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}\Psi = \Lambda^\mu{}_\nu\overline{\Psi}\gamma^\nu\Psi = \Lambda^\mu{}_\nu j^\nu$$

transforms as a Lorentz 4-vector.

$\overset{\circ}{\rightarrow}$ j^μ is the conserved Noether current corresponding to the continuous symmetry $\Psi \to e^{i\alpha}\Psi$ of the Dirac Lagrangian.

3.4 Quantization of the Dirac Field

1. Lagrangian: $\mathcal{L} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$
2. Canonical momentum: $\Pi_a = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}_a} = i\Psi_a^*$

↑ Notes

3. Hamiltonian: $H = \int d^3x\, \Psi^\dagger \underbrace{[-i\vec{\alpha}\nabla + m\beta]}_{=H_D} \Psi$ with $\vec{\alpha} = \gamma^0\vec{\gamma}$ and $\beta = \gamma^0$
   H_D is the Dirac Hamiltonian of single-particle quantum mechanics.
   → Expand Ψ in eigenmodes of H_D to diagonalize H
4. Eigenmodes: $H_D u^s(\vec{p})e^{i\vec{p}\vec{x}} = E_{\vec{p}}\square$ and $H_D v^s(\vec{p})e^{-i\vec{p}\vec{x}} = -E_{\vec{p}}\square$
   This can be seen from $[i\gamma^0\partial_0 + i\vec{\gamma}\nabla - m]\Psi = 0$.
5. Mode expansion:

$$\Psi(\vec{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[a^s_{\vec{p}}u^s(\vec{p})e^{i\vec{p}\vec{x}} + b^s_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\vec{x}}\right] \tag{3.12}$$

   $a^s_{\vec{p}}$ and $b^s_{\vec{p}}$ are operator-valued expansion coefficients. We do not yet fix their algebra!
6. Use

$$H_D\Psi(\vec{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\left[a^s_{\vec{p}}u^s(\vec{p})e^{i\vec{p}\vec{x}} - b^s_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\vec{x}}\right]$$

   then (using the orthonormality relations (3.10))

$$H = \int d^3x\, \Psi^\dagger H_D \Psi \stackrel{\circ}{=} \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}}(a^{s\dagger}_{\vec{p}}a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}}b^s_{\vec{p}})$$

   You do not need reordering of operators to show this. The algebra is still undefined!

First try: Commutator

7. Canonical quantization with equal-time commutators:

$$\left[\Psi_a(\vec{x}), \Pi_b(\vec{y})\right] = i\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \quad \Leftrightarrow \quad \left[\Psi_a(\vec{x}), \Psi^\dagger_b(\vec{y})\right] = \delta_{ab}\delta^{(3)}(\vec{x}-\vec{y})$$
$$\left[\Psi_a(\vec{x}), \Psi_b(\vec{y})\right] = 0 \tag{3.13}$$

8. $\stackrel{\circ}{\rightarrow}$ Mode algebra

$$\left[a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}\right] = \left[b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\right] = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q})$$
$$\left[a^r_{\vec{p}}, b^{s(\dagger)}_{\vec{q}}\right] = 0$$

   Show (using the mode expansion (3.12) and the spin sums (3.11)) that this is equivalent to the commutators of the fields.
   Beware: Eq. (3.89) of P&S is mathematically ill-defined since $\Psi\Psi^\dagger$ is a matrix but $\Psi^\dagger\Psi$ is not (it's just sloppy math that doesn't belong in a textbook for students). Do it right, i.e., componentwise: $[\Psi_a(\vec{x}), \Psi^\dagger_b(\vec{y})] = \delta_{ab}\delta^{(3)}(\vec{x}-\vec{y})$.
   → irreducible Representation = *bosonic* Fock space
9. *Problem:* $\left(b^{s\dagger}_{\vec{p}}\right)^n |0\rangle$ has energy $-nE_{\vec{p}} \xrightarrow{n\to\infty} -\infty$
   → no stable vacuum state (the spectrum of H is unbounded below)

↑ Notes

10. Fix (?): $b \leftrightarrow b^\dagger$
    a) $\Psi(\vec{x}) = \dots [a^s_{\vec{p}} \dots + b^{s\dagger}_{\vec{p}} \dots]$
    b) $H = \dots (a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^s_{\vec{p}} b^{s\dagger}_{\vec{p}})$
    c) $[b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q})$
    d) $H = \dots (a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}}) + \text{const}$
    e) $[H, b^{s\dagger}_{\vec{p}}] = E_{\vec{p}} b^{s\dagger}_{\vec{p}} \to b^{s\dagger}_{\vec{p}}$ creates a particle with *positive* energy! $\to H \geq 0$
    It *seems* that we solved the problem: The spectrum of the Hamiltonian is now bounded from below.
    f) *But:*

$$\|b^{s\dagger}_{\vec{p}}|0\rangle\|^2 = \langle 0| \left[b^s_{\vec{p}}, b^{s\dagger}_{\vec{p}}\right] |0\rangle = -(2\pi)^3 \delta^{(3)}(0) < 0$$

    $\to$ negative norm states (i.e., the constructed representation is not a Hilbert space)

11. *Conclusion:* (3.13) implies
    - either an instability of the vacuum
    - or a loss of unitarity

    $\to$ no consistent quantization possible!

Second try: Anticommutator

7. Canonical quantization with equal-time *anti*commutators:

$$\left\{\Psi_a(\vec{x}), \Psi^\dagger_b(\vec{y})\right\} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}) \quad \text{and} \quad \{\Psi_a(\vec{x}), \Psi_b(\vec{y})\} = 0$$

Note that these are *equal-time* anticommutators!

8. $\overset{\circ}{\to}$ Mode algebra

$$\left\{a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}\right\} = \left\{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\right\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}) \quad \text{and} \quad \left\{a^r_{\vec{p}}, b^{s(\dagger)}_{\vec{q}}\right\} = 0 \tag{3.14}$$

The proof is similar to the bosonic case above.
$\to$ irreducible Representation = *fermionic* Fock space

9. *Problem:* $b^{s\dagger}_{\vec{p}}|0\rangle$ has energy $-E_{\vec{p}}$ & infinite sum over momenta
   $\to$ still no stable vacuum state
   (the spectrum of H is still unbounded below due to the sum over momenta)

10. Fix (?): $b \leftrightarrow b^\dagger$ (we saw above that it changes the sign of the excitation energies)

↑ Notes

a) *Hamiltonian:*

$$H = \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}}(a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s - b_{\vec{p}}^s b_{\vec{p}}^{s\dagger})$$
$$= \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}}(a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s + b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s) - \infty$$

We will drop the infinite constant henceforth. Cross the $-\infty$.

b) The mode algebra (3.14) is *invariant* under $b \leftrightarrow b^\dagger$!
→ Unitarity is preserved *and* Hamiltonian is bounded from below
With anticommutation relations, quantization is consistently possible!

11. *Heisenberg picture:*
Now that we have a representation where the Hamiltonian generates a unitary time evolution, we can switch to the Heisenberg picture:
With

$$e^{iHt} a_{\vec{p}}^s e^{-iHt} \stackrel{\circ}{=} a_{\vec{p}}^s e^{-iE_{\vec{p}}t} \quad \text{and} \quad e^{iHt} b_{\vec{p}}^s e^{-iHt} \stackrel{\circ}{=} b_{\vec{p}}^s e^{-iE_{\vec{p}}t}$$

and $\Psi(x) = e^{iHt}\Psi(\vec{x})e^{-iHt}$ we find

$$\Psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}}^s u^s(p) e^{-ipx} + b_{\vec{p}}^{s\dagger} v^s(p) e^{ipx}\right]$$
$$\overline{\Psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}}^{s\dagger} \overline{u}^s(p) e^{ipx} + b_{\vec{p}}^s \overline{v}^s(p) e^{-ipx}\right]$$

These are operator-valued spinor fields, i.e., functions (more precisely: distributions) on Minkowski spacetime that assign to an event x a tuple ("spinor") of operators that act on the fermionic Fock space where the states of the quantized theory live.

Continuous symmetries & Conserved charges

- Time translation → Hamiltonian (see above)
- Spatial translations → momentum operator

$$\vec{P} \stackrel{\circ}{=} \int d^3x\, \Psi^\dagger(-i\nabla)\Psi \stackrel{\circ}{=} \sum_s \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, (a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s + b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s)$$

- Rotations → angular momentum operator $\vec{J}$

$$\vec{J} = \int d^3x\, \Psi^\dagger \left[\vec{x} \times (-i\nabla) + \frac{1}{2}\vec{\Sigma}\right]\Psi \quad \text{with} \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{3.15}$$

↑ Notes

- Global phase rotations $e^{i\alpha}\Psi \overset{\circ}{\rightarrow}$ conserved current $j^\mu = \overline{\Psi}\gamma^\mu\Psi \rightarrow$ conserved charge

$$
\begin{aligned}
Q = \int d^3x\, \Psi^\dagger\Psi &\mathrel{\hat{=}} \sum_s \int \frac{d^3p}{(2\pi)^3}(a_{\vec{p}}^{s\dagger}a_{\vec{p}}^s + b_{-\vec{p}}^s b_{-\vec{p}}^{s\dagger}) \\
&= \sum_s \int \frac{d^3p}{(2\pi)^3}(a_{\vec{p}}^{s\dagger}a_{\vec{p}}^s - b_{\vec{p}}^{s\dagger} b_{\vec{p}}^{s}) + \infty
\end{aligned}
$$

In QED we couple the fermions to the EM field; then, Q is the total EM charge of the fermion field.

The operators of conserved charges generate symmetry transformations of the Hamiltonian.

Excitations = Particles

$a_{\vec{p}}^{s\dagger}|0\rangle$: *Fermion* with energy $E_{\vec{p}}$,
momentum $\vec{p}$,
spin $J = \frac{1}{2}$ (polarization s),
and charge $Q = +1$

$b_{\vec{p}}^{s\dagger}|0\rangle$: *Antifermion* with energy $E_{\vec{p}}$,
momentum $\vec{p}$,
spin $J = \frac{1}{2}$ (polarization opposite to s),
and charge $Q = -1$

In QED, the fermions will be *electrons* and the antifermions *positrons*.

Note 3.2

- The two states for $s = 1, 2$ *suggest* a spin-$\frac{1}{2}$ representation
- To show this, the action of $\vec{J}$ (see (3.15)) on one-particle states must be studied
- One finds for particles at rest

$$
J_z\, a_{\vec{0}}^{s\dagger}|0\rangle = \pm\frac{1}{2}a_{\vec{0}}^{s\dagger}|0\rangle \quad \text{and} \quad J_z\, b_{\vec{0}}^{s\dagger}|0\rangle = \mp\frac{1}{2}b_{\vec{0}}^{s\dagger}|0\rangle
$$

with $\xi^{s=1} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\xi^{s=2} = \begin{pmatrix}0\\1\end{pmatrix}$.

Lorentz transformations

1. $\sphericalangle$ Lorentz transformation $\Lambda \in \mathrm{SO}^+(1,3)$ on single particle state $|\vec{p}, s\rangle \equiv \sqrt{2E_{\vec{p}}}\, a_{\vec{p}}^{s\dagger}|0\rangle$:

$$
|\vec{p}, s\rangle \mapsto U(\Lambda)|\vec{p}, s\rangle
$$

$U(\Lambda)$: representation of $\mathrm{SO}^+(1,3)$ on Fock space
For generic rotations/boosts, this mixes the two spin components!

↑ Notes

2. Special case: quantization axis parallel to boost and/or rotation axis
   → spin polarizations do not mix:

$$U(\Lambda)\, a^s_{\vec{p}}\, U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}\, a^s_{\Lambda\vec{p}}$$

   Note that spins mix under generic Lorentz transformations: $a^1_{\vec{p}} \leftrightarrow a^2_{\vec{q}}$.

3. Consider this special case, then:

$$\langle\vec{p},s|\vec{q},r\rangle = \underbrace{2E_{\vec{p}}(2\pi)^3\delta^{(3)}\left(\vec{p}-\vec{q}\right)\delta^{rs}}_{\text{Lorentz invariant}} = \langle\vec{p},s|U^\dagger(\Lambda)U(\Lambda)|\vec{q},r\rangle$$

   → $U(\Lambda)$ is *unitary*

4. Now we have 3 representations:

Λ	acts on 4-vectors in $\mathbb{R}^{1,3}$	D=4	not unitary
$\Lambda_{\frac{1}{2}}$	acts on bispinors in $\mathbb{C}^2 \oplus \mathbb{C}^2$	D=4	not unitary
$U(\Lambda)$	acts on states in fermionic Fock space	D=∞	unitary

5. Action by conjugation on field operators $\overset{\circ}{\to}$

$$U(\Lambda)\Psi(x)U^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\Psi(\Lambda x)$$

↑ Notes

Problem Set 4

(due **08.05.2020**)

1. Application of the Dirac equation: The relativistic hydrogen atom
2. Discrete symmetries of the Dirac equation: Parity transformation of Dirac spinors

Spin-statistics theorem

- *Observation:*

  Klein-Gordon field ϕ: Spin 0 (scalar) → commutator → bosonic excitations
  Dirac field Ψ: Spin $\frac{1}{2}$ (spinor) → anticommutator → fermionic excitations

  This is no coincidence but hints at a more fundamental connection:

- *Spin-statistics theorem:*

$$\left.\begin{array}{r}\text{Lorentz invariance}\\ \text{Causality}\\ \text{Positive energies}\\ \text{Positive norms}\end{array}\right\} \Rightarrow \left\{\begin{array}{rcl}\text{Integer spin} & \leftrightarrow & \text{Bosons}\\ \text{Half-integer spin} & \leftrightarrow & \text{Fermions}\end{array}\right.$$

  This means, whenever you quantize a relativistic field that transforms under a (projective) half-integer spin representation, the Poisson bracket must be replaced by *anti*commutators. Otherwise unitarity is lost or the vacuum becomes unstable.

- The proof is elaborate and quite technical
  (➔ http://math.ucr.edu/home/baez/spin_stat.html)

Dirac Propagator

All that follows is very similar to our discussion of the Klein-Gordon propagator.
For details, we refer the student to the corresponding notes.

1. Propagation amplitudes:

$$\langle 0|\Psi_a(x)\bar{\Psi}_b(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(x-y)}\underbrace{\sum_s u_a^s(p)\bar{u}_b^s(p)}_{(\not{p}+m)_{ab}}$$

$$\overset{x^0 \geq y^0}{=} \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\varepsilon}e^{-ip(x-y)}$$

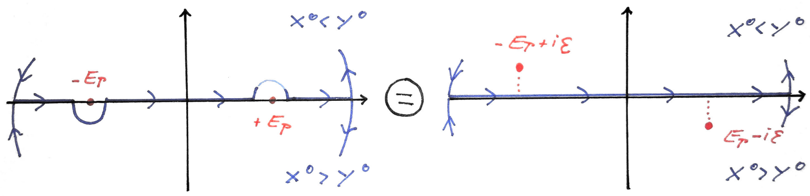
and (use colors to skip this calculation)

$$\langle 0|\bar{\Psi}_b(y)\Psi_a(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip(y-x)}\underbrace{\sum_s v_a^s(p)\bar{v}_b^s(p)}_{(\not{p}-m)_{ab}}$$

$$\overset{x^0 \leq y^0}{=} -\int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\varepsilon}e^{-ip(x-y)}$$

↑ Notes

Recall:



- $x^0 > y^0$: close contour below
- $x^0 < y^0$: close contour above

2. Therefore we define the *Feynman propagator:*

$$S_F^{ab}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not p + m)_{ab}}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$$
$$= \begin{cases} \langle 0|\Psi_a(x)\overline{\Psi}_b(y)|0\rangle & \text{for } x^0 > y^0 \\ -\langle 0|\overline{\Psi}_b(y)\Psi_a(x)|0\rangle & \text{for } x^0 < y^0 \end{cases}$$
$$\equiv \langle 0|\mathcal{T}\Psi_a(x)\overline{\Psi}_b(y)|0\rangle$$

Note: For $t_1 > t_2$ it is $\mathcal{T}\Psi(t_2)\Psi(t_1) \equiv -\Psi(t_1)\Psi(t_2)$ for fermionic fields!

3. Similarly, one can derive the Retarded Green's function:

$$S_R^{ab}(x-y) \equiv \theta(x^0 - y^0)\,\langle 0|\left\{\Psi_a(x), \overline{\Psi}_b(y)\right\}|0\rangle \mathrel{\hat=} (i\not\partial_x + m)_{ab} D_R(x-y)$$

Here, $D_R(x-y)$ is the retarded Green's function of the Klein-Gordon field; $\not\partial_x$ denotes derivatives with respect to the variables x^μ for $\mu = 1, 2, 3, 4$ and generates the $\not p$ in the integral.

Causality

- *Measurable operators*: $\hat{O}(x) = \sum \prod_{i=1}^{\text{even } N} (\Psi_i(x) \ \vee \ \partial\Psi_i(x) \ \vee \ \partial^2\Psi_i(x) \ \ldots)$
  Example: $j^\mu = \overline{\Psi}\gamma^\mu\Psi$ (but *not* $\Psi_a + \Psi_a^\dagger$!)
- $\rightarrow$ Causality for fermionic fields $\Leftrightarrow \{\Psi_a(x), \overline{\Psi}_b(y)\} = 0$ for $(x-y)^2 < 0$
  All other anticommutators vanish trivially. Note that here $x = (t, \vec{x})$ and $y = (t', \vec{y})$, i.e., we consider the anticommutator at *different* times.

  We find (using results from above)

$$\{\Psi_a(x), \overline{\Psi}_b(y)\} \mathrel{\hat=} (i\not\partial_x + m)_{ab}[D(x-y) - D(y-x)]$$
$$(x-y)^2 < 0$$
$$= (i\not\partial_x + m)_{ab}[D(x-y) - D(x-y)] = 0$$

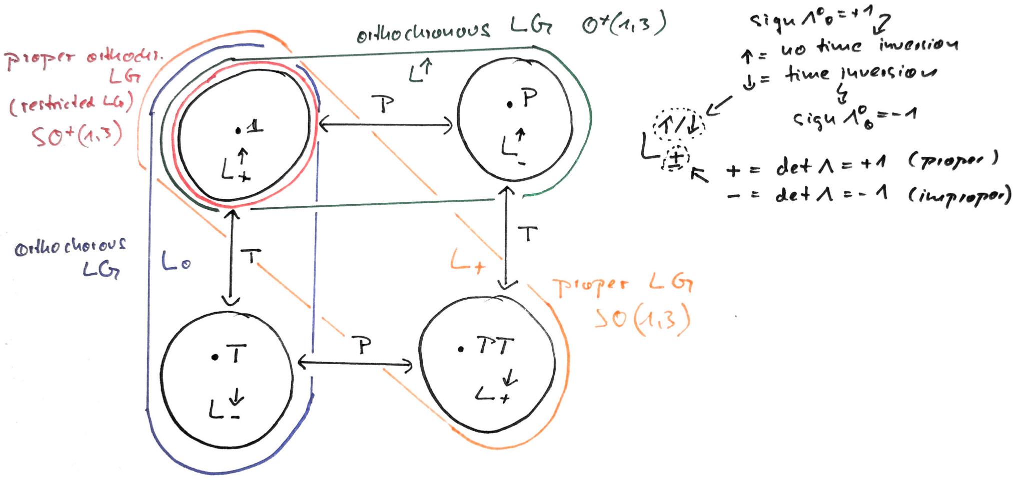
  The argument is the same as for the Klein-Gordon field.
  Recall: $D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip(x-y)}$

3.5 Discrete Symmetries of the Dirac Theory

Review of the Lorentz group

- Lorentz group O(1, 3) = Lie group with four disconnected components
- Continuous Lorentz transformations = *Proper orthochronous Lorentz group* $SO^+(1,3)$
- Four components connected by discrete transformations

$$\text{Parity} \quad P : (t,\vec{x}) \mapsto (t,-\vec{x})$$
$$\text{Time reversal} \quad T : (t,\vec{x}) \mapsto (-t,\vec{x})$$



Parity

Details: ➔ Problemset 4

- Unitary representation on the Hilbert space:

$$U(P)\,a^s_{\vec{p}}\,U^{-1}(P) = \underbrace{\eta_a}_{+1}\;a^s_{-\vec{p}} \quad\text{and}\quad U(P)\,b^s_{\vec{p}}\,U^{-1}(P) = \underbrace{\eta_b}_{-1}\;b^s_{-\vec{p}}$$

Note that we do *not* want spin to change under P because angular momentum $\vec{L} = \vec{r}\times\vec{p}$ also does not pick up a sign under inversion (it is a pseudo-vector).

Note that often $U(P)$ is simply written P.

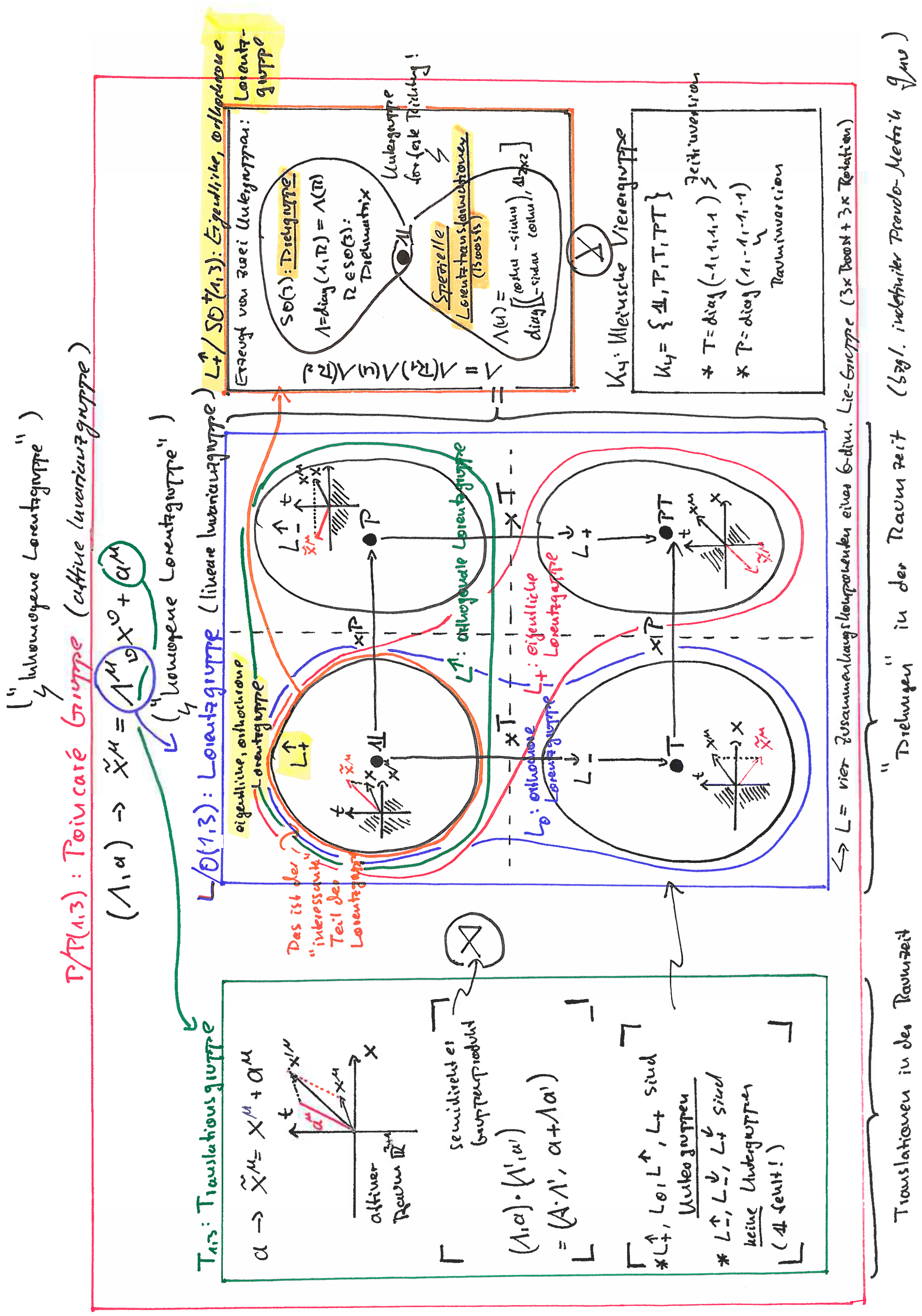
- Equivalent to

$$U(P)\Psi(t,\vec{x})U^{-1}(P) = \underbrace{\gamma^0}_{P_{\frac{1}{2}}}\,\Psi\underbrace{(t,-\vec{x})}_{Px}$$

- Dirac field bilinears (examples):

$$U(P)\overline{\Psi}\Psi U^{-1}(P) \mathrel{\hat{=}} +\overline{\Psi}\Psi(t,-\vec{x}) \qquad \to \text{scalar}$$
$$U(P)\overline{\Psi}\gamma^5\Psi U^{-1}(P) \mathrel{\hat{=}} -\overline{\Psi}\gamma^5\Psi(t,-\vec{x}) \qquad \to \text{pseudo-scalar}$$

↑ Notes


P/P(1,3): Poincaré Gruppe (affine Invarianzgruppe)
("Inhomogene Lorentzgruppe")
$(\Lambda, a) \to \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$
L/O(1,3): Lorentzgruppe (lineare Invarianzgruppe)
("Homogene Lorentzgruppe")
$L^\uparrow_+$ / $SO^+(1,3)$: Eigentliche, orthochrone Lorentzgruppe
Erzeugt von zwei Untergruppen:
SO(3): Drehgruppe
$\Lambda = \mathrm{diag}(1, R) = \Lambda(R)$
$R \in SO(3)$: Drehmatrix
Untergruppe für feste Richtung!
Spezielle Lorentztransformationen (Boosts)
$\Lambda(u) = \mathrm{diag}\left[\begin{pmatrix}\cosh u & -\sinh u \\ -\sinh u & \cosh u\end{pmatrix}, \mathbb{1}_{2\times2}\right]$
$\Lambda = \Lambda(R)\Lambda(u)\Lambda(R')$
K_4: Kleinsche Vierergruppe
$K_4 = \{\mathbb{1}, P, T, PT\}$
* $T = \mathrm{diag}(-1,1,1,1)$ Zeitinversion
* $P = \mathrm{diag}(1,-1,-1,-1)$ Rauminversion
$L^\uparrow_+$ $L^\uparrow_-$ $L^\downarrow_-$ $L^\downarrow_+$
$\mathbb{1}$ P T PT
×P ×T
eigentliche, orthochrone Lorentzgruppe
$L^\uparrow$: orthochrone Lorentzgruppe
L_+: eigentliche Lorentzgruppe
L_0: orthochore Lorentzgruppe
Das ist der "interessante" Teil der Lorentzgruppe
↳ L = vier Zusammenhangskomponenten einer 6-dim. Lie-Gruppe (3x Boost + 3x Rotation)
"Drehungen" in der Raumzeit
(bzgl. indefiniter Pseudo-Metrik $g_{\mu\nu}$)
$T_{1,3}$: Translationsgruppe
$a \to \tilde{x}^\mu = x^\mu + a^\mu$
affiner Raum $\mathbb{R}^{1,3}$
semidirektes Gruppenprodukt
$(\Lambda, a)\cdot(\Lambda', a') = (\Lambda\cdot\Lambda', a + \Lambda a')$
* $L^\uparrow_+$, L_0, $L^\uparrow$, L_+ sind Untergruppen
* $L^\uparrow_-$, $L^\downarrow_-$, $L^\downarrow_+$ sind keine Untergruppen ($\mathbb{1}$ fehlt!)
Translationen in der Raumzeit

↑ Notes

Time Reversal

1. Time reversal should …
   - $U(T)\Psi(t,\vec{x})U^{-1}(T) = T_{\frac{1}{2}}\Psi(-t,\vec{x})$,
   - $U(T)a^s_{\vec{p}}U^{-1}(T) = a^?_{-\vec{p}}$,
   - flip spins (motivated by $\vec{L} = \vec{r}\times\vec{p} \mapsto -\vec{L}$),
   - be a symmetry of the Dirac theory ($[U(T), H] = 0$),
   - obey $U^{-1}(T) = U^\dagger(T)$
     (this is required for any symmetry to preserve overlaps, ➔ Wigner's theorem)

   Note that often $U(T)$ is simply written T.

2. *Problem:*

$$
\begin{aligned}
&& \Psi(t,\vec{x}) &= e^{iHt}\Psi(\vec{x})e^{-iHt} \\
\Rightarrow && U(T)\Psi(t,\vec{x})U^{-1}(T) &= e^{iHt}U(T)\Psi(\vec{x})U^{-1}(T)e^{-iHt} \\
\Rightarrow && T_{\frac{1}{2}}\Psi(-t,\vec{x})|0\rangle &= e^{iHt}T_{\frac{1}{2}}\Psi(\vec{x})|0\rangle \\
\Rightarrow && T_{\frac{1}{2}}e^{-iHt}\Psi(\vec{x})|0\rangle &= e^{iHt}T_{\frac{1}{2}}\Psi(\vec{x})|0\rangle \\
\Rightarrow && \underbrace{e^{-2iHt}}_{\text{time-dependent!}}\; T_{\frac{1}{2}}\Psi(\vec{x})|0\rangle &= T_{\frac{1}{2}}\Psi(\vec{x})|0\rangle
\end{aligned}
$$

   Here we used that $[U(T), H] = 0$ and $H|0\rangle = 0$.
   → Not possible (for invertible $T_{\frac{1}{2}}$ and arbitrary times t)!

3. *Solution:* $U(T)$ must be *antiunitary/antilinear*:

$$
\boxed{U(T)c = c^*U(T) \quad \text{for } c\in\mathbb{C}} \tag{3.16}
$$

   Highlight the differences with colors in the derivation above:

$$
\begin{aligned}
&& \Psi(t,\vec{x}) &= e^{iHt}\Psi(\vec{x})e^{-iHt} \\
\Rightarrow && U(T)\Psi(t,\vec{x})U^{-1}(T) &= e^{-iHt}U(T)\Psi(\vec{x})U^{-1}(T)e^{iHt} \\
\Rightarrow && T_{\frac{1}{2}}\Psi(-t,\vec{x})|0\rangle &= e^{-iHt}T_{\frac{1}{2}}\Psi(\vec{x})|0\rangle \\
\Rightarrow && T_{\frac{1}{2}}e^{-iHt}\Psi(\vec{x})|0\rangle &= e^{-iHt}T_{\frac{1}{2}}\Psi(\vec{x})|0\rangle \\
\Rightarrow && \underbrace{\mathbb{1}}_{\text{time-independent!}}\; T_{\frac{1}{2}}\Psi(\vec{x})|0\rangle &= T_{\frac{1}{2}}\Psi(\vec{x})|0\rangle
\end{aligned}
$$

4. Transformation of spin:
   a) Spinors: ∢ spin basis ξ^s ($s = 1,2$) along arbitrary axis $\vec{n}$:

$$
\xi^1 = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \quad \text{and} \quad \xi^2 = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}
$$

↑ Notes

That is, $\xi^1 = |\uparrow\rangle$ and $\xi^2 = |\downarrow\rangle$.
"Time-reversed" (=flipped) spinors:

$$\overline{\xi^s} \equiv -i\sigma^2(\xi^s)^* \quad \Rightarrow \quad \begin{Bmatrix}\overline{\xi^1}\\ \overline{\xi^2}\end{Bmatrix} = \begin{Bmatrix}\xi^2\\ -\xi^1\end{Bmatrix} \tag{3.17}$$

Indeed, if $\vec{n}\cdot\vec{\sigma}\,\xi = +\xi$, we have

$$\vec{n}\cdot\vec{\sigma}\,(-i\sigma^2\xi^*) = -i\sigma^2(-\vec{n}\cdot\vec{\sigma})^*\xi^* = i\sigma^2(\xi^*) = -(-i\sigma^2\xi^*)$$

where we used $\vec{\sigma}\sigma^2 = \sigma^2(-\vec{\sigma}^*)$.

Note that $T = -i\sigma^2 K$ (where K denotes complex conjugation) is the conventional representation of time-reversal symmetry for spinful fermions that you might know from condensed matter physics where it is used to classify symmetry-protected topological phases.

b) Bispinors:

$$\begin{aligned} u^s(p) &\equiv \begin{pmatrix}\sqrt{p\sigma}\,\xi^s\\ \sqrt{p\bar{\sigma}}\,\xi^s\end{pmatrix} \quad\text{and}\quad v^s(p) \equiv \begin{pmatrix}\sqrt{p\sigma}\,\overline{\xi^s}\\ -\sqrt{p\bar{\sigma}}\,\overline{\xi^s}\end{pmatrix} \\ \overline{u^s}(p) &\equiv \begin{pmatrix}\sqrt{p\sigma}\,\overline{\xi^s}\\ \sqrt{p\bar{\sigma}}\,\overline{\xi^s}\end{pmatrix} \quad\text{and}\quad \overline{v^s}(p) \equiv \begin{pmatrix}\sqrt{p\sigma}\,\overline{\overline{\xi^s}}\\ -\sqrt{p\bar{\sigma}}\,\overline{\overline{\xi^s}}\end{pmatrix} \end{aligned} \tag{3.18}$$

Use colors to skip the second row.
Note that here $\overline{u^s}$ is *not* the Dirac adjoint $\bar{u}^s$!
Recall that the basis η^s in the definition of $v^s(p)$ was arbitrary.

c) Define the modes:

$$\begin{Bmatrix}\overline{a^1_{\vec{p}}}\\ \overline{a^2_{\vec{p}}}\end{Bmatrix} \equiv \begin{Bmatrix}a^2_{\vec{p}}\\ -a^1_{\vec{p}}\end{Bmatrix} \quad\text{and}\quad \begin{Bmatrix}\overline{b^1_{\vec{p}}}\\ \overline{b^2_{\vec{p}}}\end{Bmatrix} \equiv \begin{Bmatrix}b^2_{\vec{p}}\\ -b^1_{\vec{p}}\end{Bmatrix} \tag{3.19}$$

Skip the second part.
Note that this is analog to (3.17)!

d) Let $\tilde{p} \equiv (p^0, -\vec{p})$ and show

$$\begin{aligned} \overline{u^s}(\tilde{p}) &\overset{\circ}{=} -\gamma^1\gamma^3[u^s(p)]^* \\ \overline{v^s}(\tilde{p}) &\overset{\circ}{=} -\gamma^1\gamma^3[v^s(p)]^* \end{aligned} \tag{3.20}$$

Note: $\overline{\overline{\xi^s}} = -\xi^s$ used in $\overline{v^s}$
Use (3.18) and $\sqrt{\tilde{p}\sigma}\,\sigma^2 = \sigma^2\sqrt{p\sigma^*}$ to show this!

5. *Definition:*

$$\left.\begin{aligned} &\text{Antilinearity (3.16)} \\ U(T)\,a^s_{\vec{p}}U^{-1}(T) &\equiv \overline{a^s_{-\vec{p}}} \\ U(T)\,b^s_{\vec{p}}U^{-1}(T) &\equiv \overline{b^s_{-\vec{p}}} \end{aligned}\right\} \Rightarrow \begin{aligned} &U(T)\Psi(t,\vec{x})U^{-1}(T) \\ &\overset{\circ}{=} \underbrace{(\gamma^1\gamma^3)}_{T_{\frac{1}{2}}}\,\Psi(-t,\vec{x}) \end{aligned}$$

Use (3.20) and (3.19) and $\overline{a^2_{\vec{p}}}\,\overline{u^2}(p) = a^1_{\vec{p}}u^1(p)$ etc. to show this!

↑ Notes

6. Dirac field bilinears (example: $j^\mu = \overline{\Psi}\gamma^\mu\Psi$):

$$U(T)j^\mu(t,\vec{x})U^{-1}(T) \stackrel{\circ}{=} \begin{cases} +j^\mu(-t,\vec{x}) & \text{for } \mu = 0 \\ -j^\mu(-t,\vec{x}) & \text{for } \mu = 1,2,3 \end{cases}$$

→ As expected for density ($\mu = 0$) and 3-current ($\mu = 1,2,3$)

Charge Conjugation

1. Discrete, non-spacetime symmetry that exchanges particle and antiparticle:

$$\boxed{U(C)a^s_{\vec{p}}U^{-1}(C) = b^s_{\vec{p}} \quad \text{and} \quad U(C)b^s_{\vec{p}}U^{-1}(C) = a^s_{\vec{p}}}$$

Often $U(C)$ is simply written C. Note that there is no representation on Minkowski space as this is an "internal" symmetry.

2. Use (3.18) to show:

$$u^s(p) \stackrel{\circ}{=} -i\gamma^2(v^s(p))^* \quad \text{and} \quad v^s(p) \stackrel{\circ}{=} -i\gamma^2(u^s(p))^*$$

3. Then

$$\begin{aligned} &U(C)\Psi(x)U^{-1}(C) \\ &= \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[-i\gamma^2(v^s(p))^* b^s_{\vec{p}} e^{-ipx} - i\gamma^2(u^s(p))^* a^{s\dagger}_{\vec{p}} e^{ipx}\right] \\ &\stackrel{\circ}{=} -i\gamma^2(\Psi^\dagger)^T = -i(\overline{\Psi}\gamma^0\gamma^2)^T \end{aligned}$$

4. Therefore:

$$\boxed{\begin{aligned} &U(C)\Psi U^{-1}(C) = -i(\overline{\Psi}\gamma^0\gamma^2)^T \,(= \underbrace{-i\gamma^2}_{\equiv C_{\frac{1}{2}}} \Psi^*) \\ \text{and} \quad &U(C)\overline{\Psi}U^{-1}(C) \stackrel{\circ}{=} -i(\gamma^0\gamma^2\Psi)^T \end{aligned}}$$

Note that C essentially exchanges $\Psi \leftrightarrow \overline{\Psi}$ but is not antiunitary!
To show this, recall that γ^0 and γ^2 are symmetric matrices.

It is $C^\dagger_{\frac{1}{2}} = C_{\frac{1}{2}}$, $C^2_{\frac{1}{2}} = \mathbb{1}$ and $C_{\frac{1}{2}}\gamma^\mu C_{\frac{1}{2}} = -\gamma^{\mu *}$. Note that the expression in parantheses is only true for the transformation of *classical* (i.e. "first quantized") Dirac fields and can be used to show the symmetry of the classical Dirac equation.

5. Dirac field bilinears (examples):

$$\begin{aligned} U(C)\overline{\Psi}\Psi U^{-1}(C) &\stackrel{\circ}{=} \overline{\Psi}\Psi && \text{(Scalar)} \\ U(C)\overline{\Psi}\gamma^\mu\Psi U^{-1}(C) &\stackrel{\circ}{=} -\overline{\Psi}\gamma^\mu\Psi && \text{(Vector)} \end{aligned}$$

↑ Notes

Note 3.3

- Any relativistic QFT must be invariant under $\mathrm{SO}^+(1,3)$ $(= L_+^\uparrow)$
- The (classical) Dirac equation $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ is $\{C, P, T\}$-invariant
- The (quantized) Dirac theory is $\{C, P, T\}$-invariant: $[H, U(X)] = 0$ for $X = P, T, C$
- Weak interactions (of the standard model) violate C and P but preserve CP and T (➔ Wu experiment)
- Rare processes (decay of neutral kaons) violate CP and T but preserve CPT
- CPT seems to be a perfect symmetry of nature
- CPT *theorem*:

$$\left.\begin{array}{r}\mathrm{SO}^+(1,3)\text{ invariance}\\ \text{Causality}\\ \text{Locality}\\ \text{Stable vacuum}\end{array}\right\} \Rightarrow CPT\text{ symmetry}$$

↑ Notes

4 Interacting Fields and Feynman Diagrams

4.1 Preliminaries

- Up to now: No interactions, no scattering, Fourier modes are eigenmodes
- Now: Include non-linear terms in the Hamiltonian/Lagrangian that couple Fourier modes
- Causality → Interactions = Products of fields at same spacetime point
- In the following:

$$H_{\text{int}} = \int d^3x\, \mathcal{H}_{\text{int}}(\phi(x)) = -\int d^3x\, \mathcal{L}_{\text{int}}(\phi(x))$$

  $\mathcal{L}_{\text{int}}$ is only a function of $\phi \to \mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$.

- Examples:
  1. ϕ^4*-theory:*

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

     λ: dimensionless *coupling constant*
     Why do we choose ϕ^4 and not ϕ^3? Energy unbounded from below for ϕ^3!
     The ϕ^4-interaction arises in the standard model (Higgs field) and also in statistical mechanics.
     → Equation of motion is no longer linear:

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$$

     → Cannot be solved by Fourier modes!
  2. *Yukawa theory:*

$$\mathcal{L}_{\text{Yukawa}} = \underbrace{\overline{\Psi}(i\partial\!\!\!/ - m)\Psi}_{\text{Dirac}} + \underbrace{\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2}_{\text{Klein-Gordon}} - \underbrace{g\overline{\Psi}\Psi\phi}_{\text{Interaction}}$$

     g: dimensionless coupling constant
     Yuakawa theory = QED for a scalar field ϕ instead of a vector field A^μ.
     In the standard model, Yukawa couplings describe the coupling of the Higgs field to quarks and leptons.

↑ Notes

3. *QED (Quantum Electrodynamics):*

$$\mathcal{L}_{\text{QED}} = \underbrace{\bar\Psi(i\partial\!\!\!/ - m)\Psi}_{\text{Dirac}} - \underbrace{\frac{1}{4}(F_{\mu\nu})^2}_{\text{Maxwell}} - \underbrace{e\bar\Psi\gamma^\mu\Psi A_\mu}_{\text{Interaction}}$$
$$= \bar\Psi(iD\!\!\!/ - m)\Psi - \frac{1}{4}(F_{\mu\nu})^2$$

$e = -|e| < 0$: Electron charge
$D_\mu \equiv \partial_\mu + ieA_\mu(x)$: *Gauge covariant derivative*
The coupling via $\partial \mapsto D$ is called *minimal coupling*.
QED has a $U(1)$ gauge symmetry: $A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$ and $\Psi'(x) = e^{i\alpha(x)}\Psi(x)$.
→ Equations of motion:

$$(iD\!\!\!/ - m)\Psi(x) = 0 \quad \text{and} \quad \partial_\mu F^{\mu\nu} = ej^\nu \quad (j^\nu = \bar\Psi\gamma^\nu\Psi)$$

Quantizing the EM field is subtle due to gauge invariance. We will demonstrate one possibility at the end of this course using path integrals.

- The list of possible interaction terms is *finite* due to constraints like gauge invariance and *renormalizability* (we will discuss this in the second half of the semester).
- The standard model includes *all* of the allowed interactions. The three examples above cover nearly half of them!
- No known exactly solvable interacting QFTs in $D > 1 + 1$!
  Examples of exactly solvable interacting QFTs in $D = 1 + 1$ are *conformal field theories* which have an extensive set of symmetry generators.
  → Perturbation theory
  (we hope/assume that the coupling constants are small enough!)

4.2 Perturbation Expansion of Correlation Functions

Details: ➔ Problemset 5

1. *Goal:* Two-point Green's function $\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle$ of ϕ^4-theory
   $|\Omega\rangle$: Ground state of interacting theory
   $|0\rangle$: Ground state of free theory (free=non-interacting)
2. *Remember:* Without interactions, this is the Feynman propagator:

$$\langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle = D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon}$$

3. *Now:*

$$H_{\phi^4} = \underbrace{H_0}_{\text{KG Hamiltonian}} + \underbrace{\int d^3x\, \frac{\lambda}{4!}\phi^4(\vec{x})}_{H_{\text{int}}:\ \text{Interaction = Perturbation}}$$

→ Expand $\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle$ in powers of λ

↑ Notes

4. *Todo:*

$$\text{Express} \quad \begin{Bmatrix} \phi(x) \\ |\Omega\rangle \end{Bmatrix} \quad \text{in terms of} \quad \begin{Bmatrix} \text{free field } \phi_I(x) \\ \text{free vacuum } |0\rangle \end{Bmatrix}$$

Note that both $\phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt}$ and $|\Omega\rangle$ depend on the interaction.

5. $\sphericalangle$ reference time t_0, then

$$\phi(t_0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\vec{x}}\right)$$

This follows, because the equal-time commutation relations are still valid.
The modes $a_{\vec{p}}$ now implicitly and *non-trivially* depend on the reference time t_0! This dependence only drops out for the free theory where the Fourier modes are stationary eigenmodes.

6. *Definitions:*

$$\begin{aligned} \phi(t, \vec{x}) &\equiv e^{iH(t-t_0)}\phi(t_0, \vec{x})e^{-iH(t-t_0)} && \text{Heisenberg picture} \\ \phi_I(t, \vec{x}) &\equiv e^{iH_0(t-t_0)}\phi(t_0, \vec{x})e^{-iH_0(t-t_0)} && \text{Interaction picture} \end{aligned}$$

Then

$$\phi_I(t, \vec{x}) \stackrel{\circ}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ipx} + a^\dagger_{\vec{p}} e^{ipx}\right)$$

This is analogous to the free field!
and

$$\phi(t, \vec{x}) = U^\dagger(t, t_0)\phi_I(t, \vec{x})U(t, t_0) \quad \text{with} \quad U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{4.1}$$

Our goal is to express ϕ in terms of ϕ_I since we know its time evolution!

7. The time-evolution operator is determined by $U(t_0, t_0) = \mathbb{1}$ and the differential equation

$$i\partial_t U(t, t_0) \stackrel{\circ}{=} H_I(t)U(t, t_0) \tag{4.2}$$

with

$$H_I(t) = e^{iH_0(t-t_0)}\, H_{\text{int}}\, e^{-iH_0(t-t_0)} = \int d^3x\, \frac{\lambda}{4!}\phi_I^4(t, \vec{x}) \tag{4.3}$$

8. The solution of (4.2) is given by the *Dyson series*:

$$\begin{aligned} U(t, t_0) = \;& \mathbb{1} + (-i)\int_{t_0}^t dt_1\, H_I(t_1) \\ & + \frac{(-i)^2}{2!}\int_{t_0}^t dt_1\, dt_2\, \mathcal{T}\{H_I(t_1)H_I(t_2)\} + \dots \\ \equiv \;& \mathcal{T}\exp\left[-i\int_{t_0}^t ds\, H_I(s)\right] \end{aligned} \tag{4.4}$$

The Dyson series yields an expansion for $\phi(t, \vec{x})$ in terms of $\phi_I(t, \vec{x})$ in powers of λ.
This is the *definition* of the time-ordered exponential.

↑ Notes

9. *Properties:*

$$U(t,t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)}$$
$$U^{-1}(t,t') = U^\dagger(t,t') \tag{4.5}$$
$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$$

Here, $t \geq t'$ and $t_1 \geq t_2 \geq t_3$; the definition for $t' \neq t_0$ is given by (4.4).
➔ Problemset 5 for details.

10. Ground state $|\Omega\rangle$?
$\lambda \ll 1 \to \langle\Omega|0\rangle \neq 0$ (this is not a rigorous but a reasonable assumption)

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$
$$= e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

Then (since $E_n > E_0$ for $n \neq 0$)

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$
$$\stackrel{\circ}{=} \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(t_0+T)}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)|0\rangle \tag{4.6}$$

➔ Problemset 5 for details.
Similar:

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \tag{4.7}$$

11. *Two-point correlator:* (let $x^0 > y^0 > t_0$)

$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = (4.7)\times(4.1)\times(4.1)\times(4.6)$

$$\stackrel{(4.5)}{=} \lim_{T\to\infty(1-i\varepsilon)} \mathcal{N}_T^{-1}\langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0\rangle$$

with

$$\mathcal{N}_T \stackrel{(4.7)\times(4.6)}{=} \langle 0|U(T,t_0)U(t_0,-T)|0\rangle \stackrel{(4.5)}{=} \langle 0|U(T,-T)|0\rangle$$

For $y^0 > x^0$ we can do the same calculation for $\langle\Omega|\phi(y)\phi(x)|\Omega\rangle$ by replacing $x \leftrightarrow y$.
➔ Problemset 5 for details.

This leads to the final result:
$x^0 \gtrless y^0$ arbitrary $\to$

$$\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle \stackrel{(4.4)}{=} \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\mathcal{T}\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dt\, H_I(t)\right]\right\}|0\rangle}{\langle 0|\mathcal{T}\left\{\exp\left[-i\int_{-T}^{T}dt\, H_I(t)\right]\right\}|0\rangle} \tag{4.8}$$

The derivation goes through for arbitrary n-point correlators.

↑ Notes

4.3 Wick's Theorem

(4.3) and (4.8) → (expand the time-ordered exponential in orders of λ)

$$\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle = \sum \dots \underbrace{\langle 0|\mathcal{T}\phi_I(x_1)\phi_I(x_2)\dots\phi_I(x_n)|0\rangle}_{\text{How to evaluate this efficiently?}}\dots$$

Solution: Wick's theorem!
We could just use the mode expansion of the fields and calculate the n-point correlators the brute force way. But Wick's theorem provides a more systematic approach.

1. Define

$$\phi_I(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}a_{\vec{p}}e^{-ipx}}_{\equiv\phi_I^+(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}a_{\vec{p}}^\dagger e^{+ipx}}_{\equiv\phi_I^-(x)}$$

   Useful because $\phi_I^+|0\rangle = 0$ and $\langle 0|\phi_I^- = 0$.

2. Observation for $x^0 > y^0$ and $n = 2$:

$$\begin{aligned}\mathcal{T}\phi_I(x)\phi_I(y) &= \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)\\ &= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)\\ &\quad+[\phi_I^+(x),\phi_I^-(y)]\end{aligned}$$

   For $y^0 > x^0$ we find:

$$\begin{aligned}\mathcal{T}\phi_I(x)\phi_I(y) &= \phi_I^+(y)\phi_I^+(x) + \phi_I^+(y)\phi_I^-(x) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(y)\phi_I^-(x)\\ &= \phi_I^+(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(y)\phi_I^-(x)\\ &\quad+[\phi_I^+(y),\phi_I^-(x)]\end{aligned}$$

   Use colors to skip this step!

3. This motivates the *Definitions:*
   (We drop the subscript I henceforth as contractions always operate on interaction picture fields).

   Contraction:

$$\overbrace{\phi(x)\phi(y)} \equiv \begin{Bmatrix}[\phi^+(x),\phi^-(y)] & \text{for } x^0 > y^0\\ [\phi^+(y),\phi^-(x)] & \text{for } y^0 > x^0\end{Bmatrix} \,\hat{=}\, D_F(x-y)\cdot\mathbb{1}$$

   Normal order:

$$:a_1^{(\dagger)}\dots a_n^{(\dagger)}: \equiv \text{(creation operators)} \times \text{(annihilation operators)}$$

   Example for normal order: $:\phi^+(x)\phi^-(y): = \phi^-(y)\phi^+(x)$
   Recall that $[\phi(x),\phi(y)] = [\phi^+(x),\phi^-(y)] + [\phi^-(x),\phi^+(y)] = D(x-y) - D(y-x)$.
   Like time ordering $\mathcal{T}$, normal ordering $:\bullet:$ is a *meta operator* that acts on symbolic strings (= descriptions of operators = the free algebra of a_i and $a_i^\dagger$). In particular,

↑ Notes

normal ordering is *not* well-defined on the CCR algebra: $a^\dagger a = {:}aa^\dagger{:} \neq {:}a^\dagger a + 1{:} = {:}a^\dagger a{:} + {:}1{:} = a^\dagger a + 1$, ➔ https://physics.stackexchange.com/a/368084/45257.
The vacuum expectation value of normal-ordered products vanishes!
→

$$\mathcal{T}\phi(x)\phi(y) = {:}\phi(x)\phi(y) + \overbracket{\phi(x)\phi}(y){:}$$
$$\Rightarrow \quad \langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle = D_F(x-y)$$

4. The generalization of this statement is called *Wick's theorem:*

$$\mathcal{T}\{\phi(x_1)\dots\phi(x_n)\} = {:}\phi(x_1)\dots\phi(x_n) + \text{all possible contractions}{:}$$
$$\text{where} \quad {:}A\,\overbracket{\phi_i\,B\,\phi_j}\,C{:} \equiv D_F(x_i - x_j)\cdot{:}ABC{:}$$

Proof: ➔ Problemset 5
Wick's theorem is *not* specific to QFT but a quite generic, combinatorical statement, ➔ https://physics.stackexchange.com/a/24180/45257. For instance, in probability theory, it is well known that the expectation values of arbitrary products of Gaussian random variables are completely determined by two-point correlators, ➔ https://en.wikipedia.org/wiki/Isserlis%27_theorem.

5. *Corollary:*

$$\langle 0|\mathcal{T}\{\phi(x_1)\dots\phi(x_n)\}|0\rangle = \text{all full contractions}$$

Wick's theorem in this form is only valid for expectation values w.r.t. the non-interacting vacuum $|0\rangle$ of non-interacting fields (recall that we omit here the subscript I, i.e., $\phi = \phi_I$).

Problem Set 5

(due **15.05.2020**)

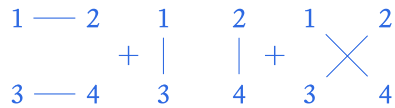
1. Recap of the lecture: Perturbation expansion of correlation functions
2. Proof of Wick's theorem by construction of a generating functional

6. Example ($\phi_i \equiv \phi(x_i)$):

$$\begin{aligned}\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\} =: \phi_1\phi_2\phi_3\phi_4 &+ \wick{\c1\phi_1\c1\phi_2\phi_3\phi_4} + \wick{\c1\phi_1\phi_2\c1\phi_3\phi_4} + \wick{\c1\phi_1\phi_2\phi_3\c1\phi_4}\\ &+ \wick{\phi_1\c1\phi_2\c1\phi_3\phi_4} + \wick{\phi_1\c1\phi_2\phi_3\c1\phi_4} + \wick{\phi_1\phi_2\c1\phi_3\c1\phi_4}\\ &+ \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} :\end{aligned}$$

Therefore

$$\begin{aligned}\langle 0|\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle &= \langle 0|\wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}|0\rangle\\ &= D_F(x_1-x_2)D_F(x_3-x_4)\\ &\quad + D_F(x_1-x_3)D_F(x_2-x_4)\\ &\quad + D_F(x_1-x_4)D_F(x_2-x_3)\end{aligned}$$

$=$ 

We associate each spacetime point x_i with a vertex and each propagator connecting two points with an edge. These are *Feynman diagrams*, here for the trivial example of free fields. We interpret edges as particles propagating from one point to another; the propagation amplitude is then the superposition of all possible ways for two particles to propagate between four points.

4.4 Feynman Diagrams

Details: ➔ Problemset 6

1. ∢ Numerator of (4.8)

$$\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle \propto \langle 0|\mathcal{T}\left\{\phi(x)\phi(y) + \phi(x)\phi(y)\left[-i\int dt\, H_I(t)\right] + \dots\right\}|0\rangle$$

We focus now on ϕ^4-theory and develop the formalism for this specific theory. ➔ Problemset 6 for an analogous treatment of the *complex* Klein-Gordon field.

2. λ^0-term: $\langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle = D_F(x-y) =$ x — y

↑ Notes

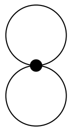
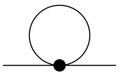
3. λ^1-term:

$$\langle 0|\mathcal{T}\left\{\phi(x)\phi(y)\frac{(-i\lambda)}{4!}\underbrace{\int d^4z}_{\int dt\,\int d^3x}\;\phi(z)\phi(z)\phi(z)\phi(z)\right\}|0\rangle$$

Wick's theorem

$$= \quad 3\cdot\frac{(-i\lambda)}{4!}D_F(x-y)\int d^4z\, D_F(z-z)D_F(z-z)$$

$$+12\cdot\frac{(-i\lambda)}{4!}\int d^4z\, D_F(x-z)D_F(y-z)D_F(z-z)$$

$= x$ — y  $+$ x  y

→ *Interpretation:*

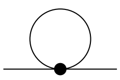
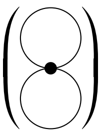
Feynman diagram {	edges	= *"propagators"*	↔	D_F	}	Analytic expres- sion
	internal points	= *"vertices"*	↔	$(-i\lambda)\int d^4z$		
	external points	= spacetime points	↔	$x, y, \ldots$		

Feynman diagram ≙ Process of particle creation & propagation & annihilation

Internal points are vertices with four emanating edges that are associated to an integration. External points are vertices that are endpoints of a single edge and associated to boundary conditions (i.e. given spacetime points $x, y, \ldots$ of the correlation function).

4. *Prefactors:*

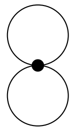
- Feynman diagram = sum of all identical terms (= prefactor)
- $\sphericalangle\ \mathcal{O}(\lambda^n)$
  → factor $\frac{1}{n!}$ and n integrals/vertices
  → $n!$ possibilities to interchange vertices cancels $\frac{1}{n!}$
  → *ignore the* $\frac{1}{n!}$
- 4 contractions at each vertex
  → 4! possibilities to interchange contractions
  → $\frac{1}{4!}$ of interaction cancels 4! (this is the reason for the $\frac{1}{4!}$ in the first place)
  → *associate* $(-i\lambda)\int d^4z$ *with each vertex*
- Symmetries of diagrams reduce the number of different contractions
  → divide expression by the *symmetry factor* S
- Examples:

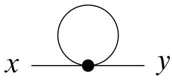
$S\Big($ x  y $\Big) = 2$ and $S\Big($  $\Big) = 2\cdot 2\cdot 2 = 8$

Imagine the diagram is made from strings pinned at external points and placed flat on the table. Strings emanating from a vertex are marked with a colored flag.

↑ Notes

Count the configurations that look the same when one forgets about the flags but are different when the flags are taken into account.
Therefore:

x — y  $= \frac{1}{8} \cdot D_F(x-y)(-i\lambda) \int d^4z\, D_F(z-z) D_F(z-z)$

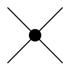


$= \frac{1}{2} \cdot (-i\lambda) \int d^4z\, D_F(x-z) D_F(y-z) D_F(z-z)$

5. Therefore

$$\langle 0|\mathcal{T}\left\{\phi(x)\phi(y)e^{-i\int dt\, H_I(t)}\right\}|0\rangle = \sum \begin{Bmatrix}\text{Feynman diagrams with two} \\ \text{external points } x \text{ and } y\end{Bmatrix}$$

with the *position/real-space Feynman rules* for the ϕ^4-theory

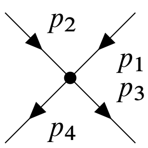
1. For each propagator, x — y $= D_F(x-y)$
2. For each vertex,  $= (-i\lambda)\int d^4z$
3. For each external point, x —— $= 1$
4. Divide by the symmetry factor, $\frac{1}{S} \times \dots$

The integration over spacetime coordinates z at each internal vertex accounts for the superposition principle: We sum over all spacetime positions where the absorption/emission of particles – represented by vertices – can occur.

6. Often calculations are simpler in momentum space:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$$

Assign arbitrary orientation to edges (since $D_F(x-y) = D_F(y-x)$) and perform vertex integrals:



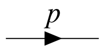
$= (-i\lambda)\int d^4z \dots = (-i\lambda)(2\pi)^4\delta(p_1 + p_2 - p_3 - p_4)$

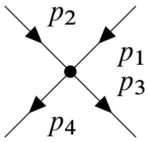
Details: ➔ Problemset 6
→ Momentum conservation at vertices
Note that it is actually

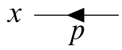
$$\int d^4z \dots = \lim_{T\to\infty(1-i\varepsilon)} \int_{-T}^{T} dz^0 \int d^3z \dots$$

↑ Notes

→ *Momentum-space Feynman rules:*

1. For each propagator, 

$= \frac{i}{p^2-m^2+i\varepsilon}$
2. For each vertex, 

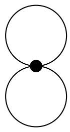
$= (-i\lambda)(2\pi)^4 \times \delta(p_1+p_2-p_3-p_4)$
3. For each external point, 

$= e^{-ipx}$
4. Integrate momenta, $\prod_i \int \frac{d^4p_i}{(2\pi)^4}\dots$
5. Divide by sym. factor, $\frac{1}{S}\times\dots$

Equivalence between momentum- and position space Feynman rules:
➔ Problemset 6

↑ Notes

7. *Problem:* Disconnected pieces of diagrams diverge!
   Example:



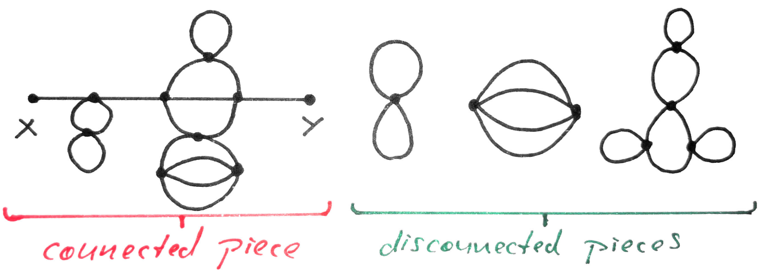
$$= \frac{1}{8} \cdot (-i\lambda) \int d^4z \underbrace{D_F(0) D_F(0)}_{\text{const}}$$

$$\propto (2T) \cdot (\text{volume of space}) = (2\pi)^4 \delta(0)$$

   Interpretation: This "detached" process can happen anytime and anywhere in an infinitely large (for $T \to \infty(1 - i\varepsilon)$) spacetime volume—and we have to sum up all these amplitudes!

8. *Exponentiation of disconnected diagrams:*
   (this is a preliminary step to cancel the divergencies with the denominator, see below)

   a) Typical diagram:



   x and y are always connected because the sum of all degrees of all vertices of a connected graph is always even (=twice the number of edges). Note that the only (graph) vertices with *odd* degree in a ϕ^4-Feynman diagram are the external points.

   b) Let

$$\mathcal{V} = \{V_1, V_2, \dots\} \equiv \left\{\begin{matrix}\text{Set of all } \textit{disconnected} \\ \text{Feynman diagrams} \\ \text{without external points}\end{matrix}\right\}$$

$$\mathcal{F}^{xy} \equiv \left\{\begin{matrix}\text{Set of all } \textit{connected} \\ \text{Feynman diagrams} \\ \text{with external points } x \\ \text{and } y\end{matrix}\right\}$$

$$\rightarrow \quad \text{Feynman diagram } F = \left\{ \underbrace{F^{xy}}_{\text{Connected part}}, \underbrace{V_1, \dots, V_1}_{\text{Multiplicity } n_1}, \underbrace{V_2, \dots, V_2}_{n_2}, V_3, \dots \right\}$$

   Abuse of notation: V_i denotes also the *value* of the corresponding diagram.

   c) Amplitude of F:

$$F = F^{xy} \cdot \prod_i \underbrace{\frac{1}{n_i!}}_{S_i} (V_i)^{n_i}$$

   S_i: Symmetry factor for exchanging the n_i copies of V_i

↑ Notes

d) Then

$$\begin{aligned}\langle 0|\mathcal{T}\left\{\phi(x)\phi(y)e^{-i\int dt\,H_I(t)}\right\}|0\rangle &= \sum_{F\in\mathcal{F}^{xy}}\sum_{n_1,n_2,\dots}\left[F\cdot\prod_i\frac{1}{n_i!}(V_i)^{n_i}\right]\\ &= \left[\sum_{F\in\mathcal{F}^{xy}}F\right]\times\left[\sum_{n_1,n_2,\dots}\prod_i\frac{1}{n_i!}(V_i)^{n_i}\right]\\ &= \left[\sum_{F\in\mathcal{F}^{xy}}F\right]\times\left[\prod_i\sum_{n_i}\frac{1}{n_i!}(V_i)^{n_i}\right]\\ &= \left[\sum_{F\in\mathcal{F}^{xy}}F\right]\times\exp\left[\sum_i V_i\right]\end{aligned}$$

In words:

(sum of all diagrams) =
(sum of all connected pieces) × exp[sum of all disconnected pieces]

→

$$\langle 0|\mathcal{T}\left\{\phi(x)\phi(y)e^{-i\int dt\,H_I(t)}\right\}|0\rangle = \Sigma(\mathcal{F}^{xy})\times e^{\Sigma(\mathcal{V})}$$

with $\Sigma(X)\equiv\sum_{x\in X}x$

9. *Denominator* of (4.8):

$$\langle 0|\mathcal{T}\left\{e^{-i\int dt\,H_I(t)}\right\}|0\rangle = e^{\Sigma(\mathcal{V})}$$

The argument runs along the same lines as for the numerator.

10. *Two-point correlator:*

$$\begin{aligned}\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle &= \Sigma(\mathcal{F}^{xy})\\ &= \begin{pmatrix}\text{Sum of all connected diagrams}\\ \text{with two external points}\end{pmatrix}\end{aligned}$$

11. Generalization to *n-point correlators:*

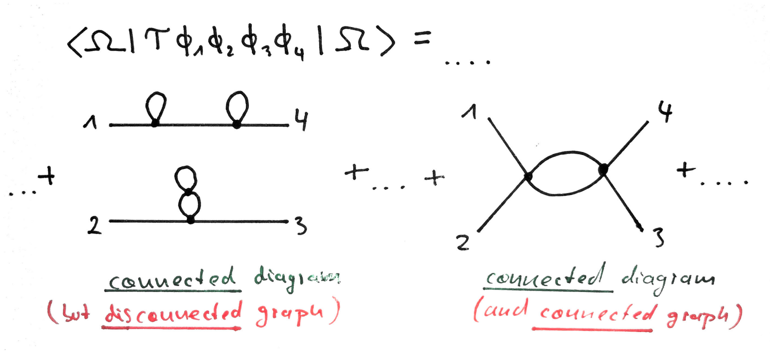
$$\begin{aligned}\langle\Omega|\mathcal{T}\phi(x_1)\dots\phi(x_n)|\Omega\rangle &= \Sigma(\mathcal{F}^{x_1\dots x_n})\\ &= \begin{pmatrix}\text{Sum of all connected diagrams}\\ \text{with } n \text{ external points}\end{pmatrix}\end{aligned}$$

In ϕ^4-theory, correlators with n odd vanish identically as the set $\mathcal{F}^{x_1\dots x_n}$ of allowed connected diagrams is empty. This follows also from Wick's theorem where full contractions are only possible with an even number of fields.

↑ Notes

Note 4.1

- *Connected diagrams* are connected to external points and not necessarily connected graphs:



- *Disconnected diagrams* = "Vacuum bubbles"
- Interpretation of vacuum bubbles:
  With (4.7) and (4.6)

$$\lim_{T\to\infty(1-i\varepsilon)} \langle 0|\mathcal{T}\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\}|0\rangle$$

$$= \underbrace{\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle}_{=\Sigma(\mathcal{F}^{xy})} \times \lim_{T\to\infty(1-i\varepsilon)}\left(|\langle 0|\Omega\rangle|^2 \underbrace{e^{-iE_0(2T)}}_{\propto\exp[\Sigma(\mathcal{V})]}\right)$$

With $V_i = \tilde{V}_i \cdot (2T\cdot V)$:

$$\frac{E_0}{V} = i\sum_j \tilde{V}_j \quad \text{(independent of } T\text{)}$$

V: Volume of space
→ total vacuum energy $E_0 \propto V$ (good!)
→ Vacuum bubbles determine *vacuum energy density*

Problem Set 6

(due **22.05.2020**)

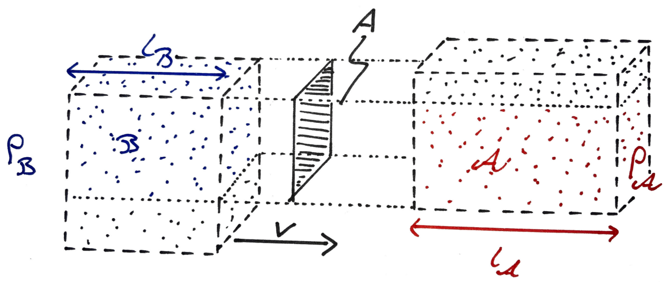
1. Application of Feynman diagrams for ϕ^4-theory
2. Derivation of Feynman rules for the complex Klein-Gordon field with arbitrary interaction potential

4.5 Cross Sections and the S-Matrix

- So far: Time-ordered correlators $\langle\Omega|\mathcal{T}\phi_1\dots\phi_n|\Omega\rangle \to$ *cannot* be directly measured
- Now: *Cross sections*
  (can be measured with scattering experiments in particle accelerators)
- Recipe: Feynman diagrams $\to$ S-matrix $\to$ Cross section
- This lecture: Define cross section and S-matrix, compute cross section from S-matrix

The Cross Section

1. $\sphericalangle$ *Scattering experiment:*
   Collide two beams of particles with well-defined momenta and observe the outcome:



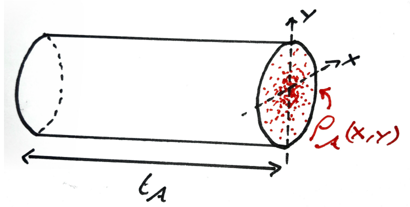
2. *Cross section:*

$$\sigma_{(X)} \equiv \frac{\#\text{ of scattering events (with outcome } X)}{\rho_\mathcal{A}\ell_\mathcal{A}\rho_\mathcal{B}\ell_\mathcal{B}A} \quad\Rightarrow\quad [\sigma] = L^2 = \text{Area}$$

   $\to$ encodes the likelihood of scattering event X
   $\to$ intrinsic property of the colliding particles
   In particular, the cross section is independent of the parameters of the experiment (like beam size, particle density etc.).

3. Real experiments: Densities not homogeneous across beam: $\rho_\mathcal{X} \mapsto \rho_\mathcal{X}(x,y)$

↑ Notes



If interaction range and wavepacket size are much smaller than the beam diameter, the densities can be taken as (locally) constant and the following derivations apply. The only difference is:

$$\begin{aligned}&\#\text{ of scattering events (with outcome } X)\\ &= \sigma_{(X)}\ell_{\mathcal{A}}\ell_{\mathcal{B}}\int_{\text{Beam cross section}} dx\, dy\, \rho_{\mathcal{A}}(x,y)\rho_{\mathcal{B}}(x,y)\end{aligned}$$

homogeneous beam: $\rho_{\mathcal{X}} = \text{const}$

$$= \frac{\sigma_{(X)}N_{\mathcal{A}}N_{\mathcal{B}}}{A}$$

$N_{\mathcal{X}}$: # of particles of type $\mathcal{X}$ in the interaction volume $\ell_{\mathcal{X}} \cdot A$

4. Typically there are many outcomes X possible, e.g.

$$e^+e^- \to \begin{cases} e^+e^- \\ \mu^+\mu^- & (\mu^-\text{: muon}) \\ \mu^+\mu^-\gamma & (\gamma\text{: photon}) \\ \dots \end{cases}$$

The possible outcomes depend on the field content of the theory and the interactions that couple them.

5. *Differential cross section:*
$\sphericalangle$ Scattering outcome X of n final particles with momenta $(\vec{p}_1, \dots, \vec{p}_n) \in \mathcal{V}_p$
$\mathcal{V}_p \subseteq \mathbb{R}^{3n}$: volume of final-state 3-momentum space $\mathbb{R}^{3n}$

$$\sigma_{X|\mathcal{V}_p} = \int_{\mathcal{V}_p} d^3p_1 \dots d^3p_n \underbrace{\frac{d\sigma}{d^3p_1 \dots d^3p_n}}_{\text{differential cross section}}$$

→ constrained by 4-momentum conservation: $\sum_i p_i = \text{const}$
(this follows from spacetime translation symmetry; there are 4 independent constraints)

Special case: $n = 2$
→ 6 DOF (degrees of freedom) $(\vec{p}_1, \vec{p}_2)$ and 4 constraints
→ 2 DOF left: scattering direction (ϕ, θ) in center-of-mass frame:

$$\frac{d\sigma}{d^3p_1\, d^3p_2} \to \frac{d\sigma}{d\Omega}$$

Here we skip another measurable quantity: the *decay rate*

$$\Gamma_{(X)} \equiv \frac{\#\text{ of decays per unit time (into state } X)}{\#\text{ of particles } \mathcal{A} \text{ present}}$$

↑ Notes

In scattering experiments, the decay of unstable intermediate particles modifies the scattering cross section according to the *Breit-Wigner* formula (a Lorentzian distribution)

$$\sigma \propto \frac{1}{(E^2 - m^2)^2 + m^2\Gamma^2}$$

with m the rest mass of the unstable intermediate particle and E the center-of-mass energy of the collision (this is called a *resonance*).

The S-Matrix

Goal: compute cross sections
Recipe: start with initial states → evolve in time → compute overlap with final states
Note: Henceforth we consider the scattering of *two* particles $\mathcal{A}$ and $\mathcal{B}$

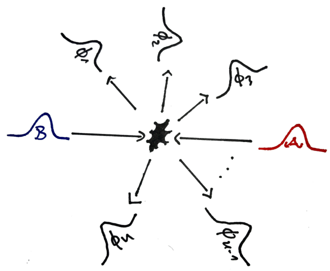
1. $\sphericalangle$ One-particle wavepacket

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \phi(\vec{k})|\vec{k}\rangle \quad \text{with} \quad \int \frac{d^3k}{(2\pi)^3} |\phi(\vec{k})|^2 = 1 = \langle\phi|\phi\rangle$$

$|\vec{k}\rangle$: one-particle state of *interacting* theory ($= \sqrt{2E_{\vec{k}}}a^\dagger_{\vec{k}}|0\rangle$ for free theory)

2. We want the probability

$$P = \underbrace{\left|{}_{\text{out}}\langle\phi_1 \dots \phi_n|\phi_{\mathcal{A}}\phi_{\mathcal{B}}\rangle_{\text{in}}\right|^2}_{\text{formal expression (definition: see below)}} \quad \widehat{=}$$



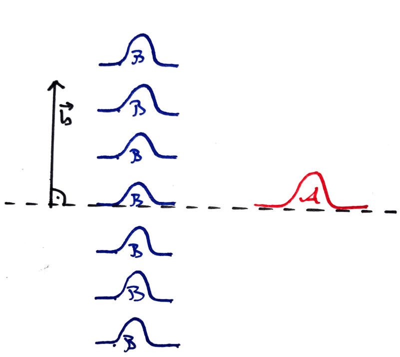
$$\begin{aligned} |\phi_{\mathcal{A}}\phi_{\mathcal{B}}\rangle_{\text{in}} &: \quad \textit{in}\text{-state} \quad \text{at } T \to -\infty \text{ of two separated wavepackets} \\ |\phi_1 \dots \phi_n\rangle_{\text{out}} &: \quad \textit{out}\text{-state} \quad \text{at } T \to +\infty \text{ of } n \text{ separated wavepackets} \end{aligned}$$

3. Fourier transform in-states (w.l.o.g.):

$$|\phi_{\mathcal{A}}\phi_{\mathcal{B}}(\vec{b})\rangle_{\text{in}} = \int \frac{d^3k_{\mathcal{A}}}{(2\pi)^3} \int \frac{d^3k_{\mathcal{B}}}{(2\pi)^3} \frac{\phi_{\mathcal{A}}(\vec{k}_{\mathcal{A}})\phi_{\mathcal{B}}(\vec{k}_{\mathcal{B}})e^{-i\vec{b}\vec{k}_{\mathcal{B}}}}{\sqrt{(2E_{\vec{k}_{\mathcal{A}}})(2E_{\vec{k}_{\mathcal{B}}})}} |\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle \tag{4.9}$$

$\vec{b}$: impact parameter
By convention, all wavefunctions constructed from $\phi_{\mathcal{B}}(\vec{k})$ are collinear; shifts by $\vec{b}$ perpendicular to the axis of incidence are then realized by $e^{-i\vec{b}\vec{k}}$:

↑ Notes



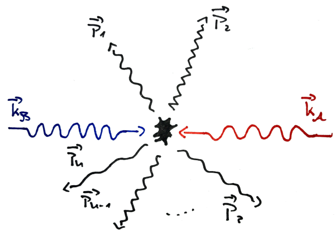
4. Simplification:

$$|\phi_1 \dots \phi_n\rangle_{\text{out}} \quad \to \quad |\vec{p}_1 \dots \vec{p}_n\rangle_{\text{out}}$$

This simplification can be justified by the characteristics of particle detectors which predominantly measure the *momentum* of scattered particles and cannot resolve positions on the scale of de Broglie wavelengths.

With (4.9), we are interested in

$${}_{\text{out}}\langle \vec{p}_1 \dots \vec{p}_n | \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \rangle_{\text{in}} \quad \hat{=}$$



5. *S-matrix:*

$$\begin{aligned} {}_{\text{out}}\langle \vec{p}_1 \dots \vec{p}_n | \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \rangle_{\text{in}} &:= \lim_{T\to\infty} \langle \vec{p}_1 \dots \vec{p}_n | \underbrace{e^{-iH(2T)}}_{e^{-iH(T-(-T))}} | \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \rangle \\ &\equiv \langle \vec{p}_1 \dots \vec{p}_n | S | \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \rangle \end{aligned}$$

Example: $S = \mathbb{1}$ for free theory

$|\vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}}\rangle$ and $|\vec{p}_1 \dots \vec{p}_n\rangle$ are momentum eigenstates of 2 respectively n particles in the asymptotic Hilbert spaces (which are Fock spaces). We stress that one-particle asymptotic states $|\vec{k}_{\mathcal{A}}\rangle$ are eigenstates of the interacting Hamiltonian H, but multi-particle states like $|\vec{k}_{\mathcal{A}}, \vec{k}_{\mathcal{B}}\rangle$ are *not* due to the interactions in H (this explains why they are not stationary and can evolve into asymptotic out-states that are different from the asymptotic in-state).

6. *T-matrix:*

$$S \equiv \underbrace{\mathbb{1}}_{\text{particles miss each other}} + \underbrace{iT}_{\text{non-trivial scattering}}$$

↑ Notes

7. 4-momentum conservation →

$$\langle \vec{p}_1 \dots \vec{p}_n | iT | \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \rangle \equiv \underbrace{(2\pi)^4 \delta^{(4)} \left(k_{\mathcal{A}} + k_{\mathcal{B}} - \textstyle\sum_f p_f \right)}_{\text{kinematics}} \times i \, \underbrace{\overbrace{\mathcal{M}(k_A k_B \mapsto \{p_f\})}^{\text{dynamics}}}_{\equiv \textit{ invariant matrix element}} \tag{4.10}$$

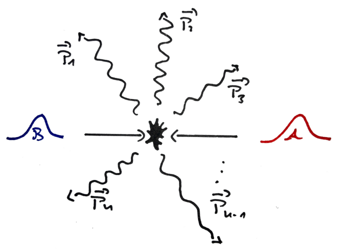
Note: here all 4-momenta are *on-shell*, i.e., $p^0 = E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$.
Invariant matrix element ≙ scattering amplitude of one-particle quantum mechanics
Two questions:

a) $\mathcal{M} = ?$ (➔ later)

b) $\sigma \overset{?}{=} \sigma(\mathcal{M})$ (➔ now)

8. ⊲ Probability to scatter in infinitesimal momentum volume $d\mathcal{V}_p = \prod_f d^3 p_f$:

$$dP\,(\mathcal{A}\mathcal{B}_{\vec{b}} \mapsto 1 \dots n) = \underbrace{\left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_{\vec{p}_f}} \right)}_{\text{for normalization}} |{}_{\text{out}}\langle \vec{p}_1 \dots \vec{p}_n | \phi_{\mathcal{A}} \phi_{\mathcal{B}}(\vec{b}) \rangle_{\text{in}}|^2$$

≙ 

Here we assume that amplitudes for different momenta do *not* interfere as the particle detector measures momentum distributions: $\left| \int_{\mathcal{V}} dp \, \langle p | \phi \rangle \right|^2 \approx \int_{\mathcal{V}} dp \, |\langle p | \phi \rangle|^2$.

9. ⊲ Single target particle $\mathcal{A}$ and many incident particles $\mathcal{B}_{\vec{b}}$:

$$d(\#\,\text{scattering events}) = \int d^2 b \, n_{\mathcal{B}} \, dP\,(\mathcal{A}\mathcal{B}_{\vec{b}} \mapsto 1 \dots n)$$

$n_{\mathcal{B}}$: Area density of $\mathcal{B}$-particles
By assumption, $n_{\mathcal{B}} \approx$ const on the interaction length scale l_0 (i.e., $dP \approx 0$ for $|\vec{b}| \gg l_0$)

↑ Notes

→

$$d\sigma = \frac{d(\#\,\text{scattering events})}{\underbrace{n_{\mathcal{B}}}_{\rho_{\mathcal{B}}\ell_{\mathcal{B}}}\;\underbrace{N_{\mathcal{A}}}_{\rho_{\mathcal{A}}\ell_{\mathcal{A}}A}} = \frac{\square}{n_{\mathcal{B}}\cdot 1} = \int d^2b\; dP\,(\mathcal{A}\mathcal{B}_{\vec{b}} \mapsto 1\dots n)$$

insert everything

$$= \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_{\vec{p}_f}}\right)\int d^2b\, \prod_{i=\mathcal{A},\mathcal{B}}\left(\int \frac{d^3k_i}{(2\pi)^3}\frac{\phi_i(\vec{k}_i)}{\sqrt{2E_{\vec{k}_i}}}\int \frac{d^3q_i}{(2\pi)^3}\frac{\phi_i^*(\vec{q}_i)}{\sqrt{2E_{\vec{q}_i}}}\right)$$

$$\times \underbrace{\frac{e^{i\vec{b}(\vec{q}_{\mathcal{B}}-\vec{k}_{\mathcal{B}})}}{}}_{(2\pi)^2\delta^{(2)}(k_{\mathcal{B}}^{\perp}-q_{\mathcal{B}}^{\perp})}\quad \underbrace{\left({}_{\text{out}}\langle\{\vec{p}_f\}|\{\vec{k}_i\}\rangle_{\text{in}}\right)}_{\substack{i\mathcal{M}(\{k_i\}\mapsto\{p_f\})\\ \times(2\pi)^4\delta^{(4)}\left(\sum k_i-\sum p_f\right)}}\quad \underbrace{\left({}_{\text{out}}\langle\{\vec{p}_f\}|\{\vec{q}_i\}\rangle_{\text{in}}\right)^*}_{\substack{-i\mathcal{M}^*(\{q_i\}\mapsto\{p_f\})\\ \times(2\pi)^4\delta^{(4)}\left(\sum q_i-\sum p_f\right)}}$$

For the matrix elements, we ignored the identity $\mathbb{1}$ in the S-matrix as we are only interested in non-trivial scattering events given by the T-matrix.

→ Evaluate the six q_i-integrals:

Only the two q_i^z-integrals are non-trivial; note that we assume w.l.o.g that $\vec{b}\perp\vec{e}_z$:

a) $q_{\mathcal{B}}^{\perp} = (q_{\mathcal{B}}^x, q_{\mathcal{B}}^y)$-integrals $\Rightarrow q_{\mathcal{B}}^{\perp} = k_{\mathcal{B}}^{\perp}$ (this follows from $\delta^{(2)}\,(k_{\mathcal{B}}^{\perp}-q_{\mathcal{B}}^{\perp})$)

b) $q_{\mathcal{A}}^{\perp} = (q_{\mathcal{A}}^x, q_{\mathcal{A}}^y)$-integrals $\Rightarrow q_{\mathcal{A}}^{\perp} = k_{\mathcal{A}}^{\perp}$
(this follows from a) in combination with the remaining two δ-functions)

c) $\sphericalangle\; q_{\mathcal{A}}^z q_{\mathcal{B}}^z$-integrals:

$$\int dq_{\mathcal{A}}^z\, dq_{\mathcal{B}}^z\, \delta\left(q_{\mathcal{A}}^z + q_{\mathcal{B}}^z - \sum p_f^z\right)\delta\left(E_{\mathcal{A}}+E_{\mathcal{B}}-\sum E_f\right)$$

$$= \int dq_{\mathcal{A}}^z\;\delta\left(\underbrace{\sqrt{q_{\mathcal{A}}^2+m_{\mathcal{A}}^2}+\sqrt{q_{\mathcal{B}}^2+m_{\mathcal{B}}^2}-\sum E_f}_{\equiv g(q_{\mathcal{A}}^z)}\right)\Bigg|_{q_{\mathcal{B}}^z=\sum p_f^z - q_{\mathcal{A}}^z}$$

$$= \frac{1}{|g'(g_{\mathcal{A}}^z)|}\Bigg|_{g(q_{\mathcal{A}}^z)=0}$$

$$\hat{=}\; \frac{1}{\left|\frac{q_{\mathcal{A}}^z}{E_{\mathcal{A}}}-\frac{q_{\mathcal{B}}^z}{E_{\mathcal{B}}}\right|} \equiv \frac{1}{|v_{\mathcal{A}}-v_{\mathcal{B}}|}$$

where a), b), and $q_{\mathcal{B}}^z = \sum p_f^z - q_{\mathcal{A}}^z$ are implied; $q_{\mathcal{A}}^z$ is a solution of $g(q_{\mathcal{A}}^z)=0 \Leftrightarrow E_{\mathcal{A}}+E_{\mathcal{B}} = \sum E_f$.

$v_{\mathcal{X}}$ is the velocity of particle $\mathcal{X}$ in the lab frame; recall: $\vec{v}_{\text{group}} = \frac{\partial E(\vec{q})}{\partial\vec{q}} = \frac{\vec{q}}{E(\vec{q})}$ for relativistic particles.

These calculations are sloppy and lack mathematical rigour. Can this be improved?

10. $\sphericalangle\;\phi_i(\vec{k}_i)$ peaked around $\vec{p}_i$ for $i=\mathcal{A},\mathcal{B}$ →

↑ Notes

(pull all continuous functions of $\vec{k}_i$ out of the integrals)

$$d\sigma = \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_{\vec{p}_f}}\right)\frac{|\mathcal{M}(p_\mathcal{A}p_\mathcal{B}\mapsto\{p_f\})|^2}{2E_{\vec{p}_\mathcal{A}}2E_{\vec{p}_\mathcal{B}}|v_\mathcal{A}-v_\mathcal{B}|}$$
$$\times\int\frac{d^3k_\mathcal{A}}{(2\pi)^3}\int\frac{d^3k_\mathcal{B}}{(2\pi)^3}|\phi_\mathcal{A}(\vec{k}_\mathcal{A})|^2\,|\phi_\mathcal{B}(\vec{k}_\mathcal{B})|^2(2\pi)^4\delta^{(4)}\left(k_\mathcal{A}+k_\mathcal{B}-\sum p_f\right)$$

11. Particle detectors project onto momentum eigenstates with *finite* resolution
→ cannot resolve momentum spread of initial wavepackets
→ $k_\mathcal{A}+k_\mathcal{B}\approx p_A+p_B$

$$d\sigma = \frac{1}{2E_{\vec{p}_\mathcal{A}}2E_{\vec{p}_\mathcal{B}}|v_\mathcal{A}-v_\mathcal{B}|}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_{\vec{p}_f}}\right)$$
$$\times|\mathcal{M}(p_\mathcal{A}p_\mathcal{B}\mapsto\{p_f\})|^2\,(2\pi)^4\delta^{(4)}\left(p_\mathcal{A}+p_\mathcal{B}-\sum p_f\right) \tag{4.11}$$

Here we used the normalization of the initial wavepackets.
Note that this result is *independent* on the shape of the initial wavepackets!
For $|\phi_i(\vec{k}_i)|^2\sim\delta^{(3)}\left(\vec{k}_i-\vec{p}_i\right)$ this approximation becomes exact.
Note that $\int d\sigma$ is *not* Lorentz invariant because the prefactor

$$\frac{1}{2E_{\vec{p}_\mathcal{A}}2E_{\vec{p}_\mathcal{B}}|v_\mathcal{A}-v_\mathcal{B}|}$$

is not (it transforms non-trivially under boosts perpendicular to the axis of incidence ($\vec{e}_z$) because of Lorentz contraction).

However, the remaining terms *are* Lorentz invariant (Li): (1) the measure is Li as shown before, (2) the invariant matrix element is Li because T commutes with the unitary representation of Lorentz transformations on the asymptotic Hilbert space [for a proof, ➔ pp. 116–121 of Weinberg's *The Quantum Theory of Fields (Vol 1)*] (note that this requires additional assumptions since the Hamiltonian does *not* commute with the generators of boosts), and (3) the δ-distribution is Li since the equation $p_\mathcal{A}+p_\mathcal{B}=\sum p_f$ is Lorentz covariant (i.e., valid in all inertial systems).

Special Cases

12. ⊲ Two final particles (p_1 and p_2) in center-of-mass frame ($\vec{p}_\mathcal{A}+\vec{p}_\mathcal{B}=0 \Leftrightarrow \vec{p}_1=-\vec{p}_2$):

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{cm}} \overset{\circ}{=} \frac{1}{2E_{\vec{p}_\mathcal{A}}2E_{\vec{p}_\mathcal{B}}|v_\mathcal{A}-v_\mathcal{B}|}\frac{|\vec{p}_1|}{(2\pi)^2 4E_{\text{cm}}}|\mathcal{M}(p_\mathcal{A}p_\mathcal{B}\mapsto p_1p_2)|^2 \tag{4.12}$$

$E_{\text{cm}}=\sqrt{(p_\mathcal{A}+p_\mathcal{B})^2}=[E_{\vec{p}_\mathcal{A}}+E_{\vec{p}_\mathcal{B}}]_{\text{cm}}$: center-of-mass energy (Lorentz invariant!)

13. If, in addition, $m_\mathcal{A}=m_\mathcal{B}=m_1=m_2$:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{cm}} \overset{\circ}{=} \frac{|\mathcal{M}(p_\mathcal{A}p_\mathcal{B}\mapsto p_1p_2)|^2}{64\pi^2E_{\text{cm}}^2}$$

↑ Notes

4.6 Computing S-Matrix Elements from Feynman Diagrams

The main result of this section will be motivated but not rigorously derived. For the proof, a technical result known as *LSZ-reduction formula* is needed. For details, see Chapter 7.2 in Peskin & Schroeder.

Motivation

1. We want

$$\langle \vec{p}_1 \dots \vec{p}_n | S | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle = \lim_{T\to\infty} \langle \vec{p}_1 \dots \vec{p}_n | e^{-iH(2T)} | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle$$

2. *Problem:*

$$\begin{aligned} |\vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}}\rangle_0 &= \sqrt{2E_{\vec{p}_{\mathcal{A}}}}\sqrt{2E_{\vec{p}_{\mathcal{B}}}}\, a^\dagger_{\vec{p}_{\mathcal{A}}} a^\dagger_{\vec{p}_{\mathcal{B}}} |0\rangle && \text{Eigenstates of } H_0 \\ |\vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}}\rangle &= \,? && |\vec{p}\rangle \text{ Eigenstate of } H = H_0 + H_{\text{int}} \end{aligned}$$

   Interactions "deform" not only the vacuum $|0\rangle \mapsto |\Omega\rangle$ but also the single-particle states $|\vec{p}\rangle_0 \mapsto |\vec{p}\rangle$ in a highly non-trivial way.

3. *Remember:* For the vacuum we found

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0 T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

4. *Assume* it holds similarly

$$|\vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}}\rangle = \lim_{T\to\infty(1-i\varepsilon)} \underbrace{\blacksquare\blacksquare}_{\text{Prefactors \& Overlaps}} e^{-iHT} |\vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}}\rangle_0$$

   This construction is not easy and we deliberately omit prefactors and overlaps! Remember that in the case of vacuum expectation values, these prefactors cancelled; here the same happens in the end.

5. If this holds, we could write

$$\begin{aligned} \langle \vec{p}_1 \dots \vec{p}_n | S | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle &\propto \lim_{T\to\infty(1-i\varepsilon)} {}_0\langle \vec{p}_1 \dots \vec{p}_n | e^{-iH(2T)} | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0 \\ &\propto \lim_{T\to\infty(1-i\varepsilon)} {}_0\langle \vec{p}_1 \dots \vec{p}_n | \mathcal{T} \exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0 \end{aligned}$$

   In the first line, we used that $\left[e^{-iHT(1-i\varepsilon)}\right]^\dagger e^{-iH2T} e^{-iHT(1-i\varepsilon)} = e^{-iH[2T(1-i\varepsilon)]}$.
   In the last line, we used that

$$U(T,-T) \stackrel{\text{def}}{=} \mathcal{T} \exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] \mathrel{\hat{=}} e^{iH_0(T-t_0)} e^{-iH(2T)} e^{-iH_0(-T-t_0)}$$

   and that $|\vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}}\rangle_0$ and ${}_0\langle \vec{p}_1 \dots \vec{p}_n|$ are eignestates of H_0.

↑ Notes

6. *Correct result*:

$$\langle \vec{p}_1 \dots \vec{p}_n | iT | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle = \lim_{T\to\infty(1-i\varepsilon)} \left\{ {}_0\langle \vec{p}_1 \dots \vec{p}_n | \mathcal{T} \exp\left[-i \int_{-T}^{T} dt\, H_I(t) \right] | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0 \right\}_{\text{fc\&a}} \tag{4.13}$$

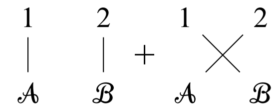
fc&a = "fully connected and amputated"
= restriction on Feynman diagrams that contribute to this amplitude (see below)

Interpretation & Application

Here: ϕ^4-theory
Details: ➔ Problemset 7

1. λ^0-order:

$$\begin{aligned} {}_0\langle \vec{p}_1 \vec{p}_2 | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0 &= \sqrt{2E_{\vec{p}_1} 2E_{\vec{p}_2} 2E_{\vec{p}_{\mathcal{A}}} 2E_{\vec{p}_{\mathcal{B}}}}\, \langle 0 | a_{\vec{p}_1} a_{\vec{p}_2} a^\dagger_{\vec{p}_{\mathcal{A}}} a^\dagger_{\vec{p}_{\mathcal{B}}} | 0 \rangle \\ &\mathrel{\hat{=}} 2E_{\vec{p}_{\mathcal{A}}} 2E_{\vec{p}_{\mathcal{B}}} (2\pi)^6 \left\{ \begin{matrix} \delta^{(3)}(\vec{p}_{\mathcal{A}} - \vec{p}_1)\, \delta^{(3)}(\vec{p}_{\mathcal{B}} - \vec{p}_2) \\ +\delta^{(3)}(\vec{p}_{\mathcal{A}} - \vec{p}_2)\, \delta^{(3)}(\vec{p}_{\mathcal{B}} - \vec{p}_1) \end{matrix} \right\} \end{aligned}$$

=  (4.14)

→ state does not change (Bosons!)
→ contributes to 1 in $S = 1 + iT$ (→ not part of fc&a diagrams)
Note that there is only one particle type in ϕ^4 theory, so all particles have naturally the same mass.

2. λ^1-order:

   a)

$${}_0\langle \vec{p}_1 \vec{p}_2 | \left(-i \frac{\lambda}{4!} \int d^4x\, \mathcal{T}\{\phi_I^4(x)\} \right) | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0$$

Wick's theorem

$$= {}_0\langle \vec{p}_1 \vec{p}_2 | \left(-i \frac{\lambda}{4!} \int d^4x : \phi_I^4(x) + \text{contractions} : \right) | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0 \tag{4.15}$$

   b) *Careful:* Not only full contractions survive because (the states contain particles)

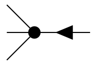
$$\phi_I^+(x) | \vec{p} \rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} a_{\vec{k}} e^{-ikx} \sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}} |0\rangle \mathrel{\hat{=}} e^{-ipx} |0\rangle$$

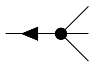
$${}_0\langle \vec{p} | \phi_I^-(x) = \langle 0 | \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \sqrt{2E_{\vec{p}}}\, a_{\vec{p}} a^\dagger_{\vec{k}} e^{+ikx} \mathrel{\hat{=}} \langle 0 | e^{+ipx}$$

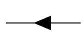
Recall that not fully contracted, normal-ordered products contain ϕ_I^+ fields on the right and ϕ_I^- fields on the left.

↑ Notes

c) *Definition:*

$$\phi_I(x)|\vec{p}\rangle \equiv e^{-ipx}|0\rangle \;\hat{=}$$  p

$$\langle\vec{p}|\phi_I(x) \equiv \langle 0|e^{+ipx} \;\hat{=}\; p$$ 

$$\langle\vec{p}|\vec{q}\rangle \equiv 2E_{\vec{p}}\,(2\pi)^3\delta^{(3)}\,(\vec{p}-\vec{q}) \;\hat{=}\; q$$  p

We omit the subscript $_0$ for states whenever it is implied by the context to lighten our notation.

Feynman diagrams for S-matrix elements contain *external lines* (labeled by momenta) instead of *external points* (labeled by positions) as compared to the diagrams for correlation functions.

d) Then

$${}_0\langle\vec{p}_1\ldots|\mathcal{T}\{\phi_a\ldots\}|\vec{p}_\mathcal{A}\ldots\rangle_0 = \left\{\begin{array}{l}\text{Sum of all } \textit{full} \text{ contractions of}\\ \text{fields } \textit{and} \text{ external-state momenta}\end{array}\right\}$$

This is a generalization of Wick's theorem for states with external momenta.

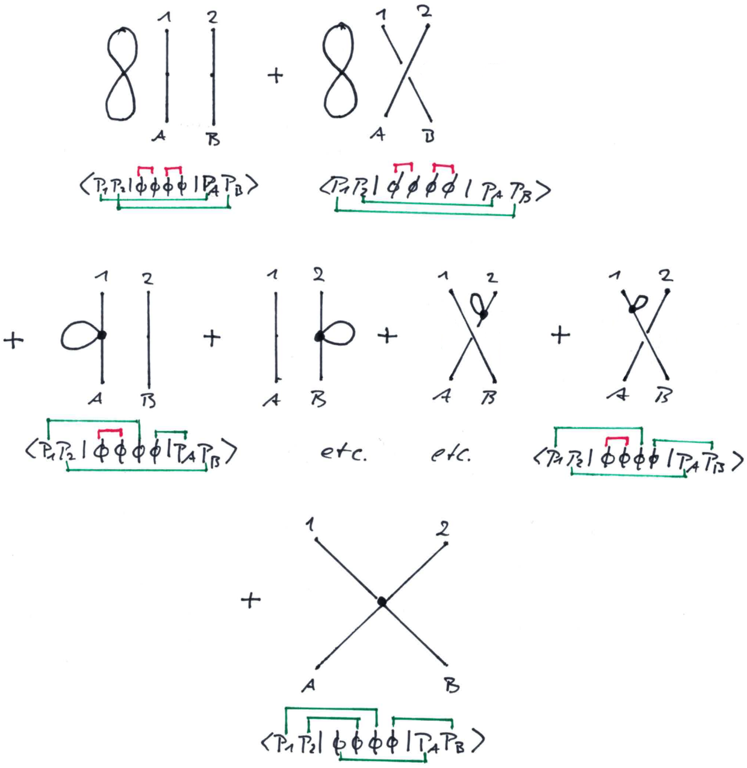
Example:

$$\begin{aligned}{}_0\langle\vec{p}_1\vec{p}_2|\vec{p}_\mathcal{A}\vec{p}_\mathcal{B}\rangle_0 &= \langle\vec{p}_1\vec{p}_2|\vec{p}_\mathcal{A}\vec{p}_\mathcal{B}\rangle + \langle\vec{p}_1\vec{p}_2|\vec{p}_\mathcal{A}\vec{p}_\mathcal{B}\rangle\\ &= (4.14)\end{aligned}$$

e) Application to (4.15):

$$-i\frac{\lambda}{4!}\int d^4x\;{}_0\langle\vec{p}_1\vec{p}_2|\mathcal{T}\{\phi_I^4(x)\}|\vec{p}_\mathcal{A}\vec{p}_\mathcal{B}\rangle_0 = \ldots$$

↑ Notes



→ Terms with $\phi\phi\phi\phi$ and $\phi\phi\phi\phi$ (red) do *not* contribute to T

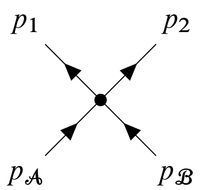
→ Only *fully connected* diagrams contribute to T

fully connected = all external lines are connected to each other

The integral in (4.15) yields a momentum-conserving δ-distribution at the vertices.

f) Therefore

$$\langle \vec p_1 \vec p_2 | iT | \vec p_{\mathcal{A}} \vec p_{\mathcal{B}} \rangle \approx$$



$$= (4!) \cdot \left(-i\frac{\lambda}{4!}\right) \int d^4x \, e^{-i(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)x}$$

$$= -i\lambda(2\pi)^4\delta^{(4)}(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2) \tag{4.16}$$

$$\overset{\text{def}}{=} i\mathcal{M}(2\pi)^4\delta^{(4)}(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)$$

→ $\mathcal{M}(p_{\mathcal{A}}p_{\mathcal{B}} \mapsto p_1 p_2) = -\lambda + \mathcal{O}(\lambda^2)$

The factor 4! comes from the 4! possibilities to contract the four external momenta with the four fields (above we show only one of these contractions exemplarily).

→ (➔ Problemset 7)

$$\sigma_{\text{total}} = \frac{\lambda^2}{32\pi E_{\text{cm}}^2}$$

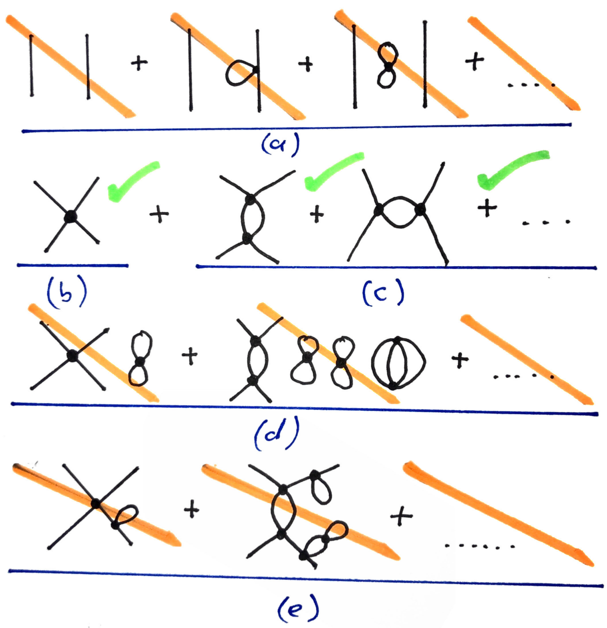
By measuring σ_{total} in a particle collider, one can determine the coupling constant λ.

↑ Notes

Note the factor $1/2$ as the two final particles are indistinguishable (that is, the final states $\langle \vec{p}_1 \vec{p}_2|$ and $\langle \vec{p}_2 \vec{p}_1|$ are physically equivalent and must not be counted twice)!

3. *Higher-order* contributions:

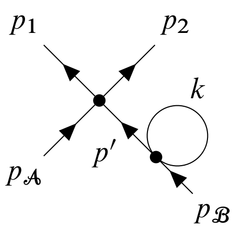
$$\langle \vec{p}_1 \vec{p}_2 | iT | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle =$$



fc&a

Which of these diagrams qualify as "fc&a"?

a) Not fully connected (see above) ✗
b) λ^1-order contribution (see above) ✓
c) Higher-order contributions ✓
d) Diagrams with vacuum bubbles
→ bubbles exponentiate & drop out (as before) ✗
e) Fully connected diagrams with "appendices to external legs" ✗/✓?



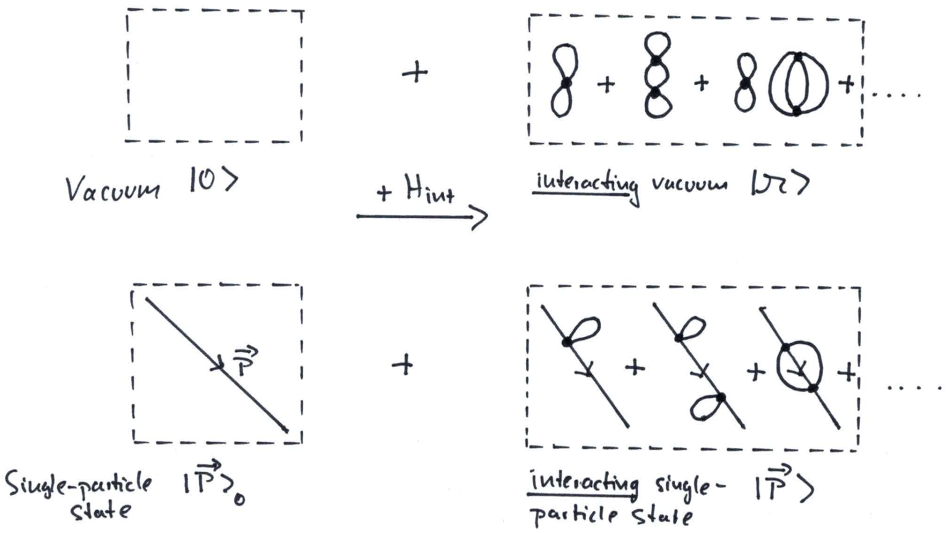
$$\hat{=} \frac{1}{2} \int \frac{d^4p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2}$$
$$\times (-i\lambda)(2\pi)^4 \delta^{(4)} \left(p_{\mathcal{A}} + p' - p_1 - p_2\right)$$
$$\times (-i\lambda)(2\pi)^4 \delta^{(4)} \left(p_{\mathcal{B}} - p'\right)$$
$$\sim \frac{1}{p_{\mathcal{B}}^2 - m^2} = \frac{1}{E_{\vec{p}_{\mathcal{B}}}^2 - \vec{p}_{\mathcal{B}}^2 - m^2} = \frac{1}{0} = \infty$$

The two momentum integrals come from the two propagators after integrating out the vertex positions. The prefactor $1/2$ is the symmetry factor of the loop.
Note that the external momenta are *on-shell*, $p^2 = m^2$, whereas the momentum integrals of internal momenta go over *off-shell* momenta, $p^2 \neq m^2$, as well.

↑ Notes

→ (4.13) makes only sense without these diagrams!

Interpretation:

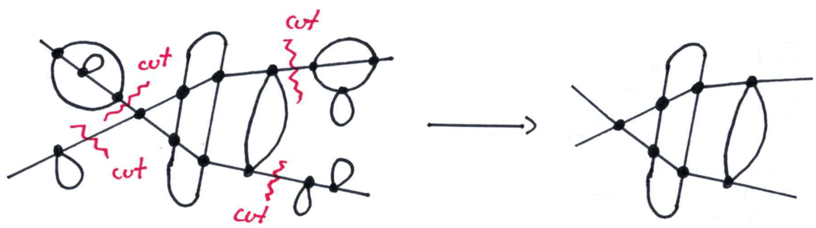


→ Not related to scattering! → ✗

→ "Amputate" legs for calculation of $\mathcal{M}$

4. *Amputation of diagrams:*
   Starting from the tip of each external leg, cut at the last point at which the diagram can be cut by removing a *single* propagator, such that this operation separates the leg from the rest of the diagram.

   Example:

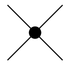
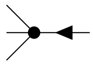


5. →

$$(4.13) = i\mathcal{M}\cdot(2\pi)^4\delta^{(4)}\left(p_{\mathcal{A}}+p_{\mathcal{B}}-\sum p_f\right)$$

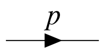
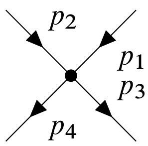
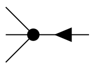
$$= \left\{\begin{array}{l}\text{Sum of all \textit{fully connected}, \textit{amputated} Feynman}\\ \text{diagrams with } p_{\mathcal{A}},\ p_{\mathcal{B}} \text{ incoming and } \{p_f\}\\ \text{outgoing}\end{array}\right\}$$

6. → *Position-space Feynman rules* for scattering amplitudes in ϕ^4-theory:

1.	For each edge,	x — y	$= D_F(x-y)$
2.	For each vertex,		$= (-i\lambda)\int d^4z$
3.	For each external line,	 p	$= e^{-ipz}$
4.	Divide by the symmetry factor,		$\frac{1}{S}\times\ldots$

Only (3) is modified as compared to Feynman rules for correlation functions.
⊲ Momentum-space representation of D_F & vertex integration
→ δ-distributions at vertices & momentum integrals over internal momenta

7. → *Momentum-space Feynman rules* for scattering amplitudes in ϕ^4-theory:

1.	For each edge,		$= \frac{i}{p^2-m^2+i\varepsilon}$
2.	For each vertex,		$= (-i\lambda)(2\pi)^4\times$ $\delta(p_1+p_2-p_3-p_4)$
3.	For each external line,	 p	$= 1$
5.	Integrate int. momenta,		$\prod_i \int \frac{d^4p_i}{(2\pi)^4}\ldots$
6.	Divide by sym. factor,		$\frac{1}{S}\times\ldots$

Only (3) is modified as compared to Feynman rules for correlation functions.

8. Because of the many δ-distributions, the expressions obtained from the momentum-space Feynman rules can be simplified considerably. On pp. 114–115 of P&S this is mentioned and, after canceling the global momentum conservation, a set of Feynman rules where only integrals over "undetermined loop momenta" are left is given. This prescription is

↑ Notes

rather obscure as they do not define what and how many of these "loop momenta" there are.

So let us think about this more carefully:

a) Consider a fully connected, amputated Feynman diagram with N_e external momenta, N_i internal momenta, and N_v vertices.

b) We can interpret the Feynman diagram as a connected graph (in the sense of graph theory) with $E = N_i + N_e$ edges and $V = N_v + N_e$ vertices (these are now "graph theory vertices", i.e., external legs terminate at vertices).

c) By variable substitutions, the N_v δ-distributions can be rewritten as follows:

$$\underbrace{\delta(\dots)\cdots\delta(\dots)}_{N_v} = \delta(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum p_f)\cdot\underbrace{\delta(\dots)\cdots\delta(\dots)}_{N_v-1}$$

(The argument of the global δ-distribution is just the sum of all N_v arguments of the original δ-distributions at vertices.) Note that the global momentum conservation cannot be used to remove a momentum integral; but it can be cancelled with the same expression in (4.10) so that the remaining expression equals $i\mathcal{M}(p_A p_B \mapsto \{p_f\})$.

d) This remaining expression has N_i momentum integrals but only $N_v - 1$ δ-distributions, so that

$$\#(\text{Loop integrals}) = N_i - N_v + 1$$

integrals remain after integrating over all δ-distributions.

e) To see why these are integrals over "loop momenta", we have to put on our graph theory goggles again: For a given (connected) graph, the set of all closed circuits (= loops = Eulerian subgraphs) forms a binary vector space (adding two loops is done modulo-2 on the edges), the so called *cycle space* $\mathcal{C}$. It is well-known that the dimension of this space (= the number of basis-loops) is given by

$$\dim\mathcal{C} = E - V + 1 = N_i - N_v + 1\,.$$

This suggests, that for each basis-loop of a given Feynman diagram, there is one undetermined "loop momentum" to integrate over. Add more details?

↑ Notes

Problem Set 7

(due **29.05.2020**)

1. Application of Feynman diagrams: Cross section of two scattering particles within ϕ^4-theory
2. Toolkit for scattering amplitudes in QED: Important relations for gamma matrices, including trace- and contraction identities

4.7 Feynman Rules for Quantum Electrodynamics

Setting the Stage

Here we leave ϕ^4-theory and switch to fermionic fields.
However, we will use and generalize the results on interactions derived for ϕ^4-theory for this new theory without detailed derivations (as these are very technical).

1. *Fields:*

$$\begin{array}{rll} \text{Fermions:} & \Psi(x) & \text{(bispinor field)} \\ \text{Photons:} & A_\mu(x) & \text{(vector field)} \end{array}$$

2. *Lagrangian:*

$$\begin{aligned} \mathcal{L}_{\text{QED}} &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{int}} \\ &= \overline{\Psi}(i\partial\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{e\overline{\Psi}\gamma^\mu\Psi}_{j^\mu}\, A_\mu \\ &= \overline{\Psi}(i D\!\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned}$$

m: mass of fermions;
e: charge of fermions (=coupling constant);
D_μ: *covariant derivative:* $D_\mu = \partial_\mu + ieA_\mu$
The replacement $\partial \mapsto D$ is called *minimal coupling* and constitutes a general recipe for coupling gauge fields to matter fields in a gauge-invariant way.

3. *Hamiltonian:*

$$\begin{aligned} H_{\text{QED}} &= H_{\text{Dirac}} + H_{\text{Maxwell}} + H_{\text{int}} \\ \text{with} \quad H_{\text{int}} &= e\int d^3x\, \overline{\Psi}\gamma^\mu\Psi\, A_\mu \end{aligned}$$

4. *Equations of motion:*

$$\begin{aligned} (iD\!\!\!\!/ - m)\Psi &= 0 \qquad \text{(gauge-covariant Dirac equation)} \\ \partial_\nu F^{\nu\mu} &= j^\mu \qquad \text{(inhomogeneous Maxwell equations)} \end{aligned}$$

↑ Notes

Note 4.2

$\mathcal{L}_{\text{QED}}$ is invariant under U(1) gauge transformations,

$$\Psi'(x) = e^{ie\alpha(x)}\Psi(x)$$
$$A'_\mu(x) = A_\mu(x) - \partial_\mu\alpha(x)$$

for arbitrary $\alpha : \mathbb{R}^{1,3} \to \mathbb{R}$.
This is the simplest example of an (abelian) gauge theory of the *Yang-Mills* form.

Note 4.3

The QED-sector of the *standard model* includes several copies of the fermion field that all couple to the same photon field,

$$\mathcal{L}_{\text{QED}}^{\text{SM}} = \sum_f \left[\overline{\Psi}_f(i\partial\!\!\!/ - m_f)\Psi_f - \underbrace{q_f\overline{\Psi}_f\gamma^\mu\Psi_f}_{j_f^\mu}\, A_\mu \right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

with mass m_f and charge q_f of fermion type

$$f \in \{\text{Leptons, Quarks}\} = \{e,\mu,\tau,\nu_e,\nu_\mu,\nu_\tau,u,d,c,s,t,b\}\,.$$

Here we restrict our discussion to a single fermion type f (think of electrons/positrons).

The situation in the standard model is actually a lot more complicated than suggested by $\mathcal{L}_{\text{QED}}^{\text{SM}}$ due to gauge symmetry constraints that forbid mass terms (a situation that is compensated by the Higgs mechanism) and electroweak symmetry breaking.

Notes on the Fermion/Dirac Sector

We have already quantized the free Dirac field $\mathcal{L}_{\text{Dirac}}$ and diagonalized the non-interacting Hamiltonian H_{Dirac}!

Remember: Feynman propagator:

$$\begin{aligned} S_F^{ab}(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{i(p\!\!\!/ + m)_{ab}}{p^2 - m^2 + i\varepsilon}e^{-ip(x-y)} \\ &= \begin{cases} \langle 0|\Psi_a(x)\overline{\Psi}_b(y)|0\rangle & \text{for } x^0 > y^0 \\ -\langle 0|\overline{\Psi}_b(y)\Psi_a(x)|0\rangle & \text{for } x^0 < y^0 \end{cases} \\ &\equiv \langle 0|\mathcal{T}\Psi_a(x)\overline{\Psi}_b(y)|0\rangle \end{aligned} \tag{4.17}$$

To deal with H_{int} perturbatively, we need *Wick's theorem* for fermions:
The proofs for all that follows are very similar to the bosonic case (except for the signs).

1. *Time ordering:* (4.17) suggests for $\psi \in \{\Psi, \overline{\Psi}\}$

$$\mathcal{T}\{\psi_{\sigma 1}\dots\psi_{\sigma N}\} \equiv (-1)^{\#}\cdot\psi_1\dots\psi_N \quad \text{for} \quad x_1^0 > \dots > x_N^0$$

σ: permutation of $\{1, 2, \dots N\}$
$(-1)^{\#}$: signum of σ with # number of operator interchanges
Note that here we suppress spinor indices!

2. *Normal order:* Define for $x \in \{a^s_{\vec{p}}, b^s_{\vec{p}}, a^{s\dagger}_{\vec{p}}, b^{s\dagger}_{\vec{p}}\}$

$$:x_1 \dots x_N: \equiv (-1)^{\#} \cdot (\text{creation operators}) \times (\text{annihilation operators})$$

#: number of operator interchanges

3. *Contraction:* Define

$$\overbrace{\psi_a(x)\psi_b}(y) \equiv \mathcal{T}\{\psi_a(x)\psi_b(y)\} - :\psi_a(x)\psi_b(y):$$

Here, a and b are spinor indices!
This definition of the contraction is analogous to the bosonic case.
and show

$$\overbrace{\Psi_a(x)\overline{\Psi}_b}(y) \mathrel{\hat{=}} \begin{Bmatrix} \{\Psi^+_a(x), \overline{\Psi}^-_b(y)\} & \text{for } x^0 > y^0 \\ -\{\overline{\Psi}^+_b(y), \Psi^-_a(x)\} & \text{for } x^0 < y^0 \end{Bmatrix} \mathrel{\hat{=}} S_F^{ab}(x-y)$$

$$\overbrace{\Psi_a(x)\Psi_b}(y) \mathrel{\hat{=}} 0$$

$$\overbrace{\overline{\Psi}_a(x)\overline{\Psi}_b}(y) \mathrel{\hat{=}} 0$$

The last two contractions vanish since $\{a^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\} = 0$.

4. *Contraction & Normal order:*

$$:A\,\overbrace{\psi_a(x)\,B\,\psi_b}(y)\,C: \equiv (-1)^{\#} \cdot \overbrace{\psi_a(x)\psi_b}(y) \cdot :ABC:$$

#: number of operator interchanges (i.e., $\psi_a(x)$ with A and $\psi_b(y)$ with AB)

5. *Wick's theorem:* For $\psi \in \{\Psi, \overline{\Psi}\}$ and $a, b, \dots$ spinor indices

$$\mathcal{T}\{\psi_a(x_1)\psi_b(x_2)\dots\} = :\psi_a(x_1)\psi_b(x_2)\dots + \text{all possible contractions}:$$

Due to the adjusted definitions of time- and normal order, Wick's theorem takes the *same form* as for bosonic fields!

Notes on the Photon/Maxwell Sector

1. *Observation:* A^μ has *four* degrees of freedom but there are only *two* photon polarizations!

↑ Notes

2. *Problem:* Gauge invariance
   → Unphysical degrees of freedom
   → Fix gauge to quantize only physical degrees of freedom

3. Different solutions:
   - Coulomb gauge $\nabla\vec{A} = 0$ (not Lorentz invariant) (➔ Advanced quantum mechanics)
   - Lorenz gauge $\partial_\mu A^\mu = 0$ (Lorentz invariant)
     (Gupta-Bleuler formalism, ➔ Itzykson & Zuber, *Quantum Field Theory*, pp. 127–134)
   - Faddeev-Popov procedure (➔ Later)

4. *Motivation:*
   a) $\sphericalangle$ Lorenz gauge: $\partial_\mu A^\mu = 0 \to$ EOMs for $\mathcal{L}_{\text{Maxwell}}$: $\partial^2 A^\nu = 0$
      Each component of $A^\nu(x)$ satisfies the Klein-Gordon equation for $m = 0$.
      Recall: $\partial_\mu F^{\mu\nu} = \partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu = 0$
      Note that the Lorenz gauge does not fix the gauge freedom completely.
   b) Expand field in classical solutions:

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=0}^{3} \left[a^r_{\vec{p}} \epsilon^r_\mu(p)\, e^{-ipx} + a^{r\dagger}_{\vec{p}} \epsilon^{r\,*}_\mu(p)\, e^{ipx} \right]$$

   with $p^2 = 0 \Leftrightarrow p^0 = E_{\vec{p}} = |\vec{p}|$ (dispersion of the massless KG equation)
   ϵ^r_μ: polarization 4-vectors (Lorenz gauge $\to p^\mu \epsilon^r_\mu = p^\mu \epsilon^{r\,*}_\mu = 0$).

5. *Results:*
   a) Impose constraints on *external (physical) photons*
      This reduces the number of degrees of freedom from 4 to 2!

$$\epsilon^\mu(p) = \begin{pmatrix} 0 \\ \vec{\epsilon}(p) \end{pmatrix} \quad \text{and} \quad \vec{p}\cdot\vec{\epsilon}(p) = 0 \quad \text{(transverse polarization)}$$

   → *Two* $r, s = 1, 2$ independent bosonic modes for each momentum $\vec{p}$:

$$\left[a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}\right] = (2\pi)^3 \delta_{rs} \delta^{(3)}(\vec{p} - \vec{q}) \quad \text{and} \quad \left[a^r_{\vec{p}}, a^s_{\vec{q}}\right] = 0 = \left[a^{r\dagger}_{\vec{p}}, a^{s\dagger}_{\vec{q}}\right]$$

   b) *Propagator (Feynman gauge):*

$$\langle 0|\mathcal{T}\{A_\mu(x) A_\nu(y)\}|0\rangle \stackrel{*}{=} \int \frac{d^4q}{(2\pi)^4} \frac{-i g_{\mu\nu}}{q^2 + i\varepsilon} e^{-iq(x-y)}$$

↑ Notes

We will derive the photon propagator using path integrals at the end of this course. As each component of A^μ satisfies the KG equation, the propagator should be similar to the massless KG propagator $D_F(x-y)$. The two-point correlator is a second-rank tensor that should be invariant under Lorentz transformations (as the theory is relativistically invariant with a unitary representation of the Lorentz group on the Hilbert space), which is realized only by $-g_{\mu\nu}$ (see (6.9) below). The sign makes the space-like components $\mu = \nu = 1, 2, 3$ positive and ensures positive norm for states of the form $A_i(x)|0\rangle$. In turn, states with $A_0(x)|0\rangle$ have negative norm – but it can be shown that these states are never produced in physical processes.

Feynman Rules

1. *Expectations:*
   a) *Two* fields (Ψ_a and A_μ)
   → Two propagators
   → Two line-types:

   Fermions (with spinor indices a and b): a  b

   Photons (with 4-vector indices μ and ν): μ ν

   The arrow for fermions denotes the (negative) charge flow, not the momentum. Since for fermion fields, particles are distinct from antiparticles, the arrow cannot be neglected: It originates at a field $\overline{\Psi}$ that creates a particle (annihilates an anti-particle) and terminates at a field Ψ that annihilates a particle (creates an antiparticle).

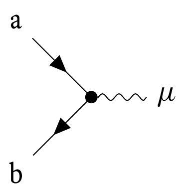
   → Two particle types: (anti-)fermions & photons
   → Two types of external states:

   Fermion/Antifermion: $|\vec{p}, s\rangle_{a/b}$ (s: Spin; a: Fermion; b: Antifermion)
   Photon: $|\vec{p}, r\rangle$ (r: Polarization)

   For each in state (ket), there is a corresponding out state (bra).

   b) Interaction with *three* fields ($H_{\text{int}} \sim \overline{\Psi}_b\, \gamma^\mu_{ba}\, \Psi_a\, A_\mu$)

   → Vertices of degree 3:



2. *Momentum-space Feynman rules* (for scattering amplitudes):
   (*Note:* In many textbooks, the colored indices are omitted.)
   The proofs are very technical but conceptually they parallel ϕ^4-theory.

   Note that there are *three* types of (graph) vertices:

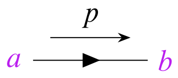
   - : internal vertex, corresponds to an interaction
   -  : external vertex, corresponds to an in- or outgoing state
   - : virtual cut of the diagram where Lorentz- or spinor-indices are summed

   Examples & Applications: ➔ next lectures

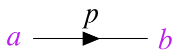
↑ Notes

Propagators

Fermions:

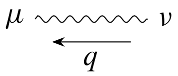


$$= \frac{i(\not{p}+m)_{ba}}{p^2-m^2+i\varepsilon} \quad \widehat{=} \quad \Psi_b(x)\overline{\Psi}_a(y)$$



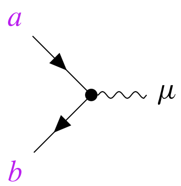
(simplified)

Photons:



$$= \frac{-ig_{\mu\nu}}{q^2+i\varepsilon} \quad \widehat{=} \quad A_\mu(x)A_\nu(y)$$

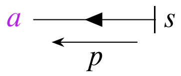
Vertices



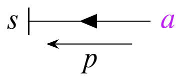
$$= -ie\gamma^\mu_{ba} \quad \widehat{=} \quad (-ie)\int d^4z\, \gamma^\mu_{ba}$$

External legs

Fermions:

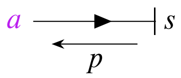


$$= u^s_a(p) \quad \widehat{=} \quad \Psi_a|\vec{p},s\rangle_a$$

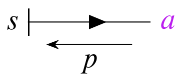


$$= \overline{u}^s_a(p) \quad \widehat{=} \quad \langle\vec{p},s|_a\overline{\Psi}_a$$

Antifermions:

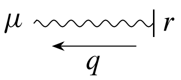


$$= \overline{v}^s_a(p) \quad \widehat{=} \quad \overline{\Psi}_a|\vec{p},s\rangle_b$$

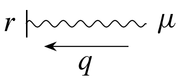


$$= v^s_a(p) \quad \widehat{=} \quad \langle\vec{p},s|_b\Psi_a$$

Photons:



$$= \epsilon^r_\mu(q) \quad \widehat{=} \quad A_\mu|\vec{q},r\rangle$$



$$= \epsilon^{r\,*}_\mu(q) \quad \widehat{=} \quad \langle\vec{q},r|A_\mu$$

Evaluation

1. Impose momentum conservation at each vertex.
2. Integrate over all undetermined momenta.
3. Compute the overall sign of the diagram.

↑ Notes

First application: The Coulomb Potential

Before we start with the computation of relativistic QED predictions in the next chapter, let us draw our first Feynman diagram and evaluate it in the non-relativistic limit to make contact with known results.

1. $\triangleleft$ Scattering process *(Møller scattering)*

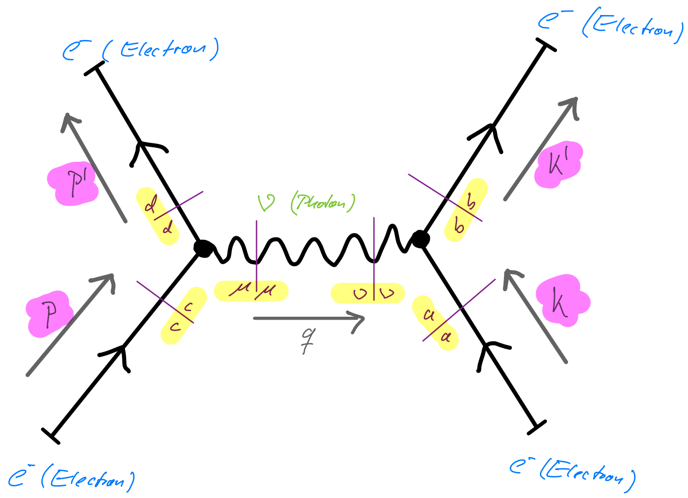
$$\text{Electron } (e^-) + \text{Electron } (e^-) \;\to\; \text{Electron } (e^-) + \text{Electron } (e^-)$$

a) Contribution to the *tree-level amplitude* (sufficient for distinguishable fermions):
"Tree-level" refers to Feynman diagrams without loops; these correspond to lowest/leading-order contributions to the scattering amplitude and do not contain integrations over undetermined momenta.

For simplicity, we omit here the spin labels s:

$$i\mathcal{M}(e^-(p)e^-(k) \to e^-(p')e^-(k'))$$

$$=$$



$$= \sigma \cdot \bar{u}_d(p')(-ie\gamma^\mu_{dc})u_c(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}_b(k')(-ie\gamma^\nu_{ba})u_a(k)$$

$$= \sigma \cdot \bar{u}(p')(-ie\gamma^\mu)u(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k')(-ie\gamma^\nu)u(k)$$

with $p - p' = q = k' - k$ (momentum conservation & integration at each vertex)
σ: sign of the diagram (see below)

Note that the order of matrix-vector chains always follows the arrows of a directed fermion path through the diagram; the different fermion paths are connected by photon lines. The terms that correspond to different fermion paths are commuting numbers indexed by as many spacetime indices as there are vertices along the path. So far we did not encounter internal fermion lines that correspond to Feynman propagators!

Typically we omit the spinor indices and imply matrix-vector products.

As electrons are *indistinguishable*, there is another tree-level diagram where the outgoing states are exchanged. This diagram has to be added with the correct sign to obtain the true tree-level scattering amplitude.

↑ Notes

b) *Nonrelativistic limit:* $|\vec{p}|^2 \ll m^2$ → keep only lowest-order terms in p
(We will discuss a full-relativistic calculation in the next chapter in detail.)

$$\rightarrow \quad u(p) = \begin{pmatrix} \sqrt{p\sigma}\xi \\ \sqrt{p\bar{\sigma}}\xi \end{pmatrix} \approx \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \quad \text{and} \quad \frac{1}{(p-p')^2} \approx \frac{-1}{|\vec{p}-\vec{p}'|^2}$$

Therefore

$$\bar{u}(p')\gamma^\mu u(p) \approx \begin{cases} 2m\xi_{p'}^\dagger \xi_p & \mu = 0 \\ 0 & \mu = 1,2,3 \end{cases} \tag{4.18}$$

and

$$i\mathcal{M} \approx \sigma \cdot \frac{-ie^2}{|\vec{p}-\vec{p}'|^2}(2m\xi_{p'}^\dagger\xi_p)(2m\xi_{k'}^\dagger\xi_k) \tag{4.19}$$

c) Compare with *nonrelativistic* scattering theory (➔ Born approximation):

$$\langle p'|iT|p\rangle \overset{*}{=} -i\hat{V}(\vec{q})(2\pi)\delta(E_{\vec{p}'}-E_{\vec{p}}) \quad (\vec{q} = \vec{p}' - \vec{p}) \tag{4.20}$$

$\hat{V}(\vec{q})$: Fourier transform of the scattering potential
→

$$\hat{V}(\vec{q}) = \sigma \cdot \frac{e^2}{|\vec{q}|^2} \quad \Rightarrow \quad V(\vec{r}) \overset{\circ}{=} \sigma \cdot \frac{e^2}{4\pi|\vec{r}|} = \sigma \cdot \frac{\alpha}{r}$$

$\alpha = e^2/4\pi \approx 1/137$: fine-structure constant (in natural units $c = \hbar = \varepsilon_0 = 1$)

The terms $(2m\xi_{p'}^\dagger\xi_p)$ etc. are due to the QFT normalization conditions and must be ignored for a sensible comparison with nonrelativistic scattering theory.

For the Fourier transform of the Coulomb potential in three dimensions, a regularization is necessary. To this end, one Fourier transforms the Yukawa potential $V(r) = \frac{e^{-mr}}{4\pi r}$ instead and sets the screening mass to zero in the end; for this integration, the residue theorem is needed.

d) *Sign of the diagram:* (here we suppress both spinor and spacetime indices)

$$_{aa}\langle \vec{p}', \vec{k}'|\,\bar{\Psi}\Psi A \bar{\Psi}\Psi A\,|\vec{p},\vec{k}\rangle_{aa}$$

$$= \langle 0|\, a_{\vec{k}'} a_{\vec{p}'} \bar{\Psi}\Psi A \bar{\Psi}\Psi A\, a_{\vec{p}}^\dagger a_{\vec{k}}^\dagger\,|0\rangle$$

→ 1+1+2=4 interchanges → $\sigma = +1$

e) → *Repulsive* Coulomb potential:

$$V_{e^-e^-}(r) = +\frac{e^2}{4\pi r}$$

→ Equal charges repel each other (As it should be!)

2. ◁ Scattering process *(Bhabha scattering)*

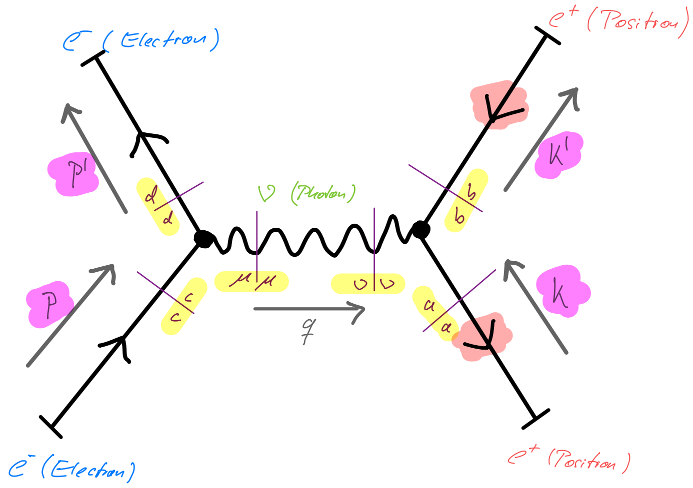
Electron (e^-) + Positron (e^+) → Electron (e^-) + Positron (e^+)

↑ Notes

a) Contribution to the *tree-level amplitude:*

$$i\mathcal{M}(e^-(p)e^+(k) \to e^-(p')e^+(k'))$$

$$=$$



$$= \sigma \cdot \bar{u}_d(p')(-ie\gamma^\mu_{dc})u_c(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{v}_b(k)(-ie\gamma^\nu_{ba})v_a(k')$$

$$= \sigma \cdot \bar{u}(p')(-ie\gamma^\mu)u(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{v}(k)(-ie\gamma^\nu)v(k')$$

with $p - p' = q = k' - k$ (Skip the spinor indices and reuse the diagram above.)

There is another tree-level contribution where an electron and a positron annihilate to a virtual photon which then decays into an electron-positron pair. The sum of both diagrams yields the tree-level scattering amplitude.

b) Nonrelativistic limit → Same result as (4.19) (with $k \leftrightarrow k'$), but what is σ?

c) *Sign of the diagram:*

$${}_{ab}\langle \vec{p}', \vec{k}' | \bar{\Psi}\Psi A \bar{\Psi}\Psi A | \vec{p}, \vec{k}\rangle_{ab}$$

$$= \langle 0 | b_{\vec{k}'} a_{\vec{p}'} \bar{\Psi}\Psi A \bar{\Psi}\Psi A a^\dagger_{\vec{p}} b^\dagger_{\vec{k}} | 0\rangle$$

$$\to \quad 2+1+2=5 \text{ interchanges} \quad \to \quad \sigma = -1$$

d) → *Attractive* Coulomb potential:

$$V_{e^+e^-}(r) = -\frac{e^2}{4\pi r}$$

→ Opposite charges attract each other (As it should be!)

These examples demonstrated four things:

- How to translate Feynman diagrams into analytical expressions.
- How to determine the sign of Feynman diagrams with fermions.

↑ Notes

- The predictions of QED seem to be reasonable!
- Signs of diagrams are important!

↑ Notes

5 Elementary Processes of Quantum Electrodynamics

In this chapter, we use the machinery developed in the last few chapters to study predictions of QED.

5.1 Cross section of $e^+e^- \to \mu^+\mu^-$ scattering

1. $\triangleleft$ Reaction

   Electron (e^-) + Positron (e^+) $\to$ Muon (μ^-) + Antimuon (μ^+)

   This process is the simplest non-trivial QED process and used to calibrate e^+e^- colliders.

2. *Note:* Both electrons and muons are spin-$\frac{1}{2}$ fermions with equal charge $q_e = q_m = e = -|e|$ but different mass $m_e \ll m_m$:
   Here, we use m to label muons since μ is already used for spacetime indices.

$$\mathcal{L}_{\text{QED}}^{e,m} = \sum_{f=e,m} \left[\overline{\Psi}_f (i\not{\partial} - m_f)\Psi_f - \underbrace{q_f \overline{\Psi}_f \gamma^\mu \Psi_f}_{j_f^\mu} A_\mu \right] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

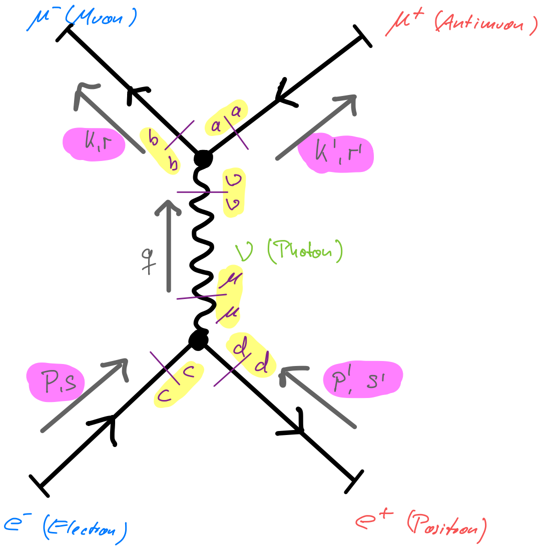
   So there is one Fermion field for electrons/positrons Ψ_e and one Fermion field for muons/antimuons Ψ_m. Mathematically, they only differ in the mass parameter m_f that enters the propagator. Note that the two Fermion fields can never couple directly but only indirectly via the photon (gauge) field A_μ!

↑ Notes

3. *Tree-level amplitude:*

$$i\mathcal{M}(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k'))$$

$=$



$$= \underbrace{(\bar{v}_e^{s'})_d(p')(-iq_e\gamma^\mu_{dc})(u_e^s)_c(p)}_{\text{Electron sector }(e)}\left(\frac{-ig_{\mu\nu}}{q^2}\right)\underbrace{(\bar{u}_m^r)_b(k)(-iq_m\gamma^\nu_{ba})(v_m^{r'})_a(k')}_{\text{Muon sector }(\mu)}$$

$$= \bar{v}_e^{s'}(p')(-iq_e\gamma^\mu)u_e^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}_m^r(k)(-iq_m\gamma^\nu)v_m^{r'}(k')$$

$$= \frac{ie^2}{q^2}\left(\bar{v}(p')\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v(k')\right)$$

with $p + p' = q = k + k'$ (momentum conservation & integration at each vertex)
Typically we omit the spinor indices and imply matrix-vector products.

In the following, we also suppress the spin superscripts and the fermion flavour subscripts.

4. We want $d\sigma \propto |\mathcal{M}|^2 \to$ need $\mathcal{M}^*$. Use $(\bar{v}\gamma^\mu u)^* \mathrel{\hat{=}} (\bar{u}\gamma^\mu v)$:

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\underbrace{\left(\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\right)}_{\square}\left(\bar{u}(k)\gamma_\mu v(k')\bar{v}(k')\gamma_\nu u(k)\right)$$

5. Typical collider setup:
   - e^+- and e^--beam unpolarized → *average* over spin polarizations of in-states
   - Muon detector cannot resolve spin → *sum* over spin polarizations of out-states

→

$$d\sigma \propto \frac{1}{4}\sum_{s,s'}\sum_{r,r'}|\mathcal{M}(s,s'\to r,r')|^2$$

6. Use spin sums (3.11) and spinor indices to evaluate □:

$$\sum_{s,s'}\bar{v}_a^{s'}(p')\gamma^\mu_{ab}u_b^s(p)\bar{u}_c^s(p)\gamma^\nu_{cd}v_d^{s'}(p') \mathrel{\hat{=}} \mathrm{Tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right]$$

Details: ➔ Problemset 8

↑ Notes

7. →

$$\frac{1}{4}\sum_{s,s',r,r'}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\operatorname{Tr}\left[(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right]\operatorname{Tr}\left[(\not{k}+m_m)\gamma_\mu(\not{k}'-m_m)\gamma_\nu\right]$$

Any squared and spin-summed QED amplitude with external fermions can be converted into a trace of products of γ-matrices.

8. *Trace technology:* (due to Feynman, for derivations ➔ Problemset 7)

Trace identities:

$$\operatorname{Tr}[\text{odd \# of } \gamma\text{'s}] = 0$$
$$\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\right] = 4g^{\mu\nu}$$
$$\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$
$$\operatorname{Tr}\left[\gamma^5\right] = 0$$
$$\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] = 0$$
$$\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right] = -4i\varepsilon^{\mu\nu\rho\sigma}$$
$$\operatorname{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\dots\right] = \operatorname{Tr}\left[\dots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right]$$

Contraction identities:

$$\gamma^\mu\gamma_\mu = 4$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$$

These identities are useful for many QED calculations!

9. →

$$\operatorname{Tr}\left[(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right] \hat{=} 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(pp'+m_e^2)\right]$$
$$\operatorname{Tr}\left[(\not{k}+m_m)\gamma_\mu(\not{k}'-m_m)\gamma_\nu\right] \hat{=} 4\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(kk'+m_m^2)\right]$$

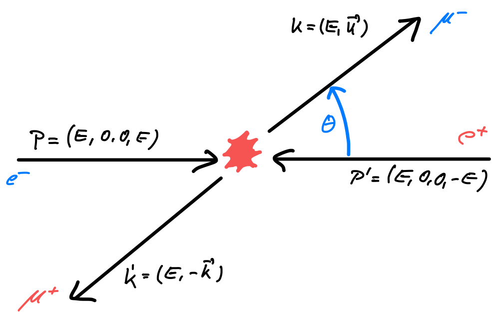
10. Since $m_e/m_m \approx 1/200$, we set $m_e = 0$ henceforth:
(➔ Problemset 8 for the general result with $m_e \neq 0$)

$$\frac{1}{4}\sum_{s,s',r,r'}|\mathcal{M}|^2 \hat{=} \frac{8e^4}{q^4}\left[(pk)(p'k') + (pk')(p'k) + m_m^2(pp')\right]$$

11. $\sphericalangle$ *Center-of-mass frame:* $\vec{p} + \vec{p}' = 0 = \vec{k} + \vec{k}'$

- w.l.o.g. $p = (E, E\hat{z})$, $p' = (E, -E\hat{z})$ (since $m_e = 0$)
- $|\vec{k}| = \sqrt{E^2 - m_m^2}$ (since $E = E_e(p) = E_e(p') = E_m(k) = E_m(k')$)
- $\vec{k}\cdot\hat{z} = |\vec{k}|\cos\theta$

↑ Notes



→

$$q^2 = (p+p')^2 = 4E^2$$
$$pp' = 2E^2$$
$$pk = p'k' = E^2 - E|\vec{k}|\cos\theta$$
$$pk' = p'k = E^2 + E|\vec{k}|\cos\theta$$

→

$$|\overline{\mathcal{M}}|^2 \equiv \frac{1}{4}\sum_{s,s',r,r'}|\mathcal{M}|^2 \triangleq e^4\left[\left(1+\frac{m_m^2}{E^2}\right)+\left(1-\frac{m_m^2}{E^2}\right)\cos^2\theta\right]$$

12. *Differential scattering cross section* from (4.12):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} = \frac{1}{2E_{\vec{p}}2E_{\vec{p}'}|v_p - v_{p'}|}\frac{|\vec{k}|}{(2\pi)^2 4E_{\text{cm}}}|\overline{\mathcal{M}}|^2$$
$$= \frac{\alpha^2}{4E_{\text{cm}}^2}\sqrt{1-\frac{m_m^2}{E^2}}\left[\left(1+\frac{m_m^2}{E^2}\right)+\left(1-\frac{m_m^2}{E^2}\right)\cos^2\theta\right]$$

It is $E_{\text{cm}} = 2E$ and $|v_p - v_{p'}| = |p^3/E_{\vec{p}} - p'^3/E_{\vec{p}'}| = 2$.

13. *Total cross section:*

$$\sigma_{\text{total}} = \underbrace{\frac{4\pi\alpha^2}{3E_{\text{cm}}^2}\sqrt{1-\frac{m_m^2}{E^2}}}_{\square}\left(1+\frac{m_m^2}{2E^2}\right)$$

14. *Discussion:*

- For $E_{\text{cm}} < 2m_m$ no pair-production is possible.
- Prediction of QED: non-trivial energy dependence of $\overline{\mathcal{M}}$
  Experimental results verify this additional dependence!
  (➔ P&S Fig. 5.2 on p. 138 or W. Bacino et. al. PRL **41**, 13 (1978))
  Recall that the energy-dependence of the prefactor □ was derived on very general grounds and is not QED-specific!
- Measuring σ_{total} as a function of E_{cm} yields the muon mass m_m.

↑ Notes

5.2 Summary of QED calculations

1. Draw relevant Feynman diagrams.
2. Use Feynman rules to calculate $\mathcal{M}$.
3. Calculate $|\overline{\mathcal{M}}|^2 = \sum_{\text{spins}} |\mathcal{M}|^2$ (use spin-sum relations).
4. Evaluate traces (use trace technology).
5. Fix a frame of reference and express all 4-momenta in terms of kinematic variables (energies, angles ...).
6. Plug in $|\overline{\mathcal{M}}|^2$ in Eq. (4.11) and integrate over phase-space variables that are not measured.

Following this procedure, one can evaluate cross sections for many other QED processes (like Compton scattering) and compare them with measurements from particle colliders (➔ P&S pp. 139–169). We will not dwell on these often very technical calculations but proceed with a more interesting question: What happens if we go beyond tree-level diagrams?

↑ Notes

6 Radiative Corrections of QED

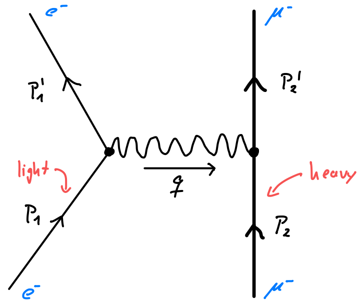
> **Problem Set 8**
>
> (due **12.06.2020**)
>
> 1. Derivation of the Rutherford formula for the differential cross section of non-relativistic charged particles
> 2. Recap of the lecture: Derivation of the cross section for electron-positron scattering in QED with finite electron mass

6.1 Overview

1. *Process:* For simplicity (see below), $\sphericalangle\, e^-$ scattering of a very heavy particle, e.g.,

$$\lim_{m_\mu \to \infty} \{\, \text{Electron } (e^-) + \text{Muon } (\mu^-) \;\to\; \text{Electron } (e^-) + \text{Muon } (\mu^-) \,\}$$

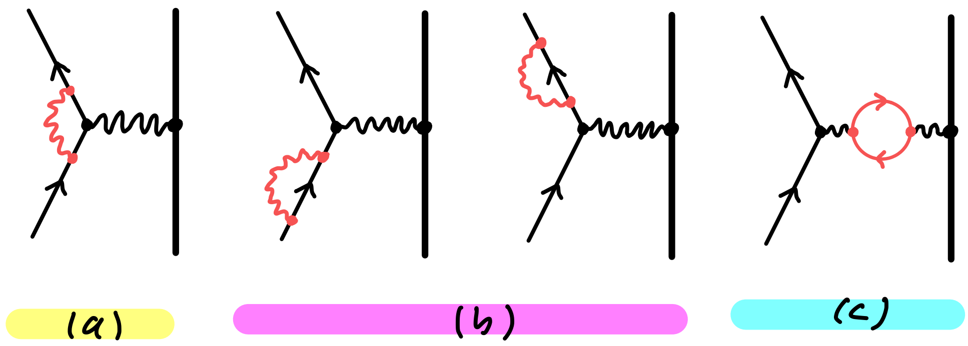
2. *Tree-level:*
   The computation runs along the same lines as for $e^-e^- \to e^-e^-$ scattering.
   In the following, however, we do not need the tree-level result.



   Alternatively, *crossing symmetry* relates the process to $e^+e^- \to \mu^+\mu^-$ and allows us to reuse the results we obtained for finite electron mass (➔ Problemset 8) with suitable substitutions.

3. *Radiative corrections* =
   Higher-order contributions to tree-level amplitudes from diagrams with …
   - loops:

↑ Notes



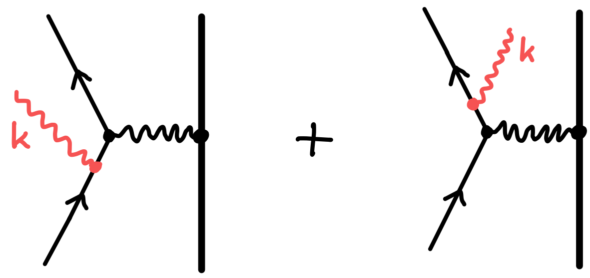
The 6 additional one-loop diagrams involving the heavy particle can be neglected as these include propagators of the heavy particle that vanish for $m_\mu \to \infty$.
Physically, the heavy particle does not accelerate much upon absorption/emission of a photon but behaves like a "static wall".

a) *Vertex correction*: UV-divergence & IR-divergence
(most interesting, ➔ below)
UV-divergence: divergence for $k \to \infty$ in integral of loop momentum
IR-divergence: divergence for $k \to 0$ in integral of loop momentum
(The vertex correction yields the anomalous magnetic moment of the electron.)

b) *External leg corrections*: UV-divergence & IR-divergence
(not amputated, ➔ later)

c) *Vacuum polarization*: UV-divergence
(complicated evaluation, ➔ later)

- extra final-state photons (*Bremsstrahlung*):



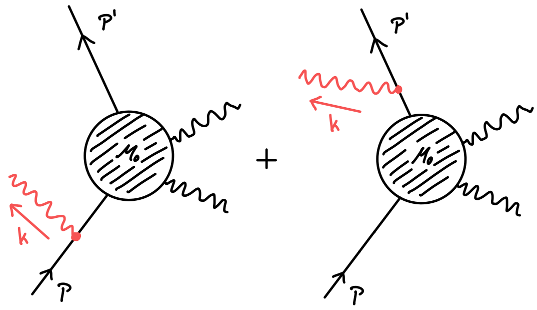
→ IR-divergence for $k \to 0$
In this limit, photons cannot be measured by detectors, so we should add these diagrams to the scattering amplitude.

4. Spoilers:
   - UV-divergences: cancel in observable quantitites
   - IR-divergences: cancel with the divergences of the bremsstrahlung diagrams
     (That is, radiative corrections are only consistent if *both* types of corrections (loops and bremsstrahlung) are included.)

6.2 Soft Bremsstrahlung

1. *Bremsstrahlung* = Electromagnetic radiation emitted by decelerated, charged particles
   Soft = Low-energy photons ($k \approx 0$)

2. Can be classically derived from Maxwell's equations (➔ P&S pp. 177–182)
3. $\sphericalangle$ Corresponding QFT process:



$\mathcal{M}_0$: (unkown) interaction amplitude
This is a 4×4-matrix with spinor indices and (potentially) multiple 4-vector indices.
→

$$i\mathcal{M} = -ie\epsilon^*_\mu(k)\bar{u}(p')\left\{\begin{array}{l}\mathcal{M}_0(p',p-k)\dfrac{i(\not p-\not k+m)}{(p-k)^2-m^2+i\varepsilon}\gamma^\mu \\ +\gamma^\mu\dfrac{i(\not p'+\not k+m)}{(p'+k)^2-m^2+i\varepsilon}\mathcal{M}_0(p'+k,p)\end{array}\right\}u(p)$$

4. Simplifications:
   - Soft photons: $|\vec{k}| \ll |\vec{p}\,'-\vec{p}|$
     $\to \mathcal{M}_0(p',p-k)\approx\mathcal{M}_0(p'+k,p)\approx\mathcal{M}_0(p',p)$ (cross ks in amplitudes)
     $\to \not p-\not k\approx\not p$ etc. (cross $\not k$s in numerators of propagators)
   - Dirac algebra →
     $$(\not p+m)\gamma^\mu\epsilon^*_\mu u(p) \mathrel{\hat=} 2p^\mu\epsilon^*_\mu u(p)$$
     $$\bar{u}(p')\gamma^\mu\epsilon^*_\mu(\not p'+m) \mathrel{\hat=} \bar{u}(p')2p'^\mu\epsilon^*_\mu$$
     Here we use the Dirac algebra and the spin-completeness relations that imply $(\not p-m)u(p)=0$.
   - Use $p^2=m^2$ and $k^2=0$:
     $$(p-k)^2-m^2=-2pk$$
     $$(p'+k)^2-m^2=2p'k$$
5. Then
   $$i\mathcal{M} = \underbrace{\bar{u}(p')\mathcal{M}_0(p',p)u(p)}_{\text{elastic scattering}}\cdot\underbrace{\left[e\left(\frac{p'\epsilon^*}{p'k}-\frac{p\epsilon^*}{pk}\right)\right]}_{\text{bremsstrahlung}}$$
6. *Scattering cross section* (cf. (4.11) for two incoming particles):
   $$d\sigma(p\to p'+\gamma) = d\sigma(p\to p')\cdot\underbrace{\int\frac{d^3k}{(2\pi)^3}\sum_r\frac{e^2}{2|\vec{k}|}\left|\frac{p'\epsilon^r}{p'k}-\frac{p\epsilon^r}{pk}\right|^2}_{\equiv d\mathcal{P}_k(p\to p')}$$

Here, we used $\epsilon = (0, \vec{\epsilon})$ for external photons.
$d\mathcal{P}_k(p \to p')$: differential probability to emit a photon into d^3k under the condition that the electron scatters from p to p'.

7. Evaluation:

$$\int d\mathcal{P}_k = \frac{\alpha}{\pi}\int_0^\infty dk\,\frac{1}{k}\underbrace{\int\frac{d\Omega_k}{4\pi}\sum_r\left|\frac{p'\epsilon^r}{p'\tilde{k}} - \frac{p\epsilon^r}{p\tilde{k}}\right|^2}_{\equiv \mathcal{I}(p,p')} \tag{6.1}$$

$$= \frac{\alpha}{\pi}\mathcal{I}(p,p')\,[\underbrace{\log(\infty)}_{\text{Problem 1}} - \underbrace{\log(0)}_{\text{Problem 2}}]$$

with $\tilde{k} = k/|\vec{k}| = (1, \hat{k})$

8. Approximations:

- *Problem 1:* Soft-photon approximation breaks down at $k \approx |\vec{q}| = |\vec{p} - \vec{p}'|$
  $\to$ Introduce upper cutoff at $|\vec{q}|$
- *Problem 2:* Probability of radiating a very soft photon is infinite!
  $\to$ IR-divergences of perturbative QED
  (note that in the limit $k \to 0$ our soft-photon approximation is exact!)
  Solution: Regularization with finite photon mass $\mu > 0$:

$$\frac{1}{k} = \frac{1}{E_{\vec{k}}} \mapsto \frac{1}{\sqrt{\mu^2 + k^2}}$$

  This is a purely mathematical ad-hoc solution to control the IR-divergence. Later we will see that in physical observables the unphysical parameter μ drops out.
  and therefore

$$\int_0^{|\vec{q}|} dk\,\frac{1}{\sqrt{\mu^2+k^2}} = \log\left(\frac{\sqrt{\mu^2+|\vec{q}|^2}+|\vec{q}|}{\mu}\right)$$
$$\sim \log\left(2\frac{|\vec{q}|}{\mu}\right) \sim \log\left(\frac{|\vec{q}|}{\mu}\right) = \frac{1}{2}\log\left(\frac{|\vec{q}|^2}{\mu^2}\right) \tag{6.2}$$

  (asymptotically for $\mu \to 0$)
- Relativistic limit ($E_{p,p'} \gg m$):

$$\mathcal{I}(p,p') \approx 2\log\left(\frac{-q^2}{m^2}\right) \quad \text{with} \quad -q^2 = -(p'-p)^2 \geq 0$$

  Proof: ➔ P&S pp. 180–182, starting at Eq. (6.12)
  Recall that for two time-like momentum vectors p and p', $p^2 = p'^2 = m^2$, their difference $q = p' - p$ is space-like, $q^2 \leq 0$ (use Cauchy-Schwarz inequality to show this). Therefore there always exists a coordinate system with $q^0 = 0$, or, equivalently, $p^0 = E = p'^0$. In this system, it is $-q^2 = |\vec{q}|^2$.

↑ Notes

9. Result:

$$d\sigma(p \to p' + \gamma) \approx d\sigma(p \to p') \cdot \frac{\alpha}{\pi} \underbrace{\log\left(\frac{-q^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right)}_{\text{Sudakov double logarithm}} \tag{6.3}$$

for $\mu \to 0$ (regularization) and $E_{p,p'} \gg m$ or $-q^2 = |\vec{q}|^2 \to \infty$ (relativistic limit)

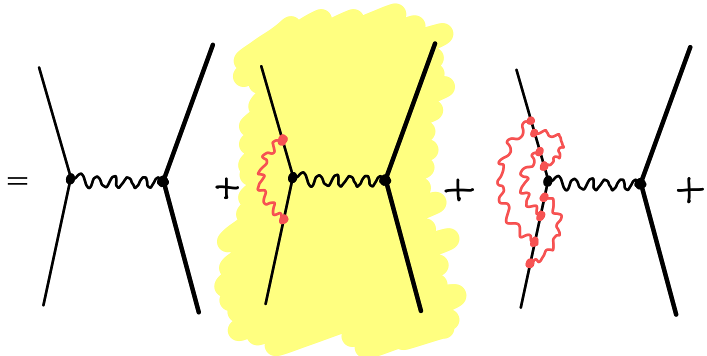
10. Two problems:
    - Dependence of unphysical photon mass μ
      (should drop out from physical predictions)
    - Logarithmic divergence for $-q^2 \to \infty$ ($\to$ cannot be interpreted as probability)
      (We will see that the correct interpretation is that of the emitted *number* of photons.)

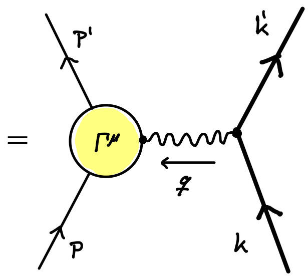
6.3 The Electron Vertex Function

6.3.1 Formal Structure

1. *Scattering amplitude:*

$$i\mathcal{M}(e^-(p)\mu^-(k) \to e^-(p')\mu^-(k'))$$





$$= ie^2 \left[\bar{u}_e(p')\Gamma^\mu(p',p)u_e(p)\right] \frac{1}{q^2} \left[\bar{u}_m(k')\gamma_\mu u_m(k)\right]$$

Note that we consider only amputated diagrams (1) without loops connecting to the heavy particle and (2) ignore the vacuum polarization diagrams as these describe corrections to the photon propagator and are not related to the interaction between fermions and gauge field.

Below we will explicitly evaluate the first loop correction (yellow).

↑ Notes

2. General form:

$$\Gamma^\mu(p',p) = f(p^\mu, p'^\mu, \gamma^\mu, m, e, \mathbb{C})$$

γ^5 is forbidden since QED does not violate parity symmetry (recall that $(\gamma^5)^2 = \mathbb{1}$ and $\gamma^5\gamma^\mu$ produces a pseudo vector and γ^5 a pseudo scalar)!

3. *Restrictions:*
All equations that follow are required to hold *if* sandwiched between bispinors $\overline{u}$ and u!
   a) *Lorentz covariance:* Γ^μ transforms like $\gamma^\mu \to$

$$\begin{aligned}\Gamma^\mu &= A\cdot\gamma^\mu + \tilde{B}\cdot p^\mu + \tilde{C}\cdot p'^\mu \\ &= A\cdot\gamma^\mu + B\cdot(p'^\mu + p^\mu) + C\cdot(p'^\mu - p^\mu)\end{aligned} \tag{6.4}$$

   Γ^μ must be a linear function of the available Lorentz vectors.
   b) Recall $\not{p}u(p) = m\cdot u(p)$ and $\overline{u}(p')\not{p}' = \overline{u}(p')\cdot m \to$

$$X = X(p^\mu, p'^\mu, m, e, \mathbb{C})\cdot\mathbb{1} \quad \text{for} \quad X = A, B, C$$

   Use the spin-sum identities (3.11) to show this.
   c) $q^2 = (p'-p)^2 = 2(m^2 - pp')$ only non-trivial scalar $\to$

$$X = X(q^2, m, e, \mathbb{C}) \quad \text{for} \quad X = A, B, C$$

   Recall that $p^2 = p'^2 = m^2$ are constants.
   d) *Ward identity* for $U(1)$ gauge-symmetry of QED Lagrangian:

$$q_\mu\Gamma^\mu(p',p) \overset{*}{=} 0 \tag{6.5}$$

   ➔ P&S pp. 238–244 for a proof and pp. 159–161 for a motivation
   This is the quantum version of the classically conserved current $\partial_\mu j^\mu(x) = 0$ in Fourier space.
   Ward identities = QFT analog of Noether's theorem
   $\to$

$$0 = q_\mu\Gamma^\mu = A\cdot\underbrace{q_\mu\gamma^\mu}_{=0} + B\cdot\underbrace{q_\mu(p'^\mu + p^\mu)}_{=0} + C\cdot q^2$$

   $\to C = 0$
   The first term vanishes only if sandwiched between bispinors,

$$\overline{u}(p')(\not{p}' - \not{p})u(p) = (m-m)\overline{u}(p')u(p) = 0$$

   the second vanishes identically since $p^2 = p'^2 = m^2$.

4. *Gordon identity:*

$$\overline{u}(p')\left[\frac{p'^\mu + p^\mu}{2m}\right]u(p) \overset{\circ}{=} \overline{u}(p')\gamma^\mu u(p) - \overline{u}(p')\left[\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \tag{6.6}$$

Absorb the first term in A. Recall that $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$ produces a second-rank Lorentz tensor.

↑ Notes

5. Therefore

$$\Gamma^\mu(p', p) = \gamma^\mu \underbrace{F_1(q^2)}_{1+\mathcal{O}(\alpha)} + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \underbrace{F_2(q^2)}_{0+\mathcal{O}(\alpha)}$$

$F_i(q^2)$: form factors
Note that we can use the Gordon identity w.l.o.g. because the vertex amplitude Γ^μ is always sandwiched between bispinors $\overline{u}$ and u.

↑ Notes

6.3.2 The Landé g-factor

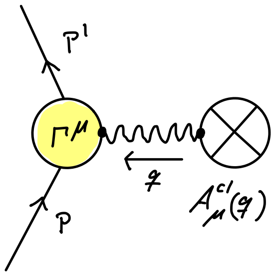
Observation: F_1 and F_2 encode the electric and magnetic response of the electron completely.
Goal: Express electric charge and magnetic moment as function of form factors.

1. *Setting:* $\sphericalangle$ Classical, external field $A_\mu^{\text{cl}}(x)$: (➔ Problemset 8)

$$H_{\text{int}} = e\int d^3x\,\overline{\Psi}(x)\gamma^\mu\Psi(x)\,A_\mu^{\text{cl}}(x)$$

$\rightarrow$

$$i\mathcal{M}\,(2\pi)\delta(p'^0 - p^0) =$$



$$= -ie\overline{u}(p')\Gamma^\mu(p',p)u(p)\cdot A_\mu^{\text{cl}}(q = p' - p)$$

Note that $A_\mu^{\text{cl}}(x)$ is a parameter and not an operator; in particular, it has no dynamics!

In general, a static potential breaks translational invariance and therefore 3-momentum is no longer conserved. However, as it is static, it does *not* break time translation invariance, so that energy is still conserved, i.e. $p'^0 = p^0$. This is like a hard wall in mechanics that can absorb momentum but not energy.

2. *Electric charge:*

   a) $\sphericalangle$ $A_\mu^{\text{cl}}(x) = (\phi(\vec{x}), \vec{0}) \quad\Rightarrow\quad A_\mu^{\text{cl}}(q) = ((2\pi)\delta(q^0)\phi(\vec{q}), \vec{0})$

   b) $i\mathcal{M} = -ie\overline{u}(p')\Gamma^0(p',p)u(p)\cdot\phi(\vec{q})$

   c) $\sphericalangle$ $\phi(\vec{x})$ slowly varying $\rightarrow$ $\phi(\vec{q})$ concentrated at $\vec{q} = 0$ $\rightarrow$ take limit $\vec{q}\rightarrow 0$:

$$i\mathcal{M} \approx -ieF_1(0)\,\overline{u}(p')\gamma^0u(p)\cdot\phi(\vec{q}) \overset{|\vec{p}|^2\ll m^2}{\approx} -ieF_1(0)\phi(\vec{q})\cdot 2m\xi'^\dagger\xi$$

   Recall Eq. (4.18) for the non-relativistic limit of bispinors.

   d) $\overset{*}{\rightarrow}$ *Born approximation* with potential

$$V(\vec{x}) = eF_1(0)\phi(\vec{x})$$

   Recall Eq. (4.20) for the Born approximation.

   e) Charge $e \overset{!}{=} eF_1(0)$ and $F_1^{(0)} = 1 \rightarrow$

$$\boxed{F_1^{(n)}(0) = 0 \quad\text{for}\quad n \geq 1}$$

   It is $F_1 = \sum_{n=0}^\infty F_1^{(n)}\,\alpha^n$ with α the fine-structure constant.

3. *Magnetic moment:*

↑ Notes

a) $\sphericalangle\ A^{\text{cl}}_\mu(x) = (0, \vec{A}(\vec{x})) \quad\Rightarrow\quad A^{\text{cl}}_\mu(q) = (0, (2\pi)\delta(q^0)\vec{A}(\vec{q}))$

b) Then

$$i\mathcal{M} = -ie\bar{u}(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p)\cdot A^{\text{cl}}_\mu(\vec{q})$$
$$= +ie\bar{u}(p')\underbrace{\left[\gamma^i F_1(q^2) + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]}_{\text{Vanishes for } q=0 \text{ and } |\vec{p}|^2 \ll m^2\text{, see (4.18)}} u(p)\cdot A^i_{\text{cl}}(\vec{q})$$

Note that $\vec{q} = 0 \;\Leftrightarrow\; q = 0$.

c) $\sphericalangle\ F_1$-term and expand bispinors in *linear* order of $\vec{p}$, $\vec{p}\,'$:

$$\bar{u}(p')\gamma^i u(p) \overset{\circ}{\approx} 2m\xi'^\dagger\left(\frac{\vec{p}\,'\vec{\sigma}}{2m}\sigma^i + \sigma^i\frac{\vec{p}\vec{\sigma}}{2m}\right)\xi$$
$$\overset{\circ}{=} \underbrace{\frac{p^i + p'^i}{2m}}_{(A)}\cdot 2m\xi'^\dagger\xi + \underbrace{2m\xi'^\dagger\left(\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k\right)\xi}_{(B)}$$

In the second step, we used $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$

Only (B) is spin-dependent and affects the magnetic moment!
Term (A) describes the kinetic energy of a charged particle in a magnetic field in nonrelativistic quantum mechanics.

d) $\sphericalangle\ F_2$-term and expand bispinors in *lowest* order of $\vec{p}$, $\vec{p}\,'$:

$$\frac{iq_\nu}{2m}\cdot\bar{u}(p')\sigma^{i\nu}u(p) \overset{\circ}{\approx} 2m\xi'^\dagger\left(\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k\right)\xi$$

Use $u(p)\approx\sqrt{m}\begin{pmatrix}\xi\\\xi\end{pmatrix}$ (➔ Eq. (4.18)), $[\sigma^i,\sigma^j] = 2i\varepsilon^{ijk}\sigma^k$ and $q_i = -q^i$.

e) In summary:

$$i\mathcal{M} = ie\bar{u}(p')\left[\gamma^i F_1(q^2) + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]u(p)\cdot A^i_{\text{cl}}(\vec{q})$$
$$\overset{q\to 0}{\approx} -ie\,\xi'^\dagger\left\{\frac{-1}{2m}\sigma^k\,[F_1(0)+F_2(0)]\right\}\xi\cdot\underbrace{\left[-i\varepsilon^{ijk}q^j A^i_{\text{cl}}(\vec{q})\right]}_{=B^k_{\text{cl}}(\vec{q})}\cdot(2m)$$

with $\vec{B}_{\text{cl}} = \nabla\times\vec{A}_{\text{cl}}$ $(B^k = \varepsilon^{ijk}\partial_i A^j \;\Rightarrow\; B^k(\vec{q}) = i\varepsilon^{ijk}q^i A^j(\vec{q}))$

f) $\overset{*}{\to}$ *Born approximation* with potential

$$V(\vec{x}) = -\langle\vec{\mu}\rangle\cdot\vec{B}_{\text{cl}}(\vec{x})$$

yields the magnetic moment

$$\langle\vec{\mu}\rangle = \frac{e}{m}[F_1(0)+F_2(0)]\cdot\xi'^\dagger\frac{\sigma^k}{2}\xi \equiv g\cdot\mu_B\cdot\langle\vec{S}\rangle$$

↑ Notes

with Bohr magneton $\mu_B = \frac{e}{2m}$ and *Landé factor*

$$g = 2\,[F_1(0) + F_2(0)] = 2 + 2F_2(0) = \underbrace{2}_{\text{Dirac equation}} + \underbrace{2\alpha F_2^{(1)}(0) + \mathcal{O}(\alpha^2)}_{\text{Anomalous magnetic moment}}$$

Here we use $F_1(0) = 1$ in all orders of α and that $F_2 = \alpha\, F_2^{(1)} + \mathcal{O}(\alpha^2)$.

This motivates our subsequent evaluation of the first loop correction $F_2^{(1)}$!

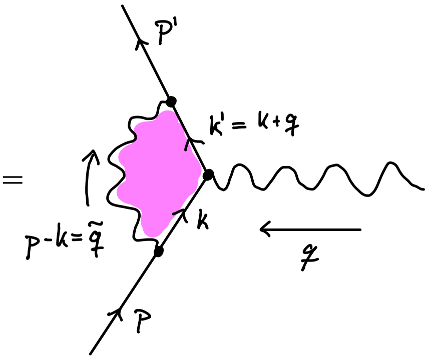
6.3.3 Evaluation

The techniques that we use below can be applied to the evaluation of all loop diagrams in QED:

1. *Scattering amplitude:*

$$\bar{u}(p')[\alpha\Gamma^{(1)}(p',p)]^\mu u(p)$$

$=$



$$= \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\nu\rho}}{\tilde{q}^2 + i\varepsilon} \bar{u}(p')(-ie\gamma^\nu) \frac{i(\not{k}' + m)}{k'^2 - m^2 + i\varepsilon} \gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon} (-ie\gamma^\rho) u(p)$$

Contraction identities: $\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu$ etc.

$$\hat{=} 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p)}{(\tilde{q}^2 + i\varepsilon)(k'^2 - m^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)} \tag{6.7}$$

In the following, the regularizations $i\varepsilon$ will be crucial to make the expressions well-defined.

2. *Feynman Parameters:*
Goal: introduce new integration variables to combine the three factors in the denominator so that we can solve the integral by completing the square.

$$\frac{1}{A_1 \dots A_n} = \left(\prod_{i=1}^n \int_0^1 dx_i\right) \delta\left(\sum_i x_i - 1\right) \frac{(n-1)!}{[x_1A_1 + \dots + x_nA_n]^n}$$

x_i: Feynman parameters
(Proof: ➲ Problemset 9)

↑ Notes

3. Application to denominator of (6.7):

$$\frac{1}{(\tilde{q}^2+i\varepsilon)(k'^2-m^2+i\varepsilon)(k^2-m^2+i\varepsilon)} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\frac{2}{D^3}$$

with (using $x+y+z=1$ and $\tilde{q}=p-k$ and $k'=k+q$)

$$D \mathrel{\hat{=}} k^2 + 2k(yq-zp) + yq^2 + zp^2 - (x+y)m^2 + i\varepsilon$$

Complete the square: $l \equiv k + yq - zp$

$$= l^2 - \Delta + i\varepsilon$$

where $\Delta \equiv -xyq^2 + (1-z)^2m^2 > 0$ ("effective mass squared") since $q^2 < 0$ (always spacelike)

4. Express the numerator of (6.7) in terms of l ($k^\mu = l^\mu - yq^\mu + zp^\mu$):

$$\bar{u}(p')\left[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p)$$

This step is only valid under the integral $\int d^4l$ (see notes below)!

$$\mathrel{\overset{\circ}{\sim}} \bar{u}(p')\begin{Bmatrix} -\frac{1}{2}\gamma^\mu l^2 + [-y\not{q}+z\not{p}]\gamma^\mu[(1-y)\not{q}+z\not{p}] \\ +m^2\gamma^\mu - 2m[(1-2y)q^\mu + 2zp^\mu] \end{Bmatrix} u(p)$$

For this step, you have to use the Dirac algebra (see notes below).

$$\mathrel{\hat{=}} \bar{u}(p')\left\{\begin{array}{l} \gamma^\mu \cdot \underbrace{\left[-\frac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2\right]}_{A} \\ +(p'+p)^\mu \cdot \underbrace{[mz(z-1)]}_{B} \\ +q^\mu \cdot \underbrace{[m(z-2)(x-y)]}_{C} \end{array}\right\} u(p)$$

This structure was expected, recall (6.4).

- The first step follows with $\int \frac{d^4l}{(2\pi)^4}\frac{l^\mu}{D(l^2)} = 0$ due to symmetry and

$$L^{\mu\nu} \equiv \int \frac{d^4l}{(2\pi)^4}\frac{l^\mu l^\nu}{D(l^2)} = \int \frac{d^4l}{(2\pi)^4}\frac{g^{\mu\nu}}{4}\frac{l^2}{D(l^2)}\,.$$

This identity can be shown as follows: First, notice that under a Lorentz transformation $\Lambda \in SO^+(1,3)$

$$L'^{\mu\nu} = \int \frac{d^4l}{(2\pi)^4}\frac{l'^\mu l'^\nu}{D(l^2)} = \int \frac{d^4l'}{(2\pi)^4}\frac{l'^\mu l'^\nu}{D(l'^2)} = L^{\mu\nu} \qquad (6.8)$$

(in the second step we used that $\det(\Lambda) = 1$ and l^2 is a scalar) and therefore

$$L^{\mu\nu} = g^{\mu\nu}C(l^2) \qquad (6.9)$$

which follows from Schur's lemma (➔ https://hal.archives-ouvertes.fr/hal-01797592) and the observation that the only scalar available is l^2. Finally, $C(l^2)$ can be determined by contracting with $g_{\mu\nu}$:

$$\overset{\times g_{\mu\nu}}{\Rightarrow} \quad g_{\mu\nu}L^{\mu\nu} = \int \frac{d^4l}{(2\pi)^4}\frac{l^2}{D(l^2)} = 4C(l^2)$$

↑ Notes

- For the second step, use
  - $\not p\gamma^\mu = 2p^\mu - \gamma^\mu \not p$
  - $\not p u(p) = mu(p)$ and $\bar{u}(p')\not p' = m\bar{u}(p')$ and therefore $\bar{u}(p')\not q u(p) = 0$
  - $x + y + z = 1$

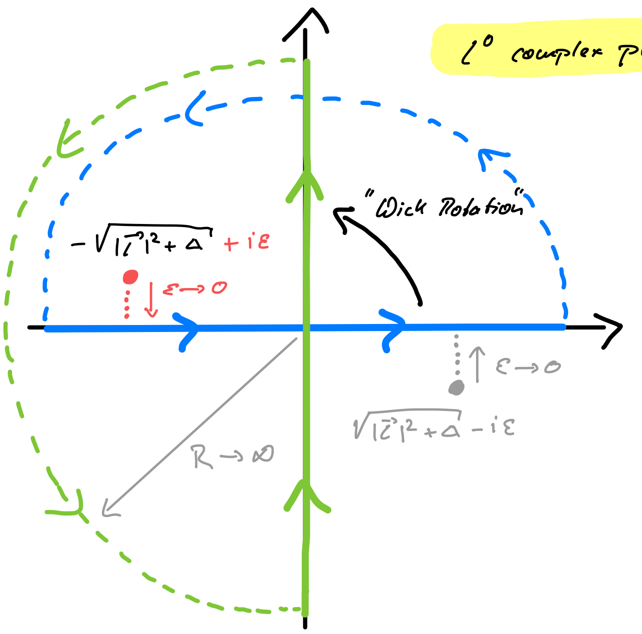
5. C is antisymmetric and D is symmetric under $x \leftrightarrow y \to$ drop C
   Formally: $\int_0^1 dx \int_0^1 dy\, C(x,y)/D(x,y) = 0$
   This result complies with the Ward identity.

6. Gordon identity (6.6) →

$$
\begin{aligned}
&\bar{u}(p')[\alpha\Gamma^{(1)}(p',p)]^\mu u(p) \\
= \;& 2ie^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \int_0^1 dy \int_0^1 dz\; \delta(x+y+z-1) \frac{2}{D(l^2)^3} \\
&\times \bar{u}(p') \left\{ \begin{aligned} &\gamma^\mu \cdot \left[-\frac{1}{2} l^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2 \right] \\ &+ \frac{i\sigma^{\mu\nu}q_\nu}{2m} \cdot \left[2m^2 z(1-z)\right] \end{aligned} \right\} u(p)
\end{aligned}
$$

   Note that the Gordon identity contributes also to the γ^μ-term, thus the modifications in the last term proportional to m^2.

7. *Momentum integral:*
   a) Problem: $l^2 = (l^0)^2 - \vec{l}^2$ cannot be integrated in four-dimensional spherical coordinates. Solution:
      Wick rotation = Evaluation of a contour integral (blue) along a rotated contour (green) that encircles the same poles (red):



      Note that this requires the integrand to vanish faster then $1/|l^0|$ so that the contribution of the half-circle vanishes for $R \to \infty$.

↑ Notes

Parametrization of the new contour:

$$l^0 \equiv i l_E^0 \quad \text{and} \quad \vec{l} \equiv \vec{l}_E \quad \text{with} \quad l_E \in \mathbb{R}^4$$
$$\Rightarrow \quad l^2 = -(l_E^0)^2 - \vec{l}_E^2 = -l_E^2$$

Here, l^2 is the squared "norm" of a four-dimensional vector in the Minkowski metric and l_E^2 in the Euclidean metric.

b) Then $(m > 2)$ (we are interested in $m = 3$)

$$\begin{aligned}\lim_{\varepsilon\to 0}\int \frac{d^4 l}{(2\pi)^4}\frac{1}{(l^2-\Delta+i\varepsilon)^m} &= \frac{i}{(-1)^m}\frac{1}{(2\pi)^4}\int d^4 l_E \frac{1}{(l_E^2+\Delta)^m}\\ &= \frac{i(-1)^m}{(2\pi)^4}\underbrace{\int d\Omega_4}_{=2\pi^2}\underbrace{\int_0^\infty dl_E \frac{l_E^3}{(l_E^2+\Delta)^m}}_{=\frac{1}{2(m-1)(m-2)\Delta^{m-2}}}\\ &= \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)\Delta^{m-2}}\end{aligned} \tag{6.10}$$

$\int d\Omega_4 = 2\pi^2$ is the surface area of the unit sphere in four dimensions.

and similarly $(m > 3)$

$$\lim_{\varepsilon\to 0}\int \frac{d^4 l}{(2\pi)^4}\frac{l^2}{(l^2-\Delta+i\varepsilon)^m} \overset{\circ}{=} \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)\Delta^{m-3}} \tag{6.11}$$

Problem: For $m = 3$ the integral diverges!
This is a UV-divergence since it occurs for $l_E^2 \sim \vec{l}^2 \sim \vec{k}^2 \to \infty$.
Note that in this case also the contribution of the half-circle does not vanish and the Wick rotation is not justified.

c) Fix: *Pauli-Villars regularization:*

$$\frac{-i g_{\mu\nu}}{\tilde{q}^2+i\varepsilon} \mapsto \frac{-i g_{\mu\nu}}{\tilde{q}^2+i\varepsilon} - \frac{-i g_{\mu\nu}}{\tilde{q}^2-\Lambda^2+i\varepsilon}$$

for large Λ (= additional, heavy photon with mass Λ)
For $\Lambda \to \infty$ we obtain the original expression. The regularization essentially introduces a UV-cutoff at momenta $k \gtrsim \Lambda$ where the difference is suppressed.

Hope: Λ does not appear in physical predictions

$\overset{\circ}{\to}$ Only change:

$$\Delta_\Lambda = -xyq^2 + (1-z)^2 m^2 + z\Lambda^2$$

d) Therefore $(m = 3)$

- (6.10) $\mapsto$ (6.10) $- \mathcal{O}(\Delta_\Lambda^{-1}) =$ (6.10) $- \mathcal{O}(\Lambda^{-2}) \approx$ (6.10)
  Drop contribution to the convergent integral since $\Lambda^{-2} \to 0$ for $\Lambda \to \infty$.

↑ Notes

- (6.11) $\mapsto$

$$\lim_{\varepsilon\to 0}\int\frac{d^4l}{(2\pi)^4}\left[\frac{l^2}{(l^2-\Delta+i\varepsilon)^3}-\frac{l^2}{(l^2-\Delta_\Lambda+i\varepsilon)^3}\right]$$

Wick rotation

$$=\frac{i}{(4\pi)^2}\int_0^\infty dl_E\left[\frac{2l_E^5}{(l_E^2+\Delta)^3}-\frac{2l_E^5}{(l_E^2+\Delta_\Lambda)^3}\right]$$

$$\mathrel{\hat{=}}\frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right)\xrightarrow{\Lambda\to\infty}\frac{i}{(4\pi)^2}\log\left(\frac{z\Lambda^2}{\Delta}\right)$$

➔ Problemset 10 for details

8. Result: (with $\Delta_\Lambda\sim z\Lambda^2$ for $\Lambda\to\infty$)

$$\bar{u}(p')[\alpha\Gamma^{(1)}(p',p)]^\mu u(p)$$
$$\mathrel{\hat{=}}\frac{\alpha}{2\pi}\int_0^1 dx\int_0^1 dy\int_0^1 dz\;\delta(x+y+z-1)$$
$$\times\,\bar{u}(p')\left\{\begin{array}{l}\gamma^\mu\cdot\underbrace{\left[\log\left(\frac{z\Lambda^2}{\Delta}\right)+\frac{(1-x)(1-y)q^2}{\Delta}+\frac{(1-4z+z^2)m^2}{\Delta}\right]}_{\hat{=}F_1^{(1)}(q^2)}\\[2ex] +\frac{i\sigma^{\mu\nu}q_\nu}{2m}\cdot\underbrace{\left[\frac{2m^2z(1-z)}{\Delta}\right]}_{\hat{=}F_2^{(1)}(q^2)}\end{array}\right\}u(p) \tag{6.12}$$

The "$\hat{=}$" signifies that the integrals over Feynman parameters and the prefactor belong to the form factors.

9. *Discussion of F_1:*

   a) Problem 1: It should be $F_1^{(1)}(0)=0$, but here $F_1^{(1)}(0)\neq 0$!
   Fix 1:

$$F_1^{(1)}(q^2)\;\mapsto\;F_1^{(1)}(q^2)-F_1^{(1)}(0) \tag{6.13}$$

   We cannot justify this substitution at this point but will do so later.
   The origin of this term can be traced back to our omission of the external leg loop corrections.

   b) Problem 2: In addition, there is a *IR-divergence* for $\tilde{q}^2\to 0$
   $\sphericalangle$ $q^2=0$ for simplicity:

$$\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(x+y+z-1)\frac{1-4z+z^2}{(1-z)^2}$$
$$=\int_0^1 dz\int_0^{1-z}dy\,\frac{-2+(1-z)(3-z)}{(1-z)^2}$$
$$=\underbrace{\int_0^1 dz\,\frac{-2}{1-z}}_{-\infty}+\text{finite terms}$$

↑ Notes

Fix 2: Add a small photon mass $\mu > 0 \to$

$$\Delta \mapsto \Delta_\mu = -xyq^2 + (1-z)^2 m^2 + z\mu^2$$

In this regularization we recover the original result for $\mu \to 0$.
We will discuss this IR-divergence later.

c) Fix 1 + Fix 2 $\to$

$$F_1(q^2) = 1 + \alpha F_1^{(1)}(q^2) + \mathcal{O}(\alpha^2) \tag{6.14}$$

with

$$F_1^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1) \times \left[\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right) + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + z\mu^2} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + z\mu^2} \right]$$

Note that we already set $\mu = 0$ in the logarithm where it is not needed to control the IR-divergence.

10. *Discussion of F_2:*
No divergences in F_2! Yay!

$$F_2(q^2) = \alpha F_2^{(1)}(q^2) + \mathcal{O}(\alpha^2)$$

with

$$F_2^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1) \times \left[\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2xy}\right]$$

11. *Landé g-factor:*

$$\begin{aligned} F_2(q^2=0) &= \frac{\alpha}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\frac{2z}{1-z} + \mathcal{O}(\alpha^2) \\ &= \frac{\alpha}{2\pi}\int_0^1 dz \int_0^{1-z} dy\, \frac{2z}{1-z} + \mathcal{O}(\alpha^2) \\ &= \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2) \end{aligned}$$

↑ Notes

Therefore the *anomalous magnetic dipole moment of the electron* is

$$a_e \equiv \frac{g-2}{2} = \frac{\alpha}{2\pi} \approx 0.0011614$$
$$a_e^{\text{exp}} \approx 0.0011597$$

Note that $\alpha^2 \sim 0.5 \cdot 10^{-4}$ so that the deviation can be explained by higher-order corrections.

Note 6.1

- Our first-order result was obtained by Schwinger in 1948
  ➔ https://doi.org/10.1103/PhysRev.73.416
  The first-order correction $\frac{\alpha}{2\pi}$ is engraved on Schwinger's tombstone:

- Modern values:

$$a_e^{\text{SM}} = 0.001\,159\,652\,182\,031(15)(15)(720)$$
$$a_e^{\text{exp}} = 0.001\,159\,652\,180\,73(28)$$

  → Agree to 11 significant digits
  This is the most accurate prediction of physics to date!
  The experimental value is from ➔ https://doi.org/10.1103/PhysRevA.83.052122 and the theoretical value is from ➔ https://doi.org/10.1103/PhysRevD.91.033006 (erratum). The theoretical result is based on numerical evaluations of contributions up to order α^5. Analytical results are known up to order α^3, ➔ https://doi.org/10.1016/0370-2693(96)00439-X. Note that the theoretical value also includes small contributions beyond QED, namely from the electroweak and hadronic sector of the standard model. The main contribution comes from higher-order QED diagrams, though.

- Our first-order result applies also to the *muon* since the mass cancels:

$$a_\mu^{(1)} = \frac{\alpha}{2\pi} = a_e^{(1)}$$

  However, in higher-order there seem to be discripancies between the standard

↑ Notes

model predictions (as for the electron, this goes beyond QED) and measurements:

$$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = 261(63)(48) \times 10^{-11}$$

For details, ➲ http://pdg.lbl.gov/2019/reviews/rpp2018-rev-g-2-muon-anom-mag-moment.pdf and references therein.

This deviation *may* hint at new physics beyond the standard model, for example contributions from supersymmetric partners.

Problem Set 9

(due **19.06.2020**)

1. Calculation of the scattering cross section of an electron off a proton including radiative corrections (Rosenbluth formula)
2. Toolkit for evaluating loop diagrams: The concept of Feynman parameters

6.3.4 The Infrared Divergence

1. Goal: Understand asymptotics of $|F_1(q^2)| \to \infty$ for $\mu \to 0$
2. Show in ➔ Problemset 10:

$$F_1(q^2) = (6.14) \overset{\mu\to 0}{\sim} 1 - \frac{\alpha}{2\pi} f_{\text{IR}}(q^2) \log\left(\frac{A}{\mu^2}\right) + \mathcal{O}(\alpha^2) \tag{6.15}$$

where $A \in \{-q^2, m^2\}$ (both are asymptotically equivalent but, depending on the additional limit $-q^2 \to 0/\infty$, one or the other must be chosen) and

$$f_{\text{IR}}(q^2) = \int_0^1 d\xi \, \frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)} - 1 \geq 0$$

Note that $-q^2 \geq 0$ and $\xi(1-\xi) \leq 1/4$ for $0 \leq \xi \leq 1$.

3. $\sphericalangle$ Cross section for electron scattering off a static potential:

$$\frac{\mathrm{d}\sigma(\vec{p} \to \vec{p}\,')}{\mathrm{d}\Omega} \sim \underbrace{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_0}_{\text{Tree-level result}} \times \left[\underbrace{1 - \frac{\alpha}{\pi} f_{\text{IR}}(q^2) \log\left(\frac{A}{\mu^2}\right)}_{\text{Problem: } \to -\infty \text{ for } \mu \to 0} + \mathcal{O}(\alpha^2) \right]$$

Recall that $d\sigma \propto |\mathcal{M}|^2 \sim [\Gamma^\mu]^2 \sim [F_1(q^2)]^2$. Just like e, F_1 is a prefactor to γ^μ so that $e \mapsto e \cdot F_1(q^2)$ is enough to obtain the contribution of F_1 to the scattering cross section.

The factor $1/2$ vanishes because the expression must be squared for the cross section. The contribution of F_2 does not affect the asymptotic behaviour as it is finite for $\mu \to 0$.

Problem: The *negative*, *diverging* $\mathcal{O}(\alpha)$ contribution to the scattering cross section is clearly unphysical!

4. $\sphericalangle$ Limit $-q^2 \to \infty$:

$$f_{\text{IR}}(q^2) \sim \int_0^1 d\xi \, \frac{-q^2/2}{-q^2\xi(1-\xi)} \overset{\circ}{\sim} \log\left(\frac{-q^2}{m^2}\right)$$

We drop the constant -1 and the masses m^2 for the asymptotic behaviour.

→

$$F_1(-q^2 \to \infty) \overset{\mu\to 0}{\sim} 1 - \frac{\alpha}{2\pi} \underbrace{\log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)}_{\text{Sudakov double logarithm}} + \mathcal{O}(\alpha^2)$$

Here we have to use $A = -q^2$ and not $A = m^2$ since $-q^2 \to \infty$.

↑ Notes

5. Comparison with *bremsstrahlung* (6.3) for $-q^2 \to \infty$:

$$\frac{d\sigma(\vec{p} \to \vec{p}')}{d\Omega} \overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]$$

$$\frac{d\sigma(\vec{p} \to \vec{p}' + \gamma)}{d\Omega} \overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[\quad + \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]$$

→ Both are divergent but their sum is finite and independent of μ!

6. *Suggested solution:* Photon detectors cannot detect photons below a lower threshold E_{min}: →

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} = \frac{d\sigma(\vec{p} \to \vec{p}')}{d\Omega} + \frac{d\sigma(\vec{p} \to \vec{p}' + \gamma(k < E_{\text{min}}))}{d\Omega}$$

To show: The cancellation does not only occur for $-q^2 \to \infty$ but for arbitrary q.

7. For general q:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[\underbrace{1 - \frac{\alpha}{\pi} f_{\text{IR}}(q^2) \log\left(\frac{A}{\mu^2}\right)}_{\text{Elastic scattering}} \underbrace{+ \frac{\alpha}{2\pi} \mathcal{I}(p, p') \log\left(\frac{E_{\text{min}}^2}{\mu^2}\right)}_{\text{Bremsstrahlung}} + \mathcal{O}(\alpha^2)\right]$$

with $\mathcal{I}(p, p')$ defined in (6.1) as

$$\mathcal{I}(p, p') = \int \frac{d\Omega_k}{4\pi} \sum_r \left|\frac{p'\epsilon^r}{p'\tilde{k}} - \frac{p\epsilon^r}{p\tilde{k}}\right|^2$$

Recall that after introducing the small photon mass μ, we found for the Bremsstrahlung cross section with (6.1) and (6.2)

$$d\sigma(p \to p' + \gamma) = d\sigma(p \to p') \cdot \frac{\alpha}{2\pi} \mathcal{I}(p, p') \log\left(\frac{|\vec{q}|^2}{\mu^2}\right)$$

where we introduced the upper cutoff $|\vec{q}|$ because there the soft-photon approximation breaks down and invalidates the result.
Here we replace this upper bound by the physically motivated cutoff $E_{\text{min}} < |\vec{q}|$ and find

$$d\sigma(p \to p' + \gamma) = d\sigma(p \to p') \cdot \frac{\alpha}{2\pi} \mathcal{I}(p, p') \log\left(\frac{E_{\text{min}}^2}{\mu^2}\right)$$

(Use colors to skip the second equation.)

8. Show (using a Feynman parameter)

$$\mathcal{I}(p, p') \overset{*}{=} 2 f_{\text{IR}}(q^2) \quad \text{for all } p, p' \tag{6.16}$$

Proof: ➲ P&S p. 201, starting at Eq. (6.69); see also P&S pp. 180–181 Eqs. (6.12)-(6.15)

↑ Notes

9. Then

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} &\overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} f_{\text{IR}}(q^2)\log\left(\frac{A}{E_{\text{min}}^2}\right) + \mathcal{O}(\alpha^2)\right] \\ &\overset{-q^2\gg m^2}{\approx} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \underbrace{\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_{\text{min}}^2}\right)}_{\text{Correction by Sudakov double logarithm}} + \mathcal{O}(\alpha^2)\right]\end{aligned} \tag{6.17}$$

→ Independent of μ but dependent on experimental conditions (E_{min}) (which is fine)

We did not evaluate the exact dependence on q (since $A \in \{-q^2, m^2\}$) but for $-q^2 \gg m^2$ (or $-q^2 \to \infty$) the result is correct.

This is an example how an unphysical regularization parameter does *not* affect measurable results.

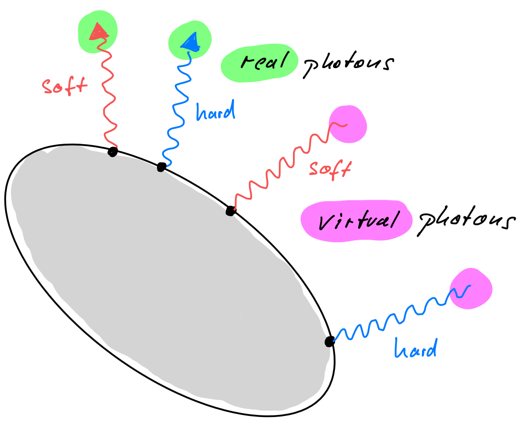
6.3.5 Summation and Interpretation of Infrared Divergences

1. *Problems:*
   - Did not show the cancellation of the IR divergences for higher orders
   - Cross section (6.17) becomes negative (and therefore unphysical) for $E_{\text{min}} \to 0$

   The solution of the second problem will follow from the solution of the first one.

   The following discussion is only a *sketch* and not mathematically rigorous as it skips several technical details that are beyond the scope of this course (and P&S).

2. *Notation:*



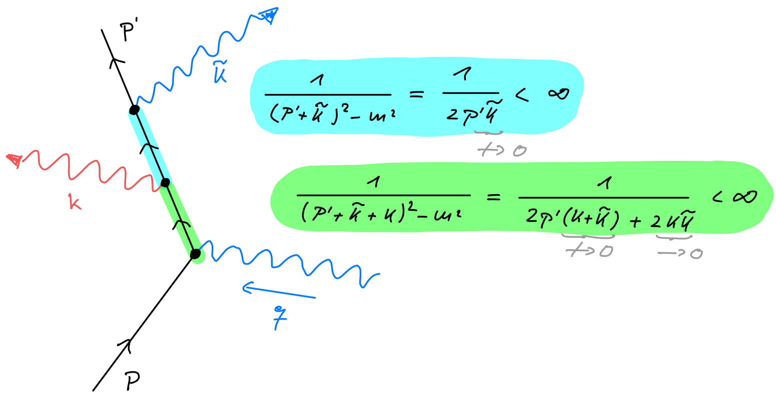
   - *Real photons* (with arrow) are on-shell, transversely polarized and are connected with only one end to the Feynman diagram
   - *Virtual photons* (without arrow) can be off-shell, longitudinally polarized and are connected with both ends to the Feynman diagram

↑ Notes

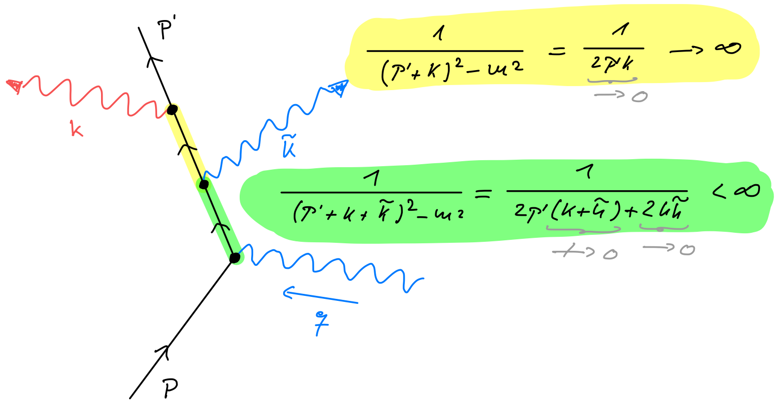
- The momentum of *soft photons* (red) is upper bounded:
  $k_E^2 < E_{\text{min}}^2$ (virtual) and $|\vec{k}| < E_{\text{min}}$ (real)
- The momentum of *hard photons* (blue) is lower boundeded:
  $k_E^2 > E_{\text{min}}^2$ (virtual) and $|\vec{k}| > E_{\text{min}}$ (real)

The subscript E denotes norms in the Euclidean norm after Wick rotation.
Virtual photons are not physical and can never be measured. Real photons can only be measured if they are hard. Soft, real photons cannot be measured due to finite detector sensitivity.

3. *Origin of IR divergences:*



→ *No* IR divergence

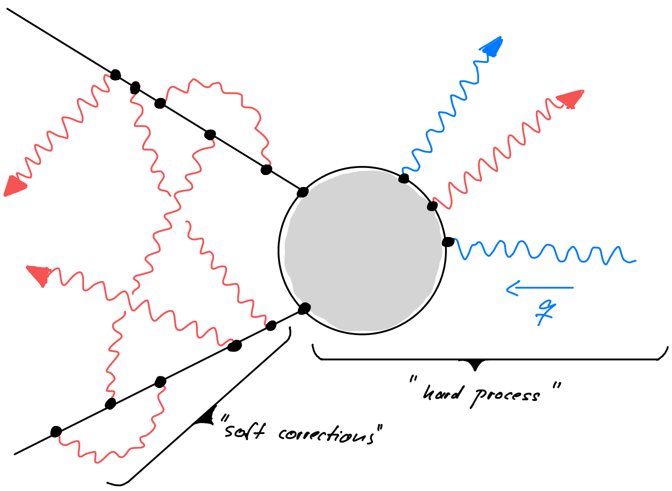


→ IR divergence (yellow)

Soft (real or virtual) photons on the legs of scattering vertices with hard photons lead to IR divergences via singular (i.e. on-shell) fermion propagators.

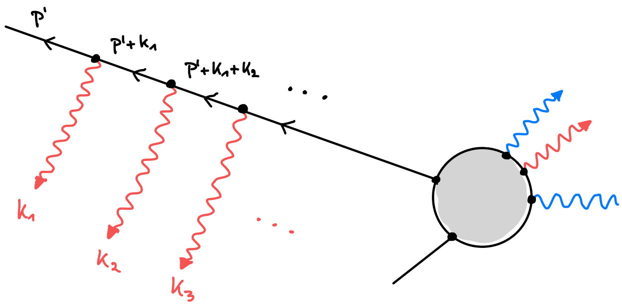
4. → ∢ Generic process

↑ Notes



Todo: Sum all such diagrams!

5. ⊲ Outgoing leg:



Here we do not care whether the soft photons are real or virtual, and, if they are virtual, whether they connect to each other (and form a leg correction) or to the incoming leg (forming a vertex correction).

a) Feynman rules →

$$\overline{u}(p')(-ie\gamma^{\mu_1})\frac{i(\not p' + \not k_1 + m)}{2p'\cdot k_1 + \mathcal{O}(k^2)}(-ie\gamma^{\mu_2})\frac{i(\not p' + \not k_1 + \not k_2 + m)}{2p'\cdot (k_1+k_2) + \mathcal{O}(k^2)}$$
$$\times\,(-ie\gamma^{\mu_n})\frac{i(\not p' + \not k_1 + \dots + \not k_n + m)}{2p'\cdot (k_1+\dots+k_n) + \mathcal{O}(k^2)}\,(i\mathcal{M}_{\text{hard}})\ \dots$$

Note that $k_i^2 = 0$ is only true for real (on-shell) photons. Since we do not specify at this point, whether we interpret the soft photons k_i as real or virtual, we cannot, strictly speaking, set $k_i^2 = 0$ in the denominators (the terms $k_i \cdot k_j$ may even be non-zero for real photons). However, in the soft-photon approximation, we drop the $\mathcal{O}(k^2)$ terms anyway and their presence is irrelevant in the end.

b) Soft-photon approximation ($k_i \to 0$)

- Drop non-singular terms $\not k_i$ in the numerators
- Drop $\mathcal{O}(k^2)$ terms in the denominators
- Use repeatedly $\gamma^\mu(\not p' + m) = (-\not p' + m)\gamma^\mu + 2p'^\mu$ (Dirac algebra)
- Use repeatedly $\overline{u}(p')(-\not p' + m) = 0$ (recall the spin sums (3.11))

↑ Notes

$\overset{\circ}{\to}$

$$\underbrace{\overline{u}(p')\left(e\frac{p'^{\mu_1}}{p'\cdot k_1}\right)\left(e\frac{p'^{\mu_2}}{p'\cdot(k_1+k_2)}\right)\cdots\left(e\frac{p'^{\mu_n}}{p'\cdot(k_1+\cdots+k_n)}\right)}_{=\square}\cdots$$

c) Sum over all orderings of $k_1,\dots,k_n$:

$$\sum_{\pi\in S_n}\square(k_i\mapsto k_{\pi(i)}) = ?$$

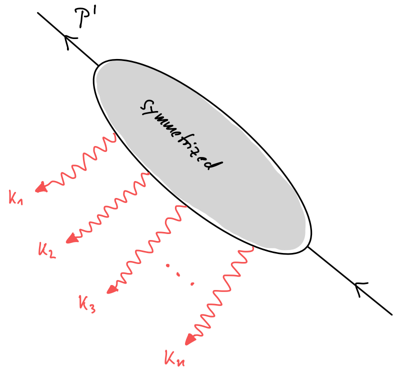
S_n denotes the permutation group of n elements and π is a particular permutation. If we connect two photons k_i and k_j to form a virtual photon (see below), the permutation $k_i\leftrightarrow k_j$ would not be necessary so that we overcount the weight of the diagram by a factor of 2 (recall that virtual photon lines are unoriented). To compensate for that, we multiply by $\frac{1}{2}$ when we form a photon loop (see below).

d) Use

$$\sum_{\pi\in S_n}\frac{1}{p\cdot k_{\pi(1)}}\frac{1}{p\cdot(k_{\pi(1)}+k_{\pi(2)})}\cdots\frac{1}{p\cdot(k_{\pi(1)}+\cdots+k_{\pi(n)})}\overset{\circ}{=}\prod_{i=1}^{n}\frac{1}{p\cdot k_i}$$

Proof by induction over n.

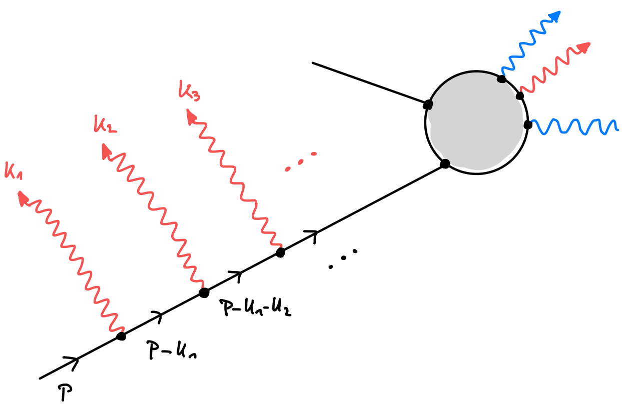
e) Then



$$\approx\overline{u}(p')\prod_i\left(e\frac{p'^{\mu_i}}{p'\cdot k_i}\right)\tag{6.18}$$

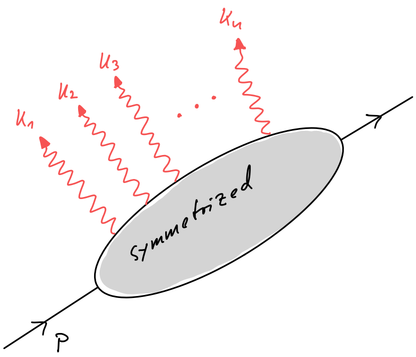
The "$\approx$" hints at the soft-photon approximation.

6. $\triangleleft$ Incoming leg:



↑ Notes

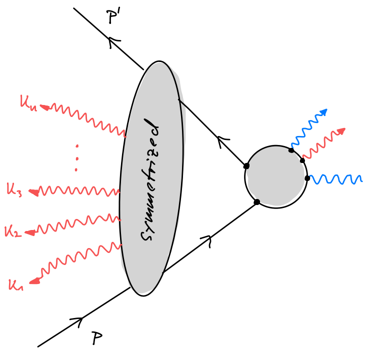
→ The same arguments yield



$$\approx \prod_i \left(-e\frac{p^{\mu_i}}{p\cdot k_i}\right) u(p) \tag{6.19}$$

Only difference: one additional minus sign per photon since $k_i \mapsto -k_i$

7. ∢ Sum over n soft photons attached *either* to the incoming *or* the outgoing leg:
(6.18) & (6.19) →



$$\approx \bar{u}(p')i\mathcal{M}_{\text{hard}}u(p)\times\prod_i e\left(\frac{p'^{\mu_i}}{p'\cdot k_i}-\frac{p^{\mu_i}}{p\cdot k_i}\right) \tag{6.20}$$

This is a process that involves only bremsstrahlung and no vertex loops.

8. *Virtual photon* between vertex i and j:
   a) Set $k_j = -k_i \equiv k$
   b) Multiply by photon propagator $\frac{-ig_{\mu\nu}}{k^2+i\varepsilon}$
   c) Integrate over k
   d) Multiply by $\frac{1}{2}$ to account for the symmetry $k_i \leftrightarrow k_j$ (see note above)

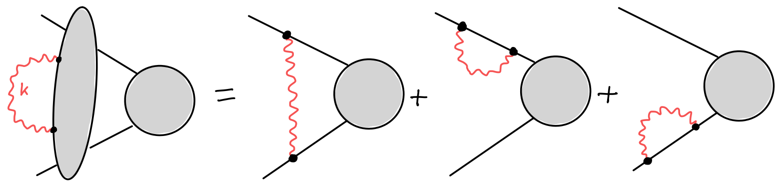
$$\frac{e^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+i\varepsilon}\left(\frac{p'^\mu}{p'\cdot k}-\frac{p^\mu}{p\cdot k}\right)\left(\frac{p'_\mu}{-p'\cdot k}-\frac{p_\mu}{-p\cdot k}\right)\equiv\mathbb{X} \tag{6.21}$$

This prescription allows us to convert two real, soft photons into a virtual soft photon which is a loop correction of either the vertex or one of the two legs.

To evaluate this integral by contour integration, as before, a regularization by introducing a small photon mass $\mu > 0$ is needed to control the IR divergence.

↑ Notes

9. Evaluation of $\mathbb{X}$: $\triangleleft$ Special case with one virtual photon:



$$\approx \bar{u}(p')i\mathcal{M}_{\text{hard}}u(p) \times \mathbb{X}$$

$$\stackrel{!}{=} \bar{u}(p')i\mathcal{M}_{\text{hard}}u(p) \times \underbrace{\left[-\frac{\alpha}{2\pi} f_{\text{IR}}(q^2)\log\left(\frac{-q^2}{\mu^2}\right)\right]}_{\text{Known IR limit of } F_1^{(1)}(q^2)\text{, see (6.15)}}$$

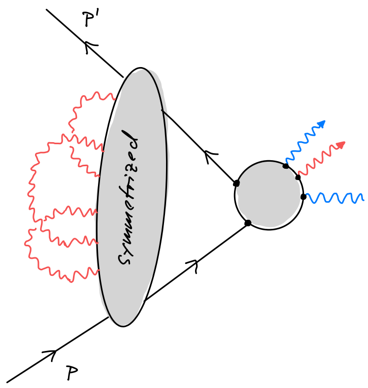
Note that the known IR limit of $F_1^{(1)}(q^2)$ followed *after* ad-hoc subtraction of $F_1^{(1)}(0)$. This is related to the fact that in $\mathbb{X}$ we also sum over the leg corrections which we ignored in our original discussion of the Form factors. The details are quite technical and beyond the scope of this course.

→

$$\boxed{\mathbb{X} = -\frac{\alpha}{2\pi} f_{\text{IR}}(q^2)\log\left(\frac{-q^2}{\mu^2}\right)} \tag{6.22}$$

This result can also be obtained by direct evaluation of the integral (6.21).

10. $\triangleleft$ Sum of arbitrary many soft, virtual photons:
(6.20) & (6.21) →

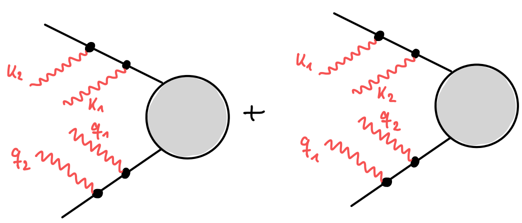


$$\approx \bar{u}(p')i\mathcal{M}_{\text{hard}}u(p) \times \sum_{m=0}^{\infty}\frac{\mathbb{X}^m}{m!}$$

$$= \bar{u}(p')i\mathcal{M}_{\text{hard}}u(p) \times \exp(\mathbb{X}) \tag{6.23}$$

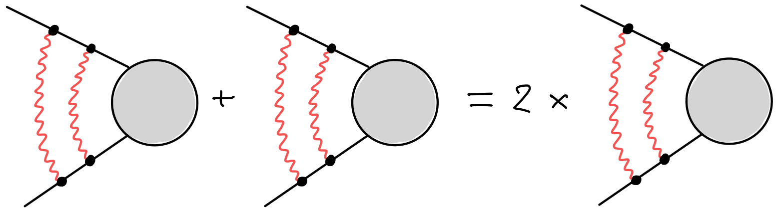
The factors $\frac{1}{m!}$ compensate for overcounting since the order of virtual photons does not matter and gives rise to equivalent diagrams.

As for the factor of $\frac{1}{2}$, this is a consequence of our "symmetrization" above. For instance, symmetrization over $n = 4$ photons includes the (distinct) summands



↑ Notes

which become identical after connecting pairs to virtual photons:



Here, $m = 2$ and $\frac{1}{2!}$ would cancel the factor of 2.

11. $\sphericalangle$ Emission of a *real* photon $k_i = k$:
    a) Multiply by polarization vector $[\epsilon_\mu^r(k)]^*$ (external outgoing photon)
    b) Square the amplitude
    c) Phase-space integration of photon momentum $\vec{k}$ (with upper cutoff $|\vec{k}| < E_{\min}$)
    d) Sum photon polarizations $r = 1, 2$

→

$$\int^{E_{\min}} \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \sum_r e^2 \left| \frac{p' \cdot \epsilon^r}{p' \cdot k} - \frac{p \cdot \epsilon^r}{p \cdot k} \right|^2 \equiv \mathbb{Y}$$

Recall the discussion of bremsstrahlung, i.e. Eqs. (6.1) and (6.2)

$$\sim \frac{\alpha}{\pi} \mathcal{I}(p, p') \log\left(\frac{E_{\min}}{\mu}\right)$$

$$\overset{(6.16)}{=} \frac{\alpha}{\pi} f_{\mathrm{IR}}(q^2) \log\left(\frac{E_{\min}^2}{\mu^2}\right) \tag{6.24}$$

Note that the complex conjugate vanishes because of the absolute value after squaring. We ignore here the amplitude of the hard process.

12. $\sphericalangle$ Cross section for emission of arbitrary number of soft photons:

$$\sum_{n=0}^{\infty} \frac{d\sigma}{d\Omega}(\vec{p} \mapsto \vec{p}' + n\gamma) = \underbrace{\frac{d\sigma}{d\Omega}(\vec{p} \mapsto \vec{p}')}_{\propto |\overline{u}(p') i \mathcal{M}_{\mathrm{hard}} u(p)|^2} \times \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{Y}^n$$

$$= \frac{d\sigma}{d\Omega}(\vec{p} \mapsto \vec{p}') \times \exp(\mathbb{Y}) \tag{6.25}$$

The prefactor $\frac{1}{n!}$ is needed since the outgoing photons are indistinguishable bosons, i.e., whether a photon originates from vertex i or any other outgoing vertex does not change the physical state. Since we treated the vertices as distinct when symmetrizing, we have to compensate for that by $\frac{1}{n!}$.

13. → Measured cross section for process

$$e^-(\vec{p}) \to e^-(\vec{p}') + (\text{Any number of photons with } |\vec{k}| < E_{\min})$$

↑ Notes

to all orders in α is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \overset{(6.23)\,\&\,(6.25)}{\approx} \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp(2\mathbb{X}+\mathbb{Y})$$

$$\overset{(6.22)\,\&\,(6.24)}{=} \left(\frac{d\sigma}{d\Omega}\right)_0 \underbrace{\exp\left[-\frac{\alpha}{\pi} f_{\text{IR}}(q^2)\log\left(\frac{-q^2}{E_{\min}^2}\right)\right]}_{\geq 0 \text{ and } \leq 1 \text{ (for } -q^2 > E_{\min})}$$

$$\overset{-q^2 \gg m^2}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \underbrace{\exp\left[-\frac{\alpha}{2\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_{\min}^2}\right)\right]}_{\text{Sudakov form factor}}$$

The exponent $2\mathbb{X}$ follows because we have to square the amplitude (6.23).

This cross section describes the combination of an *arbitrary* number of soft virtual photons with an *arbitrary* number of soft real photons.

Note 6.2

- As the result is independent of μ, it demonstrates the cancellation of IR divergences in all orders of α.
- We can recover our previous result (6.17) by expanding the exponential. However, for $E_{\min} \to 0$ the exponent becomes large (and negative) so that this expansion is no longer valid. This explains our earlier, unphysical result of purportedly negative scattering cross sections. That is, by lowering our detector sensitivty, higher-order corrections become more and more important to explain the observed cross sections.

6.4 Field-Strength Renormalization

So far, we glossed over radiative corrections to external legs twice:

- When evaluating S-matrix elements perturbatively, we considered only *amputated* Feynman diagrams. We identified the diverging contributions from loops attached to legs as modification of the propagation of particles in an interacting theory (which are not part of the scattering process itself).
- When discussing the scattering of an electron from a heavy target, we ignored the two diagrams with leg corrections, postponing their treatment to the future.

The future has come:

6.4.1 Structure of Two-Point Correlators in Interacting Theories

- Before we discuss the modification of the electron propagator due to radiative corrections in QED, let us first study the general structure of two-point correlators in interacting field theories.
- Note that the results of this discussion are exact and not built on perturbation theory.

Here: $\triangleleft$ ϕ^4-theory (later: QED)

1. Goal: Study structure of $\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle$ in an *interacting* theory
2. *Interpretation for free theory:*
   $\langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle$ = Amplitude of particle to propagate from y to x (for $x^0 > y^0$)
   $\rightarrow$ Effect of interactions?
3. *Mathematical preliminaries:*
   a) $\triangleleft$ Hilbert space of interacting theory $\mathcal{H}_{\text{int}}$
   b) Basis of $\mathcal{H}_{\text{int}}$:
   $[H,\vec{P}]=0 \rightarrow |\lambda_{\vec{p}}\rangle$ eigenstates with energy $E_{\vec{p}}(\lambda)$ and momentum $\vec{p}$
   Here, H is the *interacting* Hamiltonian and the states $|\lambda_{\vec{p}}\rangle$ can contain an arbitrary number of excitations. Note that we will refer to the vacuum state still as $|\Omega\rangle$.
   c) $\triangleleft$ Boost $\Lambda_{\vec{p}} \in \mathrm{SO}^+(1,3)$ such that

$$\Lambda_{\vec{p}}\begin{pmatrix} m_\lambda \\ \vec{0}\end{pmatrix} = \begin{pmatrix} E_{\vec{p}}(\lambda) \\ \vec{p}\end{pmatrix} \quad \text{with} \quad E_{\vec{p}}(\lambda) \equiv \sqrt{|\vec{p}|^2 + m_\lambda^2}$$

$\rightarrow \forall\, |\lambda_{\vec{p}}\rangle\ \exists\, \Lambda_{\vec{p}}\ \exists\, |\lambda_0\rangle\ :\quad |\lambda_{\vec{p}}\rangle = U(\Lambda_{\vec{p}})|\lambda_0\rangle$ with

$$\begin{aligned} H|\lambda_0\rangle &= m_\lambda|\lambda_0\rangle \quad &&\text{and} \quad \vec{P}|\lambda_0\rangle = 0 \\ H|\lambda_{\vec{p}}\rangle &= E_{\vec{p}}(\lambda)|\lambda_{\vec{p}}\rangle \quad &&\text{and} \quad \vec{P}|\lambda_{\vec{p}}\rangle = \vec{p}|\lambda_{\vec{p}}\rangle \end{aligned}$$

Note that $P^\mu = (H,\vec{P})$ transforms like a 4-vector,

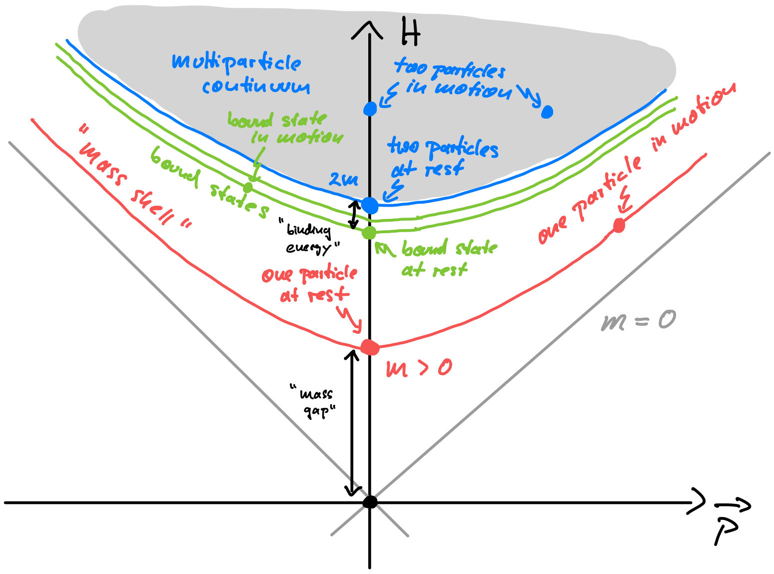
$$U^\dagger(\Lambda)P^\mu U(\Lambda) = \Lambda^\mu_\nu P^\nu$$

↑ Notes

so that from $P^\mu|\lambda_0\rangle = (m_\lambda, \vec{0})^\mu|\lambda_0\rangle$ follows

$$\begin{aligned} P^\mu|\lambda_{\vec{p}}\rangle &= P^\mu U(\Lambda_{\vec{p}})|\lambda_0\rangle \\ &= U(\Lambda_{\vec{p}})U^\dagger(\Lambda_{\vec{p}})P^\mu U(\Lambda_{\vec{p}})|\lambda_0\rangle \\ &= U(\Lambda_{\vec{p}})(\Lambda_{\vec{p}})^\mu{}_\nu P^\nu|\lambda_0\rangle \\ &= (\Lambda_{\vec{p}})^\mu{}_\nu(m_\lambda, \vec{0})^\nu U(\Lambda_{\vec{p}})|\lambda_0\rangle \\ &= (E_{\vec{p}}(\lambda), \vec{p})^\mu|\lambda_{\vec{p}}\rangle \end{aligned}$$

d) Typical spectrum of $P^\mu = (H, \vec{P})$ of an interacting theory with mass gap:



Every state $|\lambda_0\rangle$ with vanishing momentum and "mass" (=rest energy) m_λ is associated to a hyperboloid (the "mass shell") of states $\lambda_{\vec{p}}$ with energies $E_{\vec{p}}(\lambda) = \sqrt{|\vec{p}|^2 + m_\lambda^2}$ that are generated by boosts.

Note that the two-particle states occupy a *continuum* of hyperboloids because the energy of two particles can take any value $2m \leq E < \infty$ for vanishing total momentum $\vec{p} = 0$.

Depending on the interactions, *bound states* of two particles can exist where the energy $2m$ of the free particles is reduced by the binding energy. In this course, we do not discuss bound states of interacting QFTs.

e) Identity on $\mathcal{H}_{\text{int}}$:

$$(\mathbb{1})_{1-\text{particle}} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}|\vec{p}\rangle\langle\vec{p}|$$

Generalization $\rightarrow$ $$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}(\lambda)}|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|$$

Here, we choose the same normalization as for one-particle states.
The sum runs over all zero-momentum states $|\lambda_0\rangle$.

↑ Notes

4. Insert identity between fields →

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}(\lambda)}\langle\Omega|\phi(x)|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|\phi(y)|\Omega\rangle + \text{const}$$

   Here, we drop the constant term coming from $\langle\Omega|\phi(x)|\Omega\rangle$.

5. With

$$\begin{aligned}\langle\Omega|\phi(x)|\lambda_{\vec{p}}\rangle &= \langle\Omega|e^{iPx}\phi(0)e^{-iPx}|\lambda_{\vec{p}}\rangle\\ &= \langle\Omega|\phi(0)|\lambda_{\vec{p}}\rangle\, e^{-ipx}|_{p^0=E_{\vec{p}}(\lambda)}\\ &= \underbrace{\langle\Omega|U(\Lambda_{\vec{p}})}_{\langle\Omega|}\underbrace{U^\dagger(\Lambda_{\vec{p}})\phi(0)U(\Lambda_{\vec{p}})}_{\phi(\Lambda_{\vec{p}}^{-1}0)=\phi(0)}|\lambda_0\rangle\, e^{-ipx}|_{p^0=E_{\vec{p}}(\lambda)}\\ &= \langle\Omega|\phi(0)|\lambda_0\rangle\, e^{-ipx}|_{p^0=E_{\vec{p}}(\lambda)}\end{aligned}$$

   Here we use the Poincaré invariance of the (interacting) vacuum, $U(\Lambda)|\Omega\rangle = |\Omega\rangle$ and $e^{-iPx}|\Omega\rangle = |\Omega\rangle$, and the scalar nature of the field, $U(\Lambda)\phi(x)U^\dagger(\Lambda) = \phi(\Lambda x)$.

6. …we find

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}(\lambda)}\, e^{-ip(x-y)}|_{p^0=E_{\vec{p}}(\lambda)}$$

   Introduce p^0-integration [recall Eq. (2.5) ff.]

$$\stackrel{x^0\geq y^0}{=} \sum_\lambda |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \underbrace{\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_\lambda^2+i\varepsilon}\, e^{-ip(x-y)}}_{\equiv D_F(x-y;m_\lambda^2)}$$

$$\stackrel{x^0\leq y^0}{=} \langle\Omega|\phi(y)\phi(x)|\Omega\rangle$$

7. → *Källén-Lehmann spectral representation* of the two-point correlator:

$$\boxed{\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)D_F(x-y;M^2)}$$

   with *spectral density*

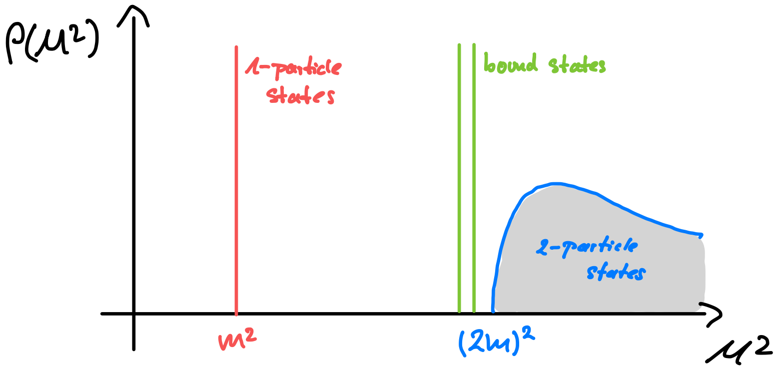
$$\boxed{\rho(M^2) = 2\pi\sum_\lambda \delta(M^2-m_\lambda^2)|\langle\Omega|\phi(0)|\lambda_0\rangle|^2}$$

   Interpretation: The propagation amplitude of a single particle in an interacting QFT is therefore the sum of propagators of all possible states with mass m_λ that can be created by a single field from the interacting vacuum: $|\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \neq 0$.

   Note that this result is exact!

8. Typical spectral density of an interacting theory with mass gap:

↑ Notes



→

$$\rho(M^2) = 2\pi\delta(M^2 - m^2)\cdot Z + \{\,\text{multi-particle states for } M^2 \gtrsim (2m)^2\,\} \tag{6.26}$$

with (We assume here that the theory has only one massive particle $\lambda = 1$.)

Field-strength renormalization	$Z = \lvert\langle\Omega\vert\phi(0)\vert\lambda_0 = 1_0\rangle\rvert^2$
Physical mass	$m = m_1$
	(given by $H\vert 1_0\rangle = m_1\vert 1_0\rangle$)
Bare mass	m_0
	(given by $H = \dots 1/2\, m_0^2\,\phi^2 \dots$)

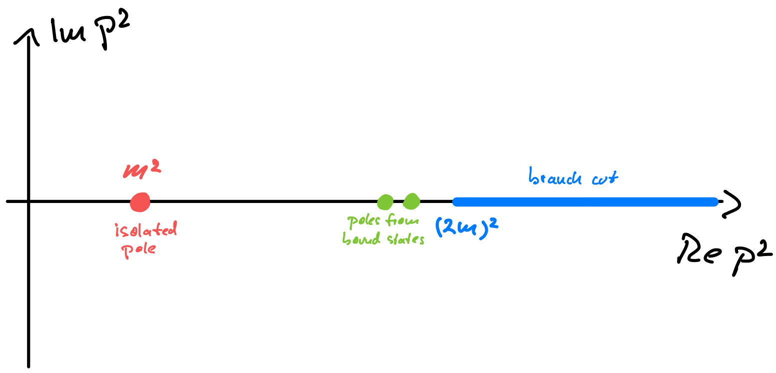
- *Free theory:* $Z = \lvert\langle 0\vert\phi(0)\vert\vec{p} = 0\rangle_0\rvert^2 = 1$ and $m = m_0$
- *Interacting theory:* $Z \neq 1$ and $m \neq m_0$
- Only m is observable
- Field-strength renormalization = Probability $\lvert\langle\Omega\vert\phi(0)\vert 1_0\rangle\rvert^2$ that the field $\phi(0)$ creates the interacting single particle state $\vert 1_0\rangle$ from the interacting vacuum $\vert\Omega\rangle$.

9. Fourier transform of the two-point correlator:

$$\begin{aligned}\int d^4x\, e^{ipx}\langle\Omega\vert\mathcal{T}\phi(x)\phi(0)\vert\Omega\rangle &= \int_0^\infty \frac{dM^2}{2\pi}\frac{i\rho(M^2)}{p^2 - M^2 + i\varepsilon}\\ &\overset{(6.26)}{=} \frac{i\cdot Z}{p^2 - m^2 + i\varepsilon} + \int_{\sim(2m)^2}^\infty \frac{dM^2}{2\pi}\frac{i\rho(M^2)}{p^2 - M^2 + i\varepsilon}\\ &\overset{\text{free}}{=} \frac{i\cdot 1}{p^2 - m_0^2 + i\varepsilon}\end{aligned}$$

→ Typical analytical structure in the complex p^2-plane:

↑ Notes



6.4.2 Application to QED: The Electron Self-Energy

Goal: Use perturbation theory to

1. verify the non-perturbative results from above for QED and
2. compute m and Z in first order of α.

For details ➔ Problemset 10.

1. ϕ^4-theory $\mapsto$ QED

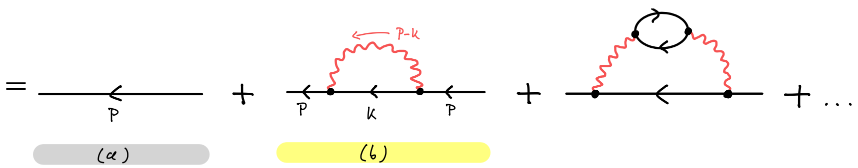
$$\int d^4x\, e^{ipx} \langle\Omega|\mathcal{T}\Psi(x)\overline{\Psi}(0)|\Omega\rangle \overset{\circ}{=} \frac{iZ_2(\not{p}+m)}{p^2-m^2+i\varepsilon} + \dots$$

   The name "Z_2" for the field-strength renormalization is conventional.
   Add details of derivation to script.

2. On the other side

$$\int d^4x\, e^{ipx} \langle\Omega|\mathcal{T}\Psi(x)\overline{\Psi}(0)|\Omega\rangle$$

   Feynman rules for correlation functions



3. α^0-order:

$$(\text{a}) = \frac{i(\not{p}+m_0)}{p^2-m_0^2+i\varepsilon}$$

4. α^1-order:

$$(\text{b}) = \frac{i(\not{p}+m_0)}{p^2-m_0^2+i\varepsilon} \times \underbrace{\left[(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k}+m_0)}{k^2-m_0^2+i\varepsilon} \gamma_\mu \frac{-i}{(p-k)^2+i\varepsilon}\right]}_{\equiv -i\Sigma_2(p)} \times \frac{i(\not{p}+m_0)}{p^2-m_0^2+i\varepsilon} \tag{6.27}$$

↑ Notes

→ IR- and UV-divergences
(Use regularization with μ, Λ and the techniques developed for the vertex corrections.)
Evaluation: ➲ Problemset 10 →

$$\Sigma_2(p) \overset{\Lambda\to\infty}{\sim} \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0 - x\not{p})\,\log\left[\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right] \tag{6.28}$$

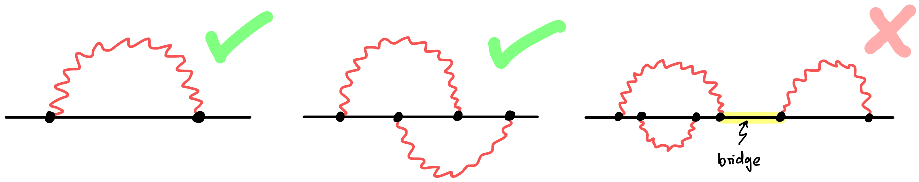
$\overset{\circ}{\to}$ This expression has a branch cut (in the complex p^2-plane) emanating from $p^2 = (m_0+\mu)^2$, i.e., at the threshold of a two-particle state consisting of an electron of mass m_0 and a photon of (artificial) mass μ. There is, however, no simple pole at $p^2 = m^2$.

5. Summation to all orders in α:
   This is needed to recover the isolate one-particle pole at $p^2 = m^2$.

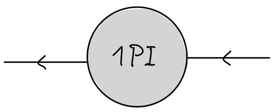
   a) *Definitions:*

   One-particle irreducible (1PI) diagram = *Bridgeless* one-particle diagram

   A *bridgeless* graph cannot be separated into two pieces by deleting a single edge.
   Examples:



   Let furthermore

   $-i\,\Sigma(p) :=$ { Sum of all 1PI diagrams } $\equiv$ 

   $= -i\,\Sigma_2(p) + \mathcal{O}(\alpha^2)$

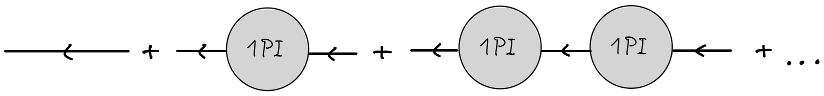
   $\Sigma(p)$ does *not* include the propagators of the two external legs, recall (6.27).

↑ Notes

b) Then

$$\int d^4x\, e^{ipx} \langle\Omega|\mathcal{T}\Psi(x)\overline{\Psi}(0)|\Omega\rangle$$

$$= \{\text{ Sum of all one-particle diagrams }\}$$

=


$$= \frac{i(\not{p}+m_0)}{p^2-m_0^2} + \frac{i(\not{p}+m_0)}{p^2-m_0^2}[-i\Sigma(p)]\frac{i(\not{p}+m_0)}{p^2-m_0^2} + \dots$$

Use $\not{p}^2 = p^2$ and $[\Sigma(p), \not{p}] = 0$ and write $\Sigma(\not{p})$ instead of $\Sigma(p)$

$$= \frac{i}{\not{p}-m_0}\sum_{n=0}^{\infty}\left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)^n$$

Geometric series (for matrices)

$$= \frac{i}{\not{p}-m_0}\cdot\frac{1}{\mathbb{1}-\frac{\Sigma(\not{p})}{\not{p}-m_0}}$$

$$= \frac{i}{\not{p}-m_0-\Sigma(\not{p})}$$

Here we omit the infinitesimals ε for the sake of simplicity.

It is $[\Sigma(p), \not{p}] = 0$ since, similar to our discussion of the general structure of the vertex function Γ^μ previously, the matrix $\Sigma(p)$ must be a Lorentz scalar, $\Lambda_{\frac{1}{2}}\Sigma(p)\Lambda_{\frac{1}{2}}^{-1} = \Sigma(\Lambda p)$, and therefore can only be constructed from contracted pairs of γ-matrices and the four-vector p, i.e.,

$$\Sigma(p) = f(\gamma^\mu p_\mu) + g(p^\mu p_\mu) + c(\gamma^\mu\gamma_\mu) = f(\not{p}) + g(\not{p}^2) + c = \Sigma(\not{p})$$

where f and g are arbitrary (analytic) functions and c is a constant (recall $\gamma^\mu\gamma_\mu = 4$); note that $\Lambda_{\frac{1}{2}}\not{p}\Lambda_{\frac{1}{2}}^{-1} = \not{p}'$. This also demonstrates that Σ can equivalently be interpreted as a function of $\not{p}$.

6. Laurent series:

$$\frac{i}{\not{p}-m_0-\Sigma(\not{p})} \stackrel{!}{=} \frac{iZ_2}{\not{p}-m} + \dots$$

→ Expect simple pole for $\not{p} = \mathbb{1}\cdot m = m$:

$$\boxed{m - m_0 = \Sigma(\not{p}=m)}$$

This is an implicit equation for the physical mass m.

→ Expand denominator around this root:

$$\not{p}-m_0-\Sigma(\not{p}) = (\not{p}-m)\cdot\left(\mathbb{1}-\frac{\mathrm{d}\Sigma}{\mathrm{d}\not{p}}\right)\bigg|_{\not{p}=m} + \mathcal{O}\left((\not{p}-m)^2\right)$$

$$\boxed{\Rightarrow\quad Z_2 = \left(1-\frac{\mathrm{d}\Sigma}{\mathrm{d}\not{p}}\bigg|_{\not{p}=m}\right)^{-1}}$$

↑ Notes

7. Results in leading order $\mathcal{O}(\alpha)$:
   a) *Physical mass:*

$$\begin{aligned}\delta m = m - m_0 = \Sigma(\not p = m) \\ &= \Sigma_2(\not p = m) + \mathcal{O}(\alpha^2) \\ &= \Sigma_2(\not p = m_0) + \mathcal{O}(\alpha^2)\end{aligned}$$

Use (6.28), ➔ Problemset 10

$$\stackrel{\Lambda\to\infty}{\sim} \frac{3\alpha}{4\pi} m_0 \log\left(\frac{\Lambda^2}{m_0^2}\right) \xrightarrow{\Lambda\to\infty} \infty$$

In the last line we expanded Σ_2 around m_0 in α and kept only the lowest order.
→ Mass shift is UV-divergent!

   b) *Field-strength renormalization:*
   Use $\frac{1}{1-x} = 1 + x + \mathcal{O}(x^2)$.

$$\begin{aligned}\delta Z_2 &= Z_2 - 1 \\ &= \left.\frac{\mathrm{d}\Sigma}{\mathrm{d}\not p}\right|_{\not p = m} + \mathcal{O}(\alpha^2) \\ &= \left.\frac{\mathrm{d}\Sigma_2}{\mathrm{d}\not p}\right|_{\not p = m} + \mathcal{O}(\alpha^2) \\ &\stackrel{\circ}{=} \frac{\alpha}{2\pi}\int_0^1 dx \left\{\begin{aligned}&-x\log\left[\frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2}\right] \\ &+2(2-x)\frac{x(1-x)m^2}{(1-x)^2m^2 + x\mu^2}\end{aligned}\right\}\end{aligned}$$

Note that the lowest order of $\Sigma = \Sigma_2 + \mathcal{O}(\alpha^2)$ is *linear* in α since we excluded the external propagators from the definition of Σ.
→ Field-strength renormalization is also UV-divergent!

Note 6.3

- The diverging mass of the electron is classically expected as it includes the energy of its electrostatic field in the vicinity of the electron. This energy diverges for a charged sphere with vanishing radius r_e as $\frac{1}{r_e} \sim \Lambda \to \infty$.
- Our results on QED processes all involved the bare mass m_0. To compare them with experiments, we should express m_0 in terms of the observed mass m, which, however, does not make sense since their difference diverges! This conceptual impasse motivates the introduction of a *renormalized perturbation theory* for QED where the physical mass m instead of m_0 shows up in the Feynman propagator (➔ later).
- It is easy to show that

$$\delta Z_2 = -F_1^{(1)}(0)$$

where $F_1^{(1)}(0)$ was the term that we subtracted from the form factor of the vertex correction to ensure that $F_1(0) = 1$, recall (6.13). An application of the *LSZ reduction formula* yields a correction to the form factor, namely

$$F_1(q^2) = 1 + F_1^{(1)}(q^2) + \delta Z_2 = 1 + F_1^{(1)}(q^2) - F_1^{(1)}(0)$$

which justifies our subtraction.

↑ Notes

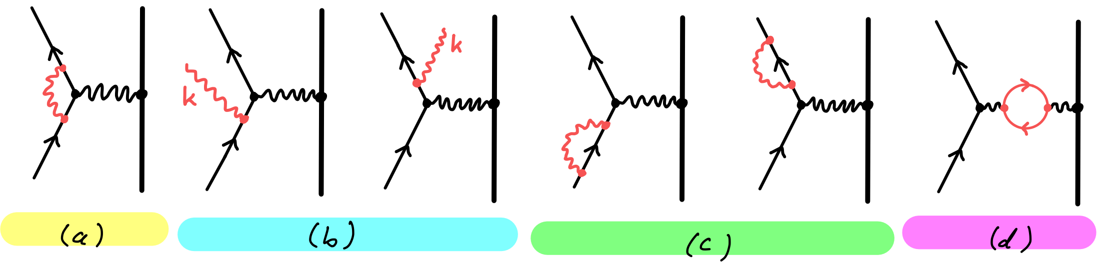
Problem Set 10

(due **26.06.2020**)

1. Infrared divergence of the electron vertex function: Technical calculations to bridge gaps in the lecture
2. The electron self-energy: One-loop corrections for the renormalized electron mass (extension to the lecture)

6.5 Electric Charge Renormalization

Remember the radiative corrections:



(a) Vertex correction
→ Form factors and anomalous magnetic moment, IR-div. and UV-div., had to subtract $F_1^{(1)}(0)$ *ad hoc*

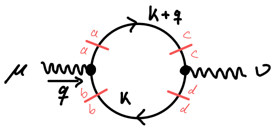
(b) Soft bremsstrahlung
→ IR-div. cancelled with IR-div. of vertex correction

(c) Electron self-energy
→ Field-strength and mass renormalization, explained subtraction of $F_1^{(1)}(0)$ and thereby removed UV-div. of vertex correction

Here $\sphericalangle$ *Vacuum polarization diagram* (d) [Mark diagram (d)] → *Photon self-energy*
This is analogous to the electron self-energy (c) which modified the electron propagation due to virtual photons. Here, the *photon propagator* will be modified due to the presence of virtual electron-positron pairs. This will lead to momentum dependent modifications of the strength of the electromagnetic field.

1. *One-loop correction*:
   As before, we exclude the photon propagators of the legs from all expressions.

↑ Notes



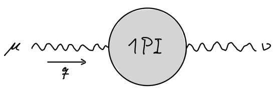
$$= \underbrace{(-1)}_{\text{Fermion loop}} \quad (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu_{ab} \frac{i(\not k + m)_{bd}}{k^2 - m^2} \gamma^\nu_{dc} \frac{i(\not k + \not q + m)_{ca}}{(k+q)^2 - m^2}$$

$$= (-1)(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i(\not k + m)}{k^2 - m^2} \gamma^\nu \frac{i(\not k + \not q + m)}{(k+q)^2 - m^2}\right]$$

$$\equiv i\Pi_2^{\mu\nu}(q) \tag{6.29}$$

The sign of the fermion loop follows from the contraction $\overline{\Psi}\Psi\overline{\Psi}\Psi = -\overline{\Psi}\Psi\overline{\Psi}\Psi$ where we used that $\overline{\Psi}\Psi = -\Psi\overline{\Psi}$ for fermionic fields (recall that $\Psi(x)\overline{\Psi}(y) = S_F(x-y)$ is the Feynman propagator).

2. ∢ Sum of all *1-particle irreducible diagrams*:



$$\equiv i\Pi^{\mu\nu}(q) = i\left[\Pi_2^{\mu\nu}(q) + \mathcal{O}(\alpha^2)\right]$$

What follows is analogous to our discussion of the generic structure of the vertex correction Γ^μ:

a) Only tensors available: $g^{\mu\nu}$ and $q^\mu q^\nu \rightarrow \Pi^{\mu\nu}(q) = A(q^2)\, g^{\mu\nu} + B(q^2)\, q^\mu q^\nu$

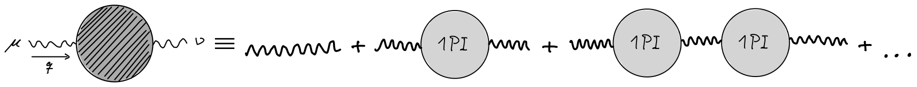
b) Ward identity (recall Eq. (6.5) and references below for the vertex correction Γ^μ):
$q_\mu \Pi^{\mu\nu}(q) \stackrel{*}{=} 0 \rightarrow B = -\frac{A}{q^2} \rightarrow \Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \cdot \frac{A}{q^2}$

c) $\stackrel{*}{\rightarrow} \Pi^{\mu\nu}(q)$ has no pole for $q^2 = 0$
Motivation: Poles at $q^2 = 0$ arise from massless intermediate states with propagator $\frac{-ig_{\mu\nu}}{q^2 + i\varepsilon}$ – but these do not occur in 1-particle irreducible diagrams. A rigorous proof of this statement is possible but non-trivial.
$\rightarrow \Pi(q^2) \equiv \frac{A(q^2)}{q^2}$ regular at $q^2 = 0$
$\rightarrow$

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu) \cdot \Pi(q^2) \tag{6.30}$$

↑ Notes

3. ⊲ Sum of *all* diagrams with two photon legs:



$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left[i(q^2 g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)\right]\frac{-ig_{\sigma\nu}}{q^2} + \dots$$

Define $\Delta^\rho_\nu \equiv \delta^\rho_\nu - q^\rho q_\nu / q^2$ and use $g^{\rho\sigma} g_{\sigma\nu} = \delta^\rho_\nu$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\nu \Pi(q^2) + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\sigma \Delta^\sigma_\nu \Pi^2(q^2) + \dots$$

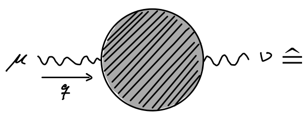
Use $\Delta^\rho_\sigma \Delta^\sigma_\nu = \Delta^\rho_\nu$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\nu \underbrace{\sum_{n=1}^{\infty} \Pi^n(q^2)}_{=\frac{1}{1-\Pi(q^2)}-1}$$

Geometric series

$$= \frac{-i}{q^2[1-\Pi(q^2)]}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right)$$

4. ⊲ $\Pi^{\mu\nu}(q)$ contracted with a vertex to form an S-matrix element:
   Ward identity $\overset{*}{\rightarrow}$



$$\hat{=} \frac{-ig_{\mu\nu}}{q^2[1-\Pi(q^2)]}$$

The "≙" signifies that this equation is only true for computations in S-matrix elements. For a proof ➔ P&S pp. 238–244, in particular Eq. (7.66)

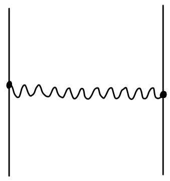
Note that this propagator has a pole at $q^2 = 0$ (to all orders in α) so that the photon remains exactly massless. Formally, this is a consequence of the Ward identity.

5. *Charge renormalization*:

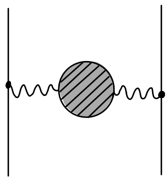
   a) Define

$$Z_3 \equiv \frac{1}{1-\Pi(0)}$$

This is a finite number since $\Pi(q^2)$ is regular at $q^2 = 0$ (and we assume $|\Pi| < 1$ as otherwise the resummation with the geometric series is not justified).
Then, for $q^2 \to 0$ (i.e. almost-on-shell photons)



$$= \dots \frac{e^2 g_{\mu\nu}}{q^2} \dots \quad \rightarrow$$



$$= \dots \frac{Z_3 e^2 g_{\mu\nu}}{q^2} \dots$$

↑ Notes

b) → *Charge renormalization:*

Bare charge	$e_0 \quad (\text{given by } \mathcal{H}_{\text{int}} = e_0 \overline{\Psi} \gamma^\mu \Psi A_\mu)$
Physical charge	$e \equiv \sqrt{Z_3} e_0$
Fine-structure constant	$\frac{e^2}{4\pi} = \alpha \equiv Z_3 \alpha_0 = Z_3 \frac{e_0^2}{4\pi}$

Note that the bare charge e_0 was previously called e. The charge measured in experiments is the physical mass $e = \sqrt{Z_3} e_0$, hence the new notation.

That is, for soft photons, $q^2 \to 0$, we can incorporate the effects of the vacuum polarization diagrams by simply replacing the bare charge e_0 by the renormalized, physical charge e in computations of S-matrix elements.

Note that in lowest order α^0, it is $Z_3 = 1$ so that $e = e_0$ and $\alpha = \alpha_0$. In general, $Z_3 = 1 + \mathcal{O}(\alpha_0)$ so that $\alpha = \alpha_0 + \mathcal{O}(\alpha_0^2)$. In particular, we can write $\mathcal{O}(\alpha^2) = \mathcal{O}(\alpha_0^2)$.

c) In addition, for $q^2 \neq 0$ and $\Pi(q^2) = \Pi_2(q^2) + \mathcal{O}(\alpha^2)$, each virtual photon line comes with (the charges come from the interaction vertices)

$$\frac{-ig_{\mu\nu}}{q^2} \cdot \frac{e_0^2}{1 - \Pi(q^2)} = \frac{-ig_{\mu\nu}}{q^2} \cdot \frac{e^2[1 - \Pi(0)]}{1 - \Pi(q^2)}$$

$$= \frac{-ig_{\mu\nu}}{q^2} \cdot \frac{e^2[1 - \Pi_2(0)]}{1 - \Pi_2(q^2)} + \mathcal{O}(\alpha^2)$$

Use that $(1 - x) = (1 + x)^{-1} + \mathcal{O}(x^2)$

$$= \frac{-ig_{\mu\nu}}{q^2} \cdot \frac{e^2}{[1 - \Pi_2(q^2)] \cdot [1 + \Pi_2(0)]} + \mathcal{O}(\alpha^2)$$

$$= \frac{-ig_{\mu\nu}}{q^2} \cdot \frac{e^2}{1 - [\Pi_2(q^2) - \Pi_2(0)]} + \mathcal{O}(\alpha^2)$$

→ q^2-dependent charge/fine-structure constant:

$$\alpha_{\text{eff}}(q^2) \equiv \frac{e_0^2/4\pi}{1 - \Pi(q^2)} = \frac{\alpha}{1 - [\Pi_2(q^2) - \Pi_2(0)]} + \mathcal{O}(\alpha^2)$$

That is, for arbitrary momenta, the effect of replacing the tree-level propagator with the full propagator is a *q^2-dependent electric charge*, or, equivalently, fine-structure constant.

6. *Computation of* Π_2:

a) From (6.29):
We use m and e instead of m_0 and e_0 since $\frac{i\alpha}{\not{k} - m} = \frac{i\alpha_0}{\not{k} - m_0} + \mathcal{O}(\alpha_0^2)$ and we are

↑ Notes

only interested in linear order corrections (note that Π_2 is already of order α_0).

$$i\Pi_2^{\mu\nu}(q) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i(\not k + m)}{k^2 - m^2}\gamma^\nu \frac{i(\not k + \not q + m)}{(k+q)^2 - m^2}\right]$$

Trace identities

$$= -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}(k\cdot(k+q) - m^2)}{(k^2-m^2)((k+q)^2-m^2)}$$

Feynman parameter, Substitution $l \equiv k + xq$, Wick rotation $l^0 \equiv i l_E^0$

$$\stackrel{\circ}{=} \quad -4ie^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \times \frac{-\frac{2}{d}g^{\mu\nu}l_E^2 + g^{\mu\nu}l_E^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)}{(l_E^2+\Delta)^2} \tag{6.31}$$

where $\Delta \equiv m^2 - x(1-x)q^2$

In the last line, the term $-\frac{2}{d}g^{\mu\nu}l_E^2$ in the numerator follows for spacetime dimension $d = 4$ from Eq. (6.8) ff. (also all terms linear in l_E have been dropped). We keep the spacetime dimension d explicitly as we need it for the dimensional regularization below.

b) Strong UV-divergence: ∢ UV-cutoff $|l_E| < \Lambda$, then

$$i\Pi_2^{\mu\nu}(q) \sim e^2\Lambda^2 g^{\mu\nu} \xrightarrow{\Lambda\to\infty} \infty$$

This follows simply by power counting.
Note that this result also *violates* the Ward identity $q_\mu \Pi_2^{\mu\nu} = 0$ as there is no corresponding term $\propto q^\mu q^\nu$; this violation produces a (infinite) photon mass!

→ To make sense of this result (and restore the Ward identity), a *regularization* is needed!

c) *Dimensional regularization:* (Proof ➲ Problemset 11)

i. Lower the spacetime dimension $d \in \mathbb{N}$ until the UV-divergence vanishes
ii. Generalize all expressions to $d \in \mathbb{R}$ (below even $d \in \mathbb{C}$ is fine)
iii. Take the limit $d \nearrow 4$ in observable quantities

We could also use *Pauli-Villars* regularization (with the same results), which, however, is in this case much more complicated than dimensional regularization.

For $d \in \mathbb{C}$ we find (Proof ➲ Problemset 11)

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{6.32}$$

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{6.33}$$

For $d \in \mathbb{N}$ these are *identities* that must be proven, for $d \in \mathbb{C}\setminus\mathbb{N}$, these are *definitions* of the left-hand side.

↑ Notes

$\sphericalangle\; n = 2$:
$\Gamma(z)$ has poles at $z = 0, -1, -2, \ldots \rightarrow \Gamma(2 - \frac{d}{2})$ has isolated poles at $d = 4, 6, \ldots$
$\sphericalangle\; d = 4 - \varepsilon$ and use

$$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma(\varepsilon/2) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon) \tag{6.34}$$

γ: Euler-Mascheroni constant

Note: $g_{\mu\nu}g^{\mu\nu} = d$, so that in invariant integrals over spacetime the substitution

$$l^\mu l^\nu \mathrel{\hat{=}} \frac{1}{d} l^2 g^{\mu\nu} \tag{6.35}$$

is valid [generalization of Eq. (6.8)]

d) Evaluate (6.31) with (6.32) & (6.33) & (6.35) (and use $z\Gamma(z) = \Gamma(1+z)$) →

$$i\Pi_2^{\mu\nu}(q) \mathrel{\mathring{=}} (q^2 g^{\mu\nu} - q^\mu q^\nu) \cdot i\Pi_2(q^2)$$

$$\text{with} \quad \Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{x(1-x)\Gamma(2-\frac{d}{2})}{\underbrace{[m^2 - x(1-x)q^2]}_{=\Delta}{}^{2-d/2}}$$

Note that $\Pi_2^{\mu\nu}(q)$ has the expected form (and satisfies the Ward identity).

e) Use (6.34) to expand in ε:

$$\Pi_2(q^2) \mathrel{\mathring{=}} \frac{-2\alpha}{\pi} \int_0^1 dx\, x(1-x) \left[\frac{2}{\varepsilon} - \log(\Delta) - \gamma + \log(4\pi)\right] + \mathcal{O}(\varepsilon) \tag{6.36}$$

To show this, use $\Delta^{-2+d/2} = 1 - \frac{\varepsilon}{2}\log(\Delta) + \mathcal{O}(\varepsilon^2)$ and $(4\pi)^{-d/2} = (4\pi)^{-2}(1 + \frac{\varepsilon}{2}\log(4\pi)) + \mathcal{O}(\varepsilon^2)$, and keep only constant and diverging terms.

7. $\mathcal{O}(\alpha)$ *charge renormalization:*

$$\frac{e^2 - e_0^2}{e_0^2} = Z_3 - 1 = \frac{\Pi(0)}{1 - \Pi(0)}$$
$$= \Pi_2(0) + \mathcal{O}(\alpha^2)$$
$$\overset{\varepsilon\to 0}{\sim} -\frac{2\alpha}{3\pi\varepsilon} \xrightarrow{\varepsilon\searrow 0} -\infty$$

→ If the observed charge is finite, $-\infty < e < 0$, the bare charge diverges, $e_0 = -\infty$
Note that e_0 is not observable so that this is not a falsifiable prediction of QED.

8. $\mathcal{O}(\alpha)$ *q^2-dependence of $\alpha_{\text{eff}}(q^2)$* depends on
(This is an experimentally observable prediction.)

$$\hat{\Pi}_2(q^2) \equiv \Pi_2(q^2) - \Pi_2(0) = \frac{-2\alpha}{\pi} \int_0^1 dx\, x(1-x) \log\left(\frac{m^2}{m^2 - x(1-x)q^2}\right)$$

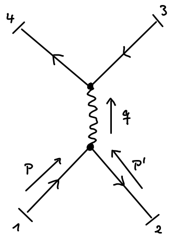
Note that the UV-divergence for $\varepsilon \to 0$ drops out!

↑ Notes

9. *Analysis & Interpretation of* $\hat{\Pi}_2(q^2)$:

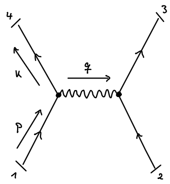
   a) *Note:*

s-channel:



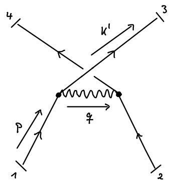
$\Rightarrow \quad s \equiv (p+p')^2 = q^2 \quad > 0$

t-channel:



$\Rightarrow \quad t \equiv (p-k)^2 = q^2 \quad < 0$

u-channel:



$\Rightarrow \quad u \equiv (p-k')^2 = q^2 \quad < 0$

The eponymous variables s, t, and u are known as *Mandelstam variables*.
To show the inequalities, use the Cauchy-Schwarz inequality.

→ *t- and u-channel:* $\hat{\Pi}_2(q^2)$ is analytic everywhere in the (complex) q^2-plane

→ *s-channel:* $\hat{\Pi}_2(q^2)$ has a branch cut on the real axis for $m^2 - x(1-x)q^2 < 0$, i.e., starting at $m^2 = q^2/4 \Leftrightarrow q^2 = (2m)^2$ where a *real* (on-shell) electron-positron pair can be produced.

   b) $\sphericalangle$ Effective potential in *nonrelativistic limit* [recall Eq. (4.18) ff.]:

$$
\begin{aligned}
V(\vec{x}) &= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{x}} \frac{-e^2}{|\vec{q}|^2[1-\hat{\Pi}_2(-|\vec{q}|^2)]} \\
&= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{x}} \left(\frac{-e^2}{|\vec{q}|^2}\right) \cdot [1 + \hat{\Pi}_2(-|\vec{q}|^2) + \mathcal{O}(\alpha^2)] \\
&\overset{q^2 \ll m^2}{=} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{x}} \left(\frac{-e^2}{|\vec{q}|^2}\right) \cdot \left[1 + \frac{\alpha}{15\pi m^2}|\vec{q}|^2\right] + \mathcal{O}(\alpha^3) \\
&\overset{\circ}{\approx} -\frac{\alpha}{|\vec{x}|} - \frac{4\alpha^2}{15m^2}\delta^{(3)}(\vec{x})
\end{aligned}
$$

Recall that $q^2 = (p-p')^2 \approx -|\vec{p}-\vec{p}'|^2$ and $|q^2| \ll m^2$ in the nonrelativistic limit. Use $\log\left(\frac{1}{1-x}\right) = x + \mathcal{O}(x^2)$ to expand the logarithm in q^2/m^2 and use that $\int_0^1 dx\, x^2(1-x)^2 = 1/30$.

→ Electromagnetic force becomes much stronger at small distances
That is, QED tells us that the Coulomb potential of charged point particles is a low-energy/large-distance *approximation*!

   c) *Experimental verification:*

↑ Notes

Energy shift of s-orbitals in the hydrogen atom (contributes to the *Lamb shift*):

$$\Delta E \approx \int d^3x\,|\psi(\vec{x})|^2 \left(-\frac{4\alpha^2}{15m^2}\delta^{(3)}(\vec{x})\right) = -\frac{4\alpha^2}{15m^2}|\psi(0)|^2 \overset{l=0}{<} 0$$

Note that the *Darwin term* $H_{\text{Darwin}} = \frac{\pi\alpha}{2m^2}\delta^{(3)}(\vec{x})$ has a similar form but follows already from the (first quantized) Dirac equation, i.e., at tree-level. In contrast, the above correction is of loop-order α^2 and contributes to the *Lamb shift* (but only 2 %), a famous prediction of QED that cannot be derived from the Dirac equation and explains the observed splitting of the hydrogen orbitals $2S_{1/2}$ and $2P_{1/2}$ with total angular momentum $j = 1/2$ (the Dirac theory result: ➔ Problemset 4).

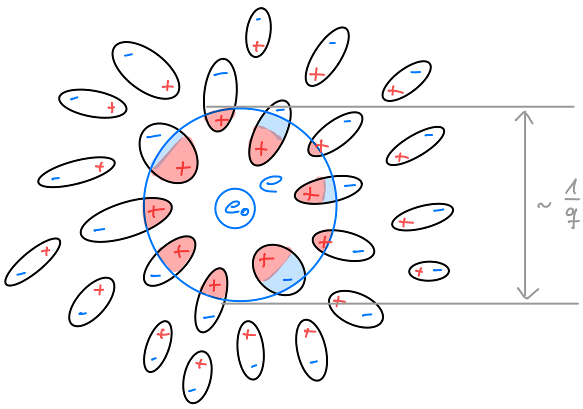
d) More generally, one finds the *Uehling potential*

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}}\frac{e^{-2mr}}{(mr)^{3/2}} + \dots\right)$$

A contour integration is needed to derive this, ➔ P&S p. 254.

Note that the range of the correction is given by the electron's Compton wavelength $\lambda_c = h/mc = 2\pi/m$. Since the length scale of variations of atomic orbitals is given by the Bohr radius, $a_0 = \lambda_c/(2\pi\alpha) \approx 22\,\lambda_c$, the nonrelativistic approximation from above is sufficient for atomic physics.

e) Interpretation: *Vacuum polarization:*



The vacuum behaves as a dielectric medium where electric dipoles of size $\sim 1/m$, formed by electron-positron pairs, screen the bare charge e_0. The energy scale q at which we observe the electron determines the size of the sphere $r = 1/q$ that contributes to the observed charge e; for $r \gtrsim 1/m$ the screening due to electron-positron pairs kicks in, for $r \lesssim 1/m$ the screening becomes weaker and the observed charge approaches the bare charge. Note that in this picture, the e^-e^+-pairs that traverse the surface of the sphere are responsible for reducing the infinite bare charge to the finite physical charge.

f) $\sphericalangle$ Relativistic limit $-q^2 \gg m^2$:

$$\begin{aligned}\hat{\Pi}_2(q^2) &= -\frac{2\alpha}{\pi}\int_0^1 dx\,x(1-x)\left[\log\left(\frac{m^2}{-q^2}\right) - \log\left(x(1-x)\right) + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right] \\ &\mathrel{\hat=} \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right]\end{aligned}$$

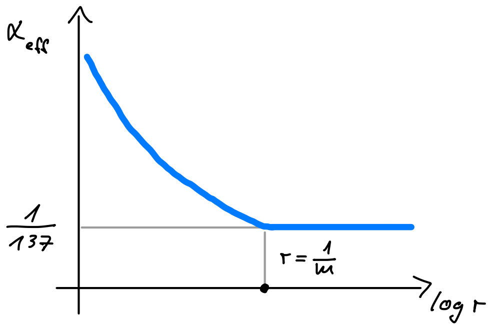
↑ Notes

Use $\log\left(\frac{x}{x+a}\right) = \log\left(\frac{x}{a}\right) + \mathcal{O}(x)$ and $\int_0^1 dx\, x(1-x)\log(x(1-x)) = -5/18$ to show this.

→ "Running" of α_{eff} with the length scale $r = \frac{1}{q} \to 0$

$$\alpha_{\text{eff}}(q^2) \overset{-q^2 \gg m^2}{\approx} \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{Am^2}\right)} \tag{6.37}$$

with $A = \exp(5/3)$:



This modification is crucial for explaining scattering cross sections at high energies, ➔ P&S p. 256 Fig. 7.7.

So when one refers to *the* fine-structure *constant*, the "constant" can only refer to constancy in time and the "the" refers to the energy/length/momentum scales that we typically observe, i.e., where $\alpha_{\text{eff}} \approx \frac{1}{137}$.

→ *Renormalization* (see next two lectures)

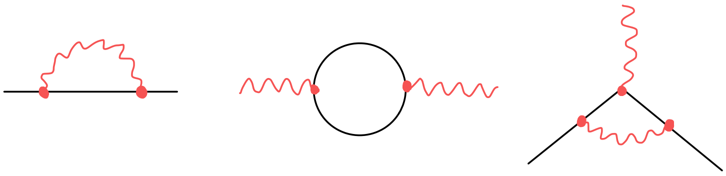
↑ Notes

7 Systematics of Renormalization

Remember:

- *IR-divergences:*
  - Due to massless particles (photons)
    (The amplitudes for $k \to 0$ real/virtual photons diverge.)
  - Regulate with small photon mass (μ)
  - Divergences from soft virtual photons (vertex correction) and soft bremsstrahlung cancel

  → Not a fundamental problem (we do not have to reinterpret/change the theory)
- *UV-divergences:*
  - Due to unbounded high momenta of particles (= unbounded small length scales) in all three radiative corrections:



  - Regulate with additional heavy particles (Λ) or dimensional regularization (ε)
  - Cancelled in several observable quantities
    (The UV-divergence of field-strength renormalization cancelled with the UV-divergence of the vertex correction. The momentum-dependence of the effective electric charge did not depend on the UV-regulator ε.)
  - Diverging differences between physical and bare quantities
    This is clearly a conceptual problem as the physical quantities are obviously finite (as given by experiments), thus the bare quantities (so far seen as fixed parameters of the microscopic Lagrangian) must then be cutoff dependent and diverge for $\Lambda \to \infty$.

  → Fundamental problem (of most interacting QFTs)

(→ UV-divergences are considered more severe problems than IR-divergences.)
→ Study UV-divergences systematically

7.1 Counting UV-Divergences

1. *Goal:* Classify UV-divergences in QED
   Which Feynman diagrams diverge and how many diverging quantitites are hidden in the amplitudes of QED?

↑ Notes

2. *Definitions:*

$$
\begin{aligned}
N_e &= \#\,\text{external electron lines}\\
N_\gamma &= \#\,\text{external photon lines}\\
P_e &= \#\,\text{electron propagators} &\rightarrow \prod_i^{P_e} \frac{1}{\not{k}_i - m}\\
P_\gamma &= \#\,\text{photon propagators} &\rightarrow \prod_i^{P_\gamma} \frac{1}{k_i^2}\\
V &= \#\,\text{vertices}\\
L &= \#\,\text{independent loops} &\rightarrow \prod_i^{L} \int \frac{d^4 k_i}{(2\pi)^4}
\end{aligned}
$$

This is valid for (diagrams of) scattering amplitudes; for (diagrams of) vertex functions, count the propagators to external points as external lines and not as propagators (they are multiplicative and therefore irrelevant for the UV-behaviour of the diagram).

3. *Superficial degree of divergence:*

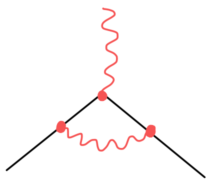
$$
D_{\text{QED}} \equiv \underbrace{(3L + L)}_{\text{Numerator}} - \underbrace{(P_e + 2P_\gamma)}_{\text{Denominator}}
$$

Note that a 4-dimensional integral diverges as Λ^4; e.g., in spherical coordinates, $3L$ comes from the jacobian and L from the integrations. D_{QED} quantifies the divergence of the *integral*, not the *integrand*.

Intuition:

$$
D_{\text{QED}} \begin{cases} > 0 & : \quad \text{Divergence with } \Lambda^D \\ = 0 & : \quad \text{Divergence with } \log\Lambda \\ < 0 & : \quad \text{No divergence} \end{cases}
$$

Example:

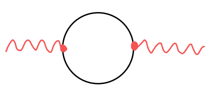


$\sim \log\Lambda$ and $D = 4\cdot 1 - (2 + 2\cdot 1) = 0$

However: Not always correct!

Reasons:

- Divergence may be weaker (or absent) if symmetries make divergent terms cancel:

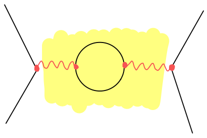


$\sim \log\Lambda$ although $D_{\text{QED}} = 4\cdot 1 - (2 + 2\cdot 0) = 2$

(Recall the restrictions on the general structure of this diagram due to the Ward identity in Eq. (6.30), see also discussion below.)

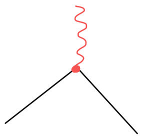
↑ Notes

- Divergence may be worse if diagram contains divergent subdiagrams (yellow):



$\sim \log \Lambda$ although $D_{\text{QED}} = 4 \cdot 1 - (2 + 2 \cdot 2) = -2$

- Tree-level diagrams with no propagators have $D = 0$ but no divergence:



~ 1 although $D_{\text{QED}} = 4 \cdot 0 - (0 + 2 \cdot 0) = 0$

4. Use (standard graph theory identities, see discussion of Fenyman rules)

$$L = \underbrace{P_e + P_\gamma}_{\#\,\text{Edges}} - V + 1 = \text{Cycle space dimension} \tag{7.1}$$

$$V = 2P_\gamma + N_\gamma = \frac{1}{2}(2P_e + N_e) \tag{7.2}$$

to show

$$\boxed{D_{\text{QED}} \overset{\circ}{=} 4 - N_\gamma - \frac{3}{2}N_e}$$

→ Independent of number of vertices!

5. Aside: *Furry's theorem:*

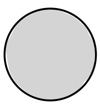
> Feynman diagrams with an *odd* number of photons as their *only* external lines vanish identically.

Proof: Follows from charge conjugation symmetry (C) of QED
(Use $C|\Omega\rangle = |\Omega\rangle$ and $Cj^\mu C^\dagger = -j^\mu$ with $j^\mu = \overline{\Psi}\gamma^\mu\Psi$, ➔ P&S p. 318.)
This tells us that a single, real photons can never be produced or absorbed by the interacting vacuum of QED.

6. Enumerate diagrams with $D_{\text{QED}} \geq 0$:
We consider only *amputated* and *one-particle irreducible* diagrams as all other diagrams are products of these. Grey blobs denote the sums of all such diagrams with specified external lines. We consider the amplitudes as functions of their external momenta and express them as power series with unkown coefficient (which may or may not diverge with the UV-cutoff Λ). In the following, "$\sim$" denotes asymptotic scaling up to regular terms. Recall that $N_e = 0, 2, 4, \ldots$ in QED.

   a) $N_e = 0$

      i. $N_\gamma = 0$ ($D_{\text{QED}} = 4$)

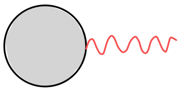


$\sim$ badly divergent

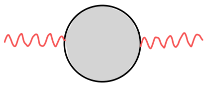
Unobservable vacuum energy shift → Ignore diagram

↑ Notes

ii. $N_\gamma = 1$ ($D_{\text{QED}} = 3$)

 $\overset{\text{Furry}}{=} 0$

iii. $N_\gamma = 2$ ($D_{\text{QED}} = 2$); Recall our first-order result in (6.36):

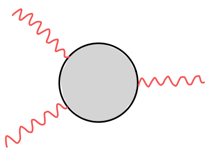


$$= (g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2)$$
$$\sim (g^{\mu\nu}q^2 - q^\mu q^\nu)\frac{\text{const}}{\varepsilon}$$
$$\sim (g^{\mu\nu}q^2 - q^\mu q^\nu)\cdot \underbrace{\text{const}\cdot\log\Lambda}_{a_0(\Lambda)} \tag{7.3}$$

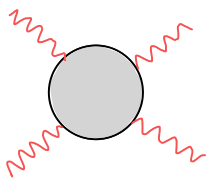
The divergence comes from $\Pi(q^2)$ and is logarithmic. Recall that we used *dimensional regularization* for our first-order calculation (6.36), so that the divergence $\log\Lambda$ with a Pauli-Villars regulator Λ is not obvious. $a_0(\Lambda)$ is a q-independent quantity diverging with Λ.

This is one of the examples where symmetry (via the Ward identity) makes the divergence less severe than superficially predicted by D: the zeroth and first-order coefficients of the q-expansion must vanish due to symmetry; the divergence left comes from the quadratic term and is only logarithmic.

iv. $N_\gamma = 3$ ($D_{\text{QED}} = 1$)

 $\overset{\text{Furry}}{=} 0$

v. $N_\gamma = 4$ ($D_{\text{QED}} = 0$)

 $\overset{\text{Ward}}{\sim} 1$

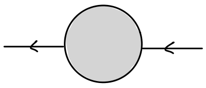
This follows from symmetry arguments (Ward identity) that make potentially diverging terms vanish identically, ➔ P&S p. 320.

Note: This diagram describes *light-by-light scattering (Halpern scattering)* in QED. The lowest-order amplitude is very weak (of order α^2/m_e^4); therefore, we do not experience this in everyday life and the linearity of classical electrodynamics is a good approximation. Nevertheless, it has consequences: In astronomy, observable γ-rays are restricted to energies below 80 TeV; above this threshold, the photons scatter at the ubiquitous microwave background and the universe becomes opaque. For direct experimental observations at LHC, see the recent paper `https://doi.org/10.1103/PhysRevLett.123.052001`.

↑ Notes

b) $N_e = 2$

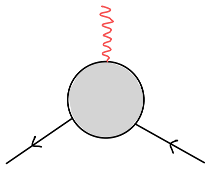
i. $N_\gamma = 0$ ($D_{\text{QED}} = 1$); Recall our first-order result in (6.28):



$$\sim \underbrace{\text{const} \cdot \log \Lambda}_{a_1(\Lambda)} + \not{p} \cdot \underbrace{\text{const} \cdot \log \Lambda}_{a_2(\Lambda)} \tag{7.4}$$

It can be shown that this scaling is true in all orders, ➲ P&S p. 319.

ii. $N_\gamma = 1$ ($D_{\text{QED}} = 0$); Recall our first-order result in (6.12):



$$\sim \underbrace{-ie\gamma^\mu \log \Lambda}_{a_3(\Lambda)} \tag{7.5}$$

It can be shown that this scaling is true in all orders, ➲ P&S p. 319.

→ Diagrams only diverge if they contain (7.3), (7.4) or (7.5) as *subdiagrams*.
→ QED contains only *four* UV-divergent numbers: a_0, a_1, a_2, a_3.

7. *Idea:* Absorb finite number of diverging quantitites in finite number of diverging but unobservable Lagrangian parameters → *Renormalization* (see below)
"Hiding" the divergences in unobservable parameters makes all other *observable* quantities (like scattering amplitudes and physical parameters) cutoff-independent and UV-finite.

8. *Generalization:* ∢ QED in d spacetime dimensions →

$$\begin{aligned} D_{\text{QED}} &= dL - P_e - 2P_\gamma \\ &\overset{\circ}{=} d + \left(\frac{d-4}{2}\right) V - \left(\frac{d-2}{2}\right) N_\gamma - \left(\frac{d-1}{2}\right) N_e \end{aligned}$$

To show this, note that the identities (7.1) and (7.2) are still valid.

→ *Observations:*

- For $d < 4$, diagrams of higher order ($V \to \infty$) are always superficially *convergent* (independent of potentially diverging subdiagrams)
- For $d = 4$, D_{QED} is *independent* of the order V (the divergence of diagrams can be traced back to a finite number of diverging amplitudes/subdiagrams)
- For $d > 4$, diagrams of higher order ($V \to \infty$) are always superficially *divergent*

This means, that the "reductionistic approach" only works in $d = 4$ dimensions where the divergence of all diagrams can be traced back to a finite number of diverging subdiagrams.

9. This is also valid for other QFTs and motivates three classes of theories:

↑ Notes

- *Super-Renormalizable theory:*
  Only a finite number of *Feynman diagrams* (not amplitudes = sums of diagrams!) superficially diverge.
  Example: QED in $d = 2 + 1$
- *Renormalizable theory:*
  Only a finite number of *amplitudes* superficially diverge.
  $\rightarrow$ Divergences at all orders in perturbation theory.
  Example: QED in $d = 3 + 1$
- *Non-Renormalizable theory:*
  All amplitudes diverge at sufficiently high order in perturbation theory.
  Example: QED in $d = 4 + 1$

Note 7.1

- There are examples in which the divergences are not as bad as superficially predicted due to symmetries that cancel diverging amplitudes.
- The diverging amplitudes of superficially renormalizable theories can always be absorbed into a finite number of unobservable Lagrangian parameters (see below).

Alternative approach

1. $\sphericalangle$ ϕ^n-theory (for simplicity)

$$\mathcal{L}_{\phi^n} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n \qquad \text{with } n \in \mathbb{N} \tag{7.6}$$

2. *Definitions:*

$$\begin{aligned} N_\phi &= \#\,\text{external lines} \\ P_\phi &= \#\,\text{propagators} \\ V &= \#\,\text{vertices} \\ L &= \#\,\text{independent loops} \end{aligned}$$

3. *Superficial degree of divergence:*

$$\boxed{\begin{aligned} D_{\phi^n} &\equiv dL - 2P_\phi \\ &\stackrel{\circ}{=} d + \left[n\left(\frac{d-2}{2}\right) - d\right]V - \left(\frac{d-2}{2}\right)N_\phi \end{aligned}} \tag{7.7}$$

Use the graph identities $L = P - V + 1$ and $nV = N_\phi + 2P_\phi$ to show this.
$\rightarrow$ For $n = 4$ in $d = 4$ independent of V $\rightarrow$ renormalizable

4. Alternative approach via *dimensional analysis*:
   a) Recall: $\hbar = c = 1$ and $\lambda_c = \frac{h}{mc} = \frac{2\pi}{m}$
   $\rightarrow$ Dimension of length: $[\lambda_c] = M^{-1}$ (M: dimension of mass)

↑ Notes

b) Dimension of action: $[S] = 1$ (since $\hbar = 1$)

c) $S = \int d^dx\, \mathcal{L}$ and $[d^dx] = M^{-d} \to$
Dimension of Lagrangian density: $[\mathcal{L}] = M^d$
As all dimensions can be expressed in M, we say that "$\mathcal{L}$ has (mass) dimension d".

d) From (7.6) follows (use $[\partial] = M$):

$$
\begin{aligned}
[\phi] &= M^{\frac{d-2}{2}} \\
[m] &= M \quad \text{(consistent!)} \\
[\lambda] &= M^{d-n(d-2)/2}
\end{aligned}
$$

e) $\sphericalangle$ Amplitude $\mathcal{M}$ of single diagram with N_ϕ external lines
→ Could arise (on tree-level) from interaction $\eta\phi^{N_\phi} \to [\eta] = M^{d-N_\phi(d-2)/2}$
$\to [\mathcal{M}] = [\eta] = M^{d-N_\phi(d-2)/2}$ (recall $\mathcal{M} = -\lambda + \mathcal{O}(\lambda^2)$ from Eq. (4.16))

f) $\sphericalangle$ Diagram with V vertices $\to \mathcal{M} \sim \lambda^V \Lambda^D$ for the UV-cutoff $\Lambda \to \infty$
(This is an implicit definition of the superficial degree of divergence D.
→ (use $[\Lambda] = M$)

$$
\begin{aligned}
[\lambda]^V [\Lambda]^D &= [\mathcal{M}] = M^{d-N_\phi(d-2)/2} \\
V \log_M[\lambda] + D &= d - N_\phi \left(\frac{d-2}{2}\right)
\end{aligned}
$$

→

$$
D_{\phi^n} = d - \underbrace{\log_M[\lambda]}_{d-n(d-2)/2} \cdot V - \left(\frac{d-2}{2}\right) N_\phi \tag{7.7}
$$

5. Therefore we find the equivalent characterization:
   - *Super-Renormalizable theory:*
     Coupling constant has positive mass dimension: $\log_M[\lambda] > 0$.
   - *Renormalizable theory:*
     Coupling constant is dimensionless: $\log_M[\lambda] = 0$.
     Example: QED with $[e] = 1$ is superficially renormalizable.
   - *Non-Renormalizable theory:*
     Coupling constant has negative mass dimension: $\log_M[\lambda] < 0$.

   This argument remains valid for other QFTs as well, in particular QED.

Aside: Why quantum gravity is special

1. *Fields:* Components of the metric tensor $g_{\mu\nu}(x)$
   Note that in general relativity, the metric is position dependent, that is, a field.

2. *Einstein-Hilbert action* of pure gravity:

$$
S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{|\det g(x)|}\, [R(g(x)) - 2\Lambda_c] \tag{7.8}
$$

↑ Notes

$R = g^{\mu\nu} R_{\mu\nu}$: Ricci scalar with Ricci tensor $R_{\mu\nu}$
Λ_c: Cosmological constant
G: Gravitational constant = Coupling constant of gravity

If matter is present, this action is extended by the covariant action of the matter fields (e.g., $\mathcal{L}_{\text{QED}}$) which then generates a non-vanishing energy-momentum tensor in the Einstein equations below.

$\overset{\circ}{\rightarrow}$ Equations of motion = *Einstein's field equations* (in vacuum):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_c g_{\mu\nu} = 0$$

3. Recall:

$$R \sim g^{\mu\nu} R_{\mu\nu} \sim g^{\mu\nu} R^{\sigma}{}_{\mu\sigma\nu} \sim g^{\mu\nu} \partial_\nu \Gamma^{\sigma}_{\mu\sigma} \sim g^{\mu\nu} \partial_\nu (g^{\sigma\kappa} \partial_\mu g_{\sigma\kappa})$$
$$\Rightarrow \quad [R] = [g]^3 [\partial]^2$$

$\Gamma^{\sigma}_{\mu\nu}$ are the *Christoffel symbols of the second kind*, $R^{\rho}{}_{\mu\rho\nu}$ is the Riemann curvature tensor.
and

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$
$$\Rightarrow \quad L^2 = [ds^2] = [g][dx]^2 = [g] L^2$$
$$\Rightarrow \quad [g] = 1$$

ds^2 is the squared length element of the Riemannian spacetime.
such that

$$[R] = [\partial]^2 = L^{-2} = M^2$$

4. From (7.8) it follows $[G]^{-1}[dx]^4[R] = [G]^{-1} M^{-4} M^2 = [S] \overset{!}{=} 1$, i.e.,

$$\boxed{\log_M [G] = -2 < 0}$$

→ Einstein gravity is superficially non-renormalizable!
Recall that $G = \frac{\hbar c}{m_P^2} = \frac{1}{m_P^2}$ with the Planck mass m_P, consistent with our result.
Superficial non-renormalizability does not *prove* non-renormalizability as there may still be non-trivial cancellations that make the theory UV-finite.

Note 7.2

- At one-loop level, *pure* Einstein gravity (no matter fields) is – quite unexpectedly! – UV-finite, ➔ http://www.numdam.org/article/AIHPA_1974__20_1_69_0.pdf.
- However, when matter is coupled to gravity, the one-loop diagrams are UV-divergent, ➔ http://www.numdam.org/article/AIHPA_1974__20_1_69_0.pdf for the example of a scalar field and references in https://doi.org/10.1016/0550-3213(86)90193-8.
- At two-loop level, pure Einstein gravity is proven to be UV-divergent, ➔ https://doi.org/10.1016/0550-3213(86)90193-8. That is, no unexpected cancellations occur.

↑ Notes

- Therefore it is widely believed (though, to my knowledge, not proven) that no unexpected cancellations occur beyond two-loop order; therefore, Einstein gravity is perturbatively not renormalizable.

7.2 Renormalized Perturbation Theory

Goal: Compute finite predictions from given physical parameters m and e for $\Lambda \to \infty$

Recipe:
Historically, this was the first widely accepted "fix" for the UV-problems of QFTs.

(i) Compute UV-divergent amplitude with UV-regulator Λ to some order in α_0:

$$\mathcal{M} = \mathcal{M}(m_0, e_0; \Lambda) + \mathcal{O}(\alpha_0^*)$$

(ii) Compute physical mass, physical charge and field-strength renormalization:

$$m = m(m_0, e_0; \Lambda) + \mathcal{O}, \quad e = e(m_0, e_0; \Lambda) + \mathcal{O}, \quad Z = Z(m_0, e_0; \Lambda) + \mathcal{O}$$

The order $\mathcal{O}$ of these computations should be consistent with the order of $\mathcal{M}$.

The field-strength renormalization Z is only needed for the computation of S-matrix elements (where we sum only over *amputated* and fully connected diagrams), but not for correlation functions (where we sum over *all* connected diagrams). This follows from the LSZ reduction formula (which we did not discuss, ➔ P&S pp. 222–230, in particular Eq. (7.45) on p. 229).

(iii) *Renormalization:*
Eliminate m_0 and e_0 in favour of m and e (which are fixed and given by experiments):

$$e_0 = e_0(m, e; \Lambda), \quad m_0 = m_0(m, e; \Lambda)$$

We did this previously when discussing the charge renormalization where we replaced m_0 and e_0 by m and e in lowest order.

(iv) Then

$$\mathcal{M}(m, e) \equiv \lim_{\Lambda \to \infty} \mathcal{M}(m_0(m, e; \Lambda), e_0(m, e; \Lambda); \Lambda)$$

is finite and indepenent of Λ in all orders of α.
This is a remarkable, non-trivial observation! Note that this requires the bare parameters to be cutoff dependent and divergent for $\Lambda \to \infty$, i.e., we *change* the "microscopic" theory parametrically with Λ. This interpretation is justified by the numerous extremely precise predictions of QED like the anomalous magnetic moment (where we – somewhat naively – used the *physical* value α and not the bare value α_0 to evaluate the numerical correction for g_e.).

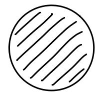
→ *Bare* perturbation theory (since the Feynman rules involve *bare* parameters)
→ Works for all renormalizable QFTs (but can be cumbersome)
→ Alternative (but equivalent!) formalism: *Renormalized perturbation theory* (➔ today)

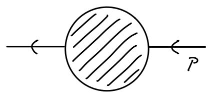
1. $\sphericalangle$ ϕ^4-theory in $d = 3 + 1$ dimensions (QED below):

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!}\phi^4$$

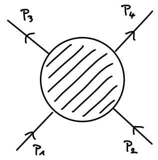
↑ Notes

2. With $D_{\phi^4} = 4 - N_\phi$ [➔ (7.7)] and $N_\phi = 0, 2, 4, \dots$ one finds the divergent amplitudes:
Note that all vertices of ϕ^4-theory have degree 4 so that only an even number of external legs is possible.

$D_{\phi^4} = 4$  unobservable vacuum energy shift

$D_{\phi^4} = 2$ 

$\sim \Lambda^2 + p^2 \log \Lambda$

$D_{\phi^4} = 0$ 

$\sim \log \Lambda$

→ 3 divergent quantities
→ Absorb in 3 unobservable parameters: bare mass m_0, bare coupling λ_0, fields ϕ

3. Recall:

$$\int d^4x\, e^{ipx} \langle \Omega | \mathcal{T} \phi(x)\phi(0) | \Omega \rangle = \frac{iZ}{p^2 - m^2} + \dots$$

The dots denote terms regular at $p^2 = m^2$. Absorb unobservable Z in rescaled fields:

$$\boxed{\phi_r \equiv \frac{1}{\sqrt{Z}} \phi}$$

Then

$$\int d^4x\, e^{ipx} \langle \Omega | \mathcal{T} \phi_r(x)\phi_r(0) | \Omega \rangle = \frac{i}{p^2 - m^2} + \dots$$

Note that this expression is no longer affected by $Z \to \infty$ for $\Lambda \to \infty$ since we rescale the field strength of ϕ_r accordingly.

4. → Lagrangian in new fields:

$$\mathcal{L}_{\phi^4} = \frac{1}{2} Z (\partial_\mu \phi_r)^2 - \frac{1}{2} m_0^2 Z \phi_r^2 - \frac{\lambda_0}{4!} Z^2 \phi_r^4$$

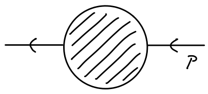
5. *Split* terms into observable parameters and unobservable ones:

$$\mathcal{L}_{\phi^4} = \overbrace{\frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!} \phi_r^4}^{\text{Physical parameters (fixed)}} \\ \underbrace{+ \frac{1}{2} \overbrace{(Z-1)}^{\equiv \delta_Z} (\partial_\mu \phi_r)^2 - \frac{1}{2} \overbrace{(m_0^2 Z - m^2)}^{\equiv \delta_m} \phi_r^2 - \frac{1}{4!} \overbrace{(\lambda_0 Z^2 - \lambda)}^{\equiv \delta_\lambda} \phi_r^4}_{\textit{Counterterms} \text{ (cutoff-dependent)}} \tag{7.9}$$

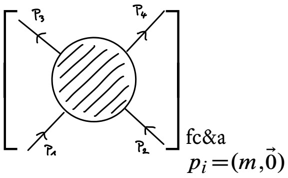
→ δ_Z, δ_m, and δ_λ absorb unobservable, diverging shifts of bare and physical quantities
So far, we only redefined quantities and shuffled them around! No magic here.

↑ Notes

6. Experimental input → *Renormalization conditions:*
We need to *force* the theory to match the observed, physical parameters m and λ to *extrapolate* from these and make non-trivial predictions.



$$\stackrel{!}{=} \frac{i}{p^2-m^2} + \dots \tag{7.10}$$

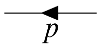
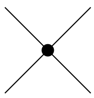


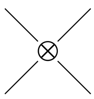
$$\stackrel{!}{=} -i\lambda \tag{7.11}$$

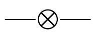
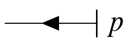
Motivation:

- (7.10) includes *two* conditions: it fixes the *pole* of the propagator at the physical mass m and the *residue* (and thereby the field strength) at 1. This enforces the scaled fields ϕ_r from above.
- Recall that in *bare perturbation theory* for the amplitude $i\mathcal{M}(p_1p_2 \mapsto p_3p_4) = -i\lambda_0 + \mathcal{O}(\lambda_0^2)$ as shown in (4.16). This *motivates* (7.11) which then is an operational *definition* of the physical parameter λ as the *measured amplitude* for the depicted scattering process at zero momentum. Note that the choice of momenta (playing the role of experimental settings) is arbitrary. Changing these would change the interpretation and the numerical value of λ, but not the predictions of the theory.

7. Perturbation theory of (7.9) →
Feynman rules for renormalized perturbation theory
of ϕ^4-theory in momentum space for S-matrix elements:

1. **Edges:**  $= \frac{i}{p^2-m^2+i\varepsilon}$
2. **Vertices:**  $= -i\lambda$

    $= -i\delta_\lambda$

    $= i(p^2\delta_Z - \delta_m)$
3. **External lines:**  $= 1$
4. Impose momentum conservation at all vertices
5. Integrate over all undetermined momenta
6. Divide by the symmetry factor

The propagator and the first vertex are the same as before, only that now the *physical mass* and the *physical coupling* enter the perturbation series. Note that the counterterms give rise to two additional vertices. To understand the term for the two-leg vertex, retrace

↑ Notes

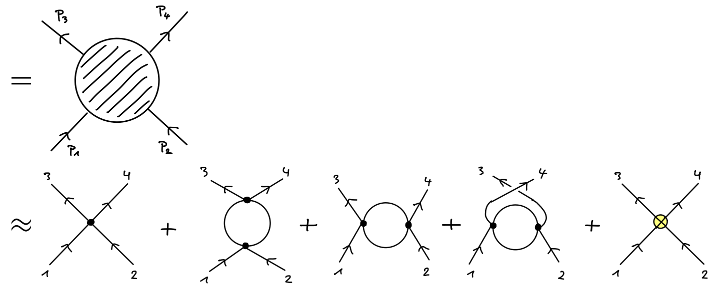
our derivation of Feynman rules in Section 4.4 and recall that in momentum space, the derivatives translate to p^2.

8. *Procedure* for computing amplitudes:
   a) Sum all relevant diagrams built from the Feynman rules above.
   b) If loop integrals diverge, introduce a regulator.
   c) The results depend on the (yet undetermined) parameters $\{\delta_\bullet\}$, the fixed physical parameters m and e, and the regulator (Λ or ε).
   d) Choose ("renormalize") the parameters $\{\delta_\bullet\}$ such that the renormalization conditions (7.10) and (7.11) are satisfied.
   e) With these $\{\delta_\bullet\}$, the amplitude is finite, independent of the regulator, and depends only on the physical parameters.

9. *Bare perturbation theory* (➔ beginning of the lecture) and *renormalized perturbation theory* are equivalent and yield the same results.
   Which one to choose depends on personal preference and the application.

10. *Example* for renormalized perturbation theory in one-loop order:
    a) ⊲ Amplitude

$\mathcal{M}(p_1 p_2 \mapsto p_3 p_4) =$

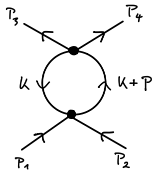


$$\hat{=} -i\lambda + (-i\lambda)^2\,[iV(s) + iV(t) + iV(u)] - i\delta_\lambda$$

with *Mandelstam variables* $s = (p_1 + p_2)^2$, $t = (p_3 - p_1)^2$, and $u = (p_4 - p_1)^2$.
Note that we include all one-loop diagrams with *two* physical vertices $\propto \lambda$ but only the tree-level diagram with *one* counterterm $\propto \delta_\lambda$. This is consistent because $\delta_\lambda = \mathcal{O}(\lambda^2)$ as we will see below.
To construct the three one-loop diagrams, enumerate all possibilities to connect two external momenta p_i with $i = 1, 2, 3, 4$ at a one vertex.

↑ Notes

b) Evaluate loop integral with dimensional regularization:

$$(-i\lambda)^2 \cdot iV(s) =$$ 

$p^2 = (p_1 + p_2)^2 = s$

$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}$$

Feynman parameter, Substitution, Wick rotation, Dimensional regularization

$$\overset{\varepsilon \to 0}{\sim} -(-i\lambda)^2 \cdot \frac{i}{32\pi^2} \int_0^1 dx \left\{ \begin{array}{l} \frac{2}{\varepsilon} - \gamma + \log(4\pi) \\ -\log\left[m^2 - x(1-x)p^2\right] \end{array} \right\}$$

c) Enforce renormalization condition (7.11) to determine δ_λ:

$$i\mathcal{M}|_{s=4m^2,t=u=0} \overset{!}{=} -i\lambda$$

solved by

$$\delta_\lambda := -\lambda^2 \left[V(4m^2) + 2V(0)\right]$$

$$\overset{\varepsilon \to 0}{\sim} \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left\{ \begin{array}{l} \frac{6}{\varepsilon} - 3\gamma + 3\log(4\pi) \\ -\log\left[m^2 - x(1-x)4m^2\right] - 2\log\left[m^2\right] \end{array} \right\}$$

Here, the *amplitude* we want to calculate is the same that we need for the *renormalization condition*. This is a special case! Note that δ_λ depends only on λ, m and ε; it is *quadratic* in the physical coupling λ which explains our perturbative expansion above.

d) → Amplitude

$$i\mathcal{M}(p_1 p_2 \mapsto p_3 p_4) = -i\lambda - i\lambda^2 \cdot \mathcal{F}(\{p_i\}; m)$$

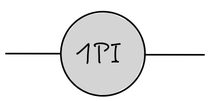
$\mathcal{F}$: finite function of the momenta $\{p_i\}$, parametrized by the physical mass m.
Important: the regulator ε drops out!

Note that $\mathcal{F} = 0$ for $s = 4m^2$ and $t = u = 0$, as demanded by the renormalization condition.

The prediction of the theory is *not* the amplitude for zero momentum $[p_i = (m, \vec{0})]$ but the non-trivial dependency on $\{p_i\}$ for non-zero momenta!

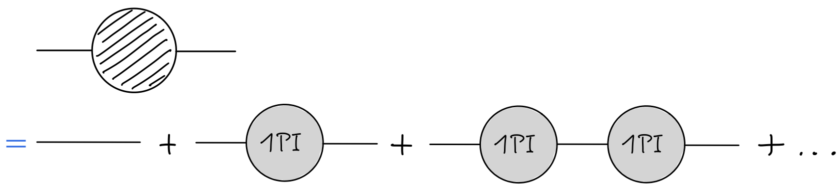
e) Enforce renormalization condition (7.10) to determine δ_Z and δ_m:

i. Define

$$-iM^2(p^2) :=$$ 

↑ Notes

ii. It follows along the same lines as for the electron self-energy



$$= \frac{i}{p^2 - m^2 - M^2(p^2)}$$

$$\overset{!}{=} \frac{i \cdot 1}{p^2 - m^2} + \dots$$

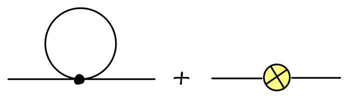
iii. → (7.10) is equivalent to

$$M^2(p^2)\big|_{p^2=m^2} \overset{!}{=} 0 \quad \text{and} \quad \frac{\mathrm{d}M^2(p^2)}{\mathrm{d}p^2}\bigg|_{p^2=m^2} \overset{!}{=} 0 \tag{7.12}$$

The first relation fixes the pole at $p^2 = m^2$, the second relation fixes the residue of this simple pole at 1, i.e., $\frac{\mathrm{d}}{\mathrm{d}p^2}\left[p^2 - m^2 - M^2(p^2)\right]_{p^2=m^2} \overset{!}{=} 1^{-1}$.

iv. In one-loop order:

$$-iM^2(p^2) \approx$$



$$= (-i\lambda) \cdot \frac{1}{2} \cdot \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} + i(p^2\delta_Z - \delta_m)$$

Wick rotation, Dimensional regularization

$$\overset{\circ}{=} -\frac{i\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1 - d/2)}{(m^2)^{1-d/2}} + i(p^2\delta_Z - \delta_m)$$

→ (7.12) solved by

$$\delta_Z := 0 \quad \text{and} \quad \delta_m := -\frac{\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1 - d/2)}{(m^2)^{1-d/2}}$$

Note that $\delta_Z = 0$ in one-loop order is a special case of ϕ^4-theory since the first term does not depend on p^2. Note that δ_m is a diverging (for $d \to 4$) function of the physical parameters (m and λ) and the UV-regulator (d).

Application to QED

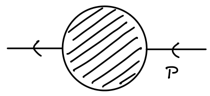
We briefly summarize the analogous results for the renormalized perturbation theory of QED.

1. Original Lagrangian:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}(F_{\mu\nu})^2 + \overline{\Psi}(i\partial\!\!\!/ - m_0)\Psi - e_0\overline{\Psi}\gamma^\mu\Psi A_\mu$$

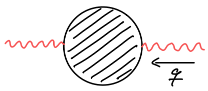
↑ Notes

2. Interacting propagators:



$$= \frac{iZ_2}{\not{p}-m} + \dots$$

and



$$= \frac{-iZ_3 g_{\mu\nu}}{q^2} + \dots$$

3. Absorb Z_2 and Z_3 → Renormalized fields:

$$\Psi_r := \frac{1}{\sqrt{Z_2}}\Psi \quad \text{and} \quad A_r^\mu := \frac{1}{\sqrt{Z_3}}A^\mu$$

4. →

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}Z_3(F_r^{\mu\nu})^2 + Z_2\overline{\Psi}_r(i\not{\partial}-m_0)\Psi_r - e_0 Z_2 Z_3^{1/2}\,\overline{\Psi}_r\gamma^\mu\Psi_r(A_r)_\mu$$

5. Define $Z_1 := Z_2 Z_3^{1/2}\frac{e_0}{e}$ with physical charge e
The physical charge e is defined by measurements at large distances, i.e., for $q \to 0$ (see below).

6. →

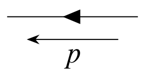
$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}(F_r^{\mu\nu})^2 + \overline{\Psi}_r(i\not{\partial}-m)\Psi_r - e\,\overline{\Psi}_r\gamma^\mu\Psi_r(A_r)_\mu$$
$$\underbrace{-\frac{1}{4}\delta_3(F_r^{\mu\nu})^2 + \overline{\Psi}_r(i\delta_2\not{\partial}-\delta_m)\Psi_r - e\delta_1\,\overline{\Psi}_r\gamma^\mu\Psi_r(A_r)_\mu}_{\text{4 counterterms}}$$

with

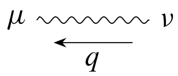
$$\delta_i := Z_i - 1 \quad \text{for} \quad i = 1,2,3 \qquad \text{and} \qquad \delta_m := Z_2 m_0 - m$$

7. → *Feynman rules* (we omit external lines etc.):

Edges:

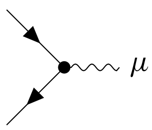


$$= \frac{i}{\not{p}-m+i\varepsilon}$$

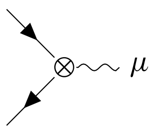


$$= \frac{-ig_{\mu\nu}}{q^2+i\varepsilon}$$

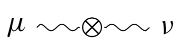
Vertices:



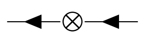
$$= -ie\gamma^\mu$$



$$= -ie\gamma^\mu\delta_1$$



$$= -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3$$



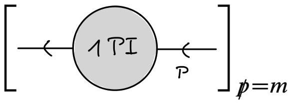
$$= i(\not{p}\delta_2 - \delta_m)$$

There are three additional counterterm vertices. The counterterm for the photon two-leg vertex follows, similarly to the two-leg vertex of ϕ^4-theory, with integration by parts.

↑ Notes

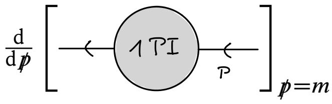
8. 4 counterterm coefficients → 4 *renormalization conditions*:

1. Fix electron mass to m:



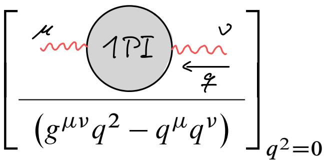
$$= -i\Sigma(\not p = m) \stackrel{!}{=} 0$$

2. Fix residue of electron propagator to 1 (choose Ψ_r):



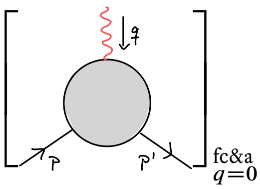
$$= -i \left.\frac{d\Sigma(\not p)}{d\not p}\right|_{\not p = m} \stackrel{!}{=} 0$$

3. Fix residue of photon propagator to 1 (choose A_r):



$$= i\Pi(q^2 = 0) \stackrel{!}{=} 0$$

4. Fix electron charge to e:



$$= -ie\Gamma^\mu(q = 0) \stackrel{!}{=} -ie\gamma^\mu$$

These are redefinitions of Σ, Π and Γ in terms of the renormalized Feynman rules above. The definition of Γ involves now the *physical* charge e.

↑ Notes

8 Functional Methods

> **Problem Set 11**
>
> (due **03.07.2020**)
>
> 1. Dimensional regularization: Technical details used in the lecture for QED
> 2. Thomas-Fermi screening: An application of renormalization in the context of condensed matter physics

- **So far:**

  Hamiltonian → Canonical quantization → Feynman rules

  The Hamiltonian is *not* Lorentz invariant (generates translations in time direction)!

- **Alternative:**

  Lagrangian → Path integral → Feynman rules

  The Lagrangian *is* Lorentz invariant (for a relativistic field theory)!

- Two descriptions of the same physics
- Application: Derivation of the *photon propagator* (easier with path integrals)

8.1 Path Integrals in Quantum Mechanics

1. $\sphericalangle$ Nonrelativistic particle in 1D: $H = \frac{p^2}{2m} + V(x)$
2. Time evolution operator: $U(x_a, x_b; T) = \langle x_b | e^{-\frac{i}{\hbar} HT} | x_a \rangle$
   Known from canonical quantization in the Hamiltonian formalism.
3. *Path integral formalism* → Alternative expression for U:

$$U(x_a, x_b; T) = \underbrace{\sum_{\text{all paths } x(t)}}_{\text{Superposition principle}} \underbrace{e^{i \overbrace{F[x(t)]}^{\text{Functional}}}}_{\text{Pure phase}} = \underbrace{\int \mathcal{D}x(t)\, e^{iF[x(t)]}}_{\text{Functional integral}}$$

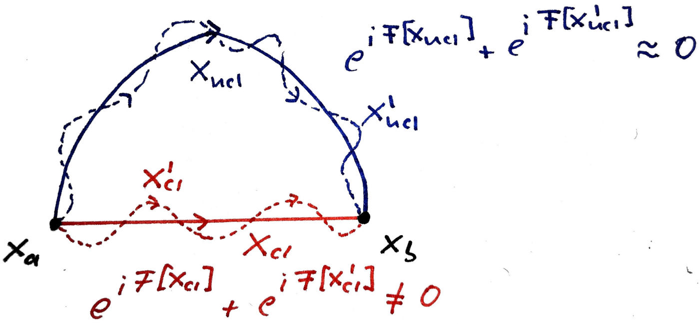
   Paths are weighted with pure phases → interference (all paths are equivalent)
   Functional integral = integration over space of *functions*

4. Conditions on F:
   a) Describes the system

b) Functional of path $x(t)$

c) Classical path $x_{\mathrm{cl}}(t)$ dominates (for $\hbar \to 0$):

$$U(x_a, x_b; T) \approx \sum_{\text{paths close to } x_{\mathrm{cl}}(t)} e^{iF[x(t)]}$$



Therefore

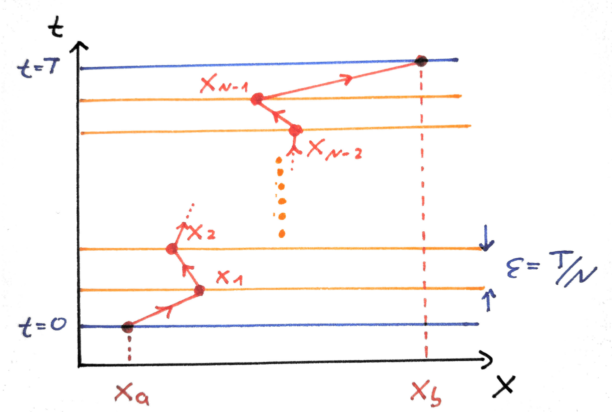
$$\left.\frac{\delta F}{\delta x}\right|_{x=x_{\mathrm{cl}}} \overset{!}{=} 0 \quad \Rightarrow \quad F = \frac{S}{\hbar} = \frac{1}{\hbar}\int dt\, L(x(t))$$

5. Propagation amplitude (*Propagator*):

$$U(x_a, x_b; T) = \int \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x(t)]} \overset{?}{=} \langle x_b | e^{-\frac{i}{\hbar}HT} | x_a \rangle \tag{8.1}$$

We show the equivalence to canonical quantization for a free particle below.
So far, the PI is just a sketchy idea and not a well-defined mathematical concept!

6. Definition of PI via *time slices*:



$$\int \mathcal{D}x(t) := \lim_{N\to\infty} \frac{1}{C_\varepsilon} \int \frac{dx_1}{C_\varepsilon} \cdots \int \frac{dx_{N-1}}{C_\varepsilon} = \lim_{N\to\infty} \frac{1}{C_\varepsilon} \prod_{k=1}^{N-1} \int \frac{dx_k}{C_\varepsilon}$$

with $\varepsilon = \frac{T}{N}$ and C_ε a constant (see below)
C_ε determines the *measure* of the functional integral.

↑ Notes

Example 8.1: Particle in potential $V(x)$

1. Lagrangian: $L = \frac{m}{2}\dot{x}^2 - V(x)$
2. Action:

$$S = \int_0^T dt\, L \approx \sum_{k=0}^{N-1}\left[\frac{m}{2}\frac{(x_{k+1}-x_k)^2}{\varepsilon} - \varepsilon V\left(\frac{x_{k+1}+x_k}{2}\right)\right]$$

3. Recursion:

$$U(x_a, x_b; T) = \int_{-\infty}^{\infty}\frac{dx'}{C_\varepsilon}\quad \exp\left[\frac{i}{\hbar}\frac{m(x_b-x')^2}{2\varepsilon} - \frac{i}{\hbar}\varepsilon V\left(\frac{x_b+x'}{2}\right)\right] \times U(x_a, x'; T-\varepsilon)$$

Use $V([x_b+x']/2) = V(x_b) + \mathcal{O}(\varepsilon)$ under the Gaussian integral

$$= \int_{-\infty}^{\infty}\frac{dx'}{C_\varepsilon}\quad \exp\left[\frac{i}{\hbar}\frac{m(x_b-x')^2}{2\varepsilon}\right] \times \left[1 - \frac{i}{\hbar}\varepsilon V(x_b) + \dots\right] \times\left[1 + (x'-x_b)\frac{\partial}{\partial x_b} + \frac{(x'-x_b)^2}{2}\frac{\partial^2}{\partial x_b^2} + \dots\right] \times U(x_a, x_b; T-\varepsilon)$$

Compute Gaussian integrals with regularization

Note that terms with odd powers of $(x'-x_b)$ vanish

$$\mathrel{\hat{=}} \underbrace{\left(\frac{1}{C_\varepsilon}\sqrt{\frac{2\pi\hbar\varepsilon}{-im}}\right)}_{\stackrel{!}{=}1} \times$$

To see this, consider both sides of the equation for $\varepsilon \to 0$

$$\left[1 - \frac{i}{\hbar}\varepsilon V(x_b) + \frac{i\hbar}{2m}\varepsilon\frac{\partial^2}{\partial x_b^2} + \mathcal{O}(\varepsilon^2)\right] U(x_a, x_b; T-\varepsilon)$$

4. → PI measure:

$$C_\varepsilon = \sqrt{\frac{2\pi\hbar\varepsilon}{-im}}$$

This is not generic but depends on $\mathcal{L}$!

5. Use $U(x_a, x_b; T-\varepsilon) \approx U(x_a, x_b; T) - \varepsilon\, \partial_T U + \mathcal{O}(\varepsilon^2)$ and compare terms of order ε:

$$i\hbar\frac{\partial}{\partial T}U = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2} + V(x_b)\right]U = HU$$

(Schrödinger equation)

↑ Notes

Behold: We *derived* the Schrödinger equation and the quantized form of the Hamiltonian from first principles (namely, the concept of weighting paths with phases proportional to their classical action)!

6. Initial condition: set $N = 1$ (No integral!) →

$$\begin{aligned} U(x_a, x_b; \varepsilon) &= \frac{1}{C_\varepsilon} \exp\left[\frac{i}{\hbar}\frac{m}{2\varepsilon}(x_b - x_a)^2 + \mathcal{O}(\varepsilon)\right] \\ &\approx \sqrt{\frac{-im}{2\pi\hbar\varepsilon}}\, e^{\frac{i}{\hbar}\frac{m}{2\varepsilon}(x_b - x_a)^2} \\ &\xrightarrow{\varepsilon\to 0} \delta(x_a - x_b) = U(x_a, x_b; 0) = \langle x_b | x_a\rangle \end{aligned}$$

7. The last two steps conclude the proof of the second equality in (8.1) for $H = \frac{p^2}{2m} + V(x)$.

Generalization

Now we reverse the reasoning:
We start with canonical quantization and derive the path integral.
For details ➔ Problemset 12.

1. $\sphericalangle$ Coordinates q_i, conjugate momenta p_i, Hamiltonian $H(\vec{q}, \vec{p})$
2. Canonical quantization: $[q_i, p_j] = i\hbar\delta_{ij} \to U(\vec{q}_a, \vec{q}_b; T) = \langle \vec{q}_b | e^{-i\hat{H}T} | \vec{q}_a\rangle$
   We set $\hbar = 1$ to simplify equations.
3. Time slicing: $e^{-i\hat{H}T} = \underbrace{e^{-i\hat{H}\varepsilon} \cdots e^{-i\hat{H}\varepsilon}}_{\times N}$
4. Insert $N-1$ identities $\mathbb{1}_k = \int d\vec{q}_k\, |\vec{q}_k\rangle\langle\vec{q}_k|$ $(k = 1, \dots, N-1)$
   →
   $$\langle \vec{q}_{k+1} | e^{-i\hat{H}\varepsilon} | \vec{q}_k\rangle = \langle \vec{q}_{k+1} | \mathbb{1} - i\hat{H}\varepsilon + \mathcal{O}(\varepsilon^2) | \vec{q}_k\rangle$$
5. For $\hat{H} = \hat{H}_1(\vec{q}) + \hat{H}_2(\vec{p})$ (Proof ➔ Problemset 12):
   $$\langle \vec{q}_{k+1} | \hat{H} | \vec{q}_k\rangle \mathrel{\hat{=}} \int \frac{d\vec{p}_k}{2\pi} H\left(\frac{\vec{q}_{k+1} + \vec{q}_k}{2}, \vec{p}_k\right) \exp\left[i\vec{p}_k \cdot (\vec{q}_{k+1} - \vec{q}_k)\right]$$
   Note that $\hat{H}$ is an operator whereas H is a function!
   This is more complicated for generic Hamiltonians $H = H(p, q)$ with terms like $q^2 p^2$ where ordering is important, ➔ Problemset 12.

↑ Notes

6. → *Hamiltonian phase-space path integral:* (We restore $\hbar$)

$$U(\vec{q}_a,\vec{q}_b;T) \mathrel{\hat{=}} \underbrace{\int_{\vec{q}_a}^{\vec{q}_b} \mathcal{D}\vec{q}(t)\; \mathcal{D}\vec{p}(t)}_{\lim_{N\to\infty}\prod_k \int \frac{d\vec{q}_k\, d\vec{p}_k}{2\pi\hbar}} \exp\left[\frac{i}{\hbar}\underbrace{\int_0^T dt\, \overbrace{\left(\vec{p}\cdot\dot{\vec{q}} - H(\vec{q},\vec{p})\right)}^{\hat{=}L(\vec{q},\dot{\vec{q}})}}_{\hat{=}S[\vec{q}]}\right] \tag{8.2}$$

- The functional integral measure is called *canonical measure* and does not depend on the system.
- In most cases (when H depends quadratically on $\vec{p}$), the functional integration over $\vec{p}$ can be evaluated. Then one ends up with the simpler form (8.1) that sums only over trajectories. The integration over momentum trajectories $\vec{p}(t)$ yields the PI measure C_ε.
- The Hamiltonian PI over phase space (8.2) is more general than the Lagrangian PI over trajectories (8.1).
- $\vec{q}$ and $\vec{p}$ do *not* satisfy the Hamiltonian EOMs! → Heisenberg uncertainty principle

8.2 Path Integrals for scalar fields

Identification: $q_i \leftrightarrow \phi(x)$

Example 8.2: Real scalar field

$$\langle\phi_a|e^{-i\hat{H}T}|\phi_a\rangle = \int_{\phi_a}^{\phi_b} \mathcal{D}\phi\; \mathcal{D}\pi\; \exp\left[\frac{i}{\hbar}\int_0^T d^4x\left(\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right)\right]$$

Evaluate π-integration

$$= \int \mathcal{D}\phi\; \exp\left[\frac{i}{\hbar}\int_0^T d^4x\, \mathcal{L}(\phi,\partial_\mu\phi)\right] \tag{8.3}$$

- *Lagrangian:* $\mathcal{L}(\phi,\partial_\mu\phi) = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)$
- *Boundaries:* $\phi(\vec{x},0) \equiv \phi_a(\vec{x})$ and $\phi(\vec{x},T) \equiv \phi_b(\vec{x})$

- All symmetries of $\mathcal{L}$ are manifest in the PI formalism
  This is not true for the Hamiltonian formalism which singles out a time direction!
- Abandon the Hamiltonian formalism and use (8.3) to *define* the time evolution
- Goal: Derive *correlation functions* & *Feynman rules* directly from PIs
  Here we only discuss correlation functions, for Feynman rules ➲ P&S pp. 284–289.

↑ Notes

Correlation functions

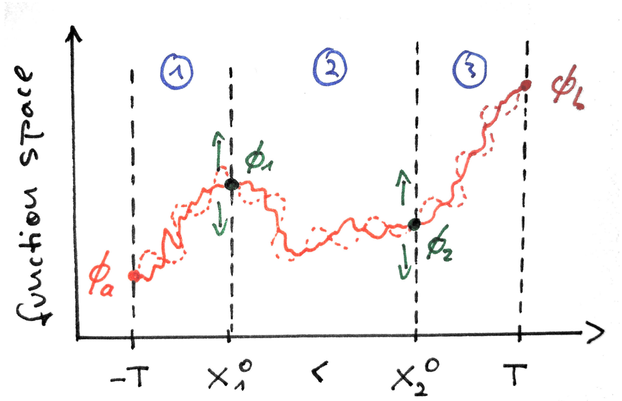
1. We would like to evaluate the two-point correlator with path integrals:

$$\langle\Omega|\mathcal{T}\underbrace{\phi_H(x_1)\phi_H(x_2)}_{\text{Operators}}|\Omega\rangle \overset{?}{\leftrightarrow} \int_{\phi(-T)=\phi_a}^{\phi(+T)=\phi_b} \mathcal{D}\phi\,\underbrace{\phi(x_1)\phi(x_2)}_{\text{Numbers}}\, e^{i\int_{-T}^{+T} d^4x\,\mathcal{L}(\phi)} \tag{8.4}$$

   ϕ_H are interacting Heisenberg field operators.

2. Split functional integral:

$$\int \mathcal{D}\phi(x) = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\phi(x_1^0,\vec{x})=\phi_1(\vec{x})}^{\phi(x_2^0,\vec{x})=\phi_2(\vec{x})} \mathcal{D}\phi(x)$$



3. →

$$(8.4) = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x})\, \underbrace{\phi_1(\vec{x}_1)\phi_2(\vec{x}_2)}_{0}$$

$$\times \underbrace{\langle\phi_b|e^{-iH(T-x_2^0)}|\phi_2\rangle}_{3}\, \underbrace{\langle\phi_2|e^{-iH(x_2^0-x_1^0)}|\phi_1\rangle}_{2}\, \underbrace{\langle\phi_1|e^{-iH(x_1^0+T)}|\phi_a\rangle}_{1}$$

   Here we use only the definition of the propagator!

4. Use $\int \mathcal{D}\phi_1(\vec{x})\,|\phi_1\rangle\langle\phi_1| = \mathbb{1}$ and $\phi_S(\vec{x}_1)|\phi_1\rangle = \phi_1(\vec{x}_1)|\phi_1\rangle$:
   (Compare this to $\hat{q}|q\rangle = q|q\rangle$.)

$$0\times 3\times 2\times 1 \overset{x_2^0>x_1^0}{=} \langle\phi_b|e^{-iH(T-x_2^0)}\phi_S(\vec{x}_2)e^{-iH(x_2^0-x_1^0)}\phi_S(\vec{x}_1)e^{-iH(x_1^0+T)}|\phi_a\rangle$$

$$= \langle\phi_b|\underbrace{e^{-iHT}}_{\to\propto|\Omega\rangle\langle\Omega|}\mathcal{T}\{\phi_H(x_1)\phi_H(x_2)\}\underbrace{e^{-iHT}}_{\to\propto|\Omega\rangle\langle\Omega|}|\phi_a\rangle$$

$$\xrightarrow{T\to\infty(1-i\varepsilon)} C\cdot\langle\Omega|\mathcal{T}\phi_H(x_1)\phi_H(x_2)|\Omega\rangle$$

5. Result ($\hbar = 1$):

$$\langle\Omega|\mathcal{T}\phi_H(x_1)\phi_H(x_2)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\int\mathcal{D}\phi\,\phi(x_1)\phi(x_2)\,\exp\left[i\int_{-T}^{+T} d^4x\,\mathcal{L}(\phi)\right]}{\int\mathcal{D}\phi\,\exp\left[i\int_{-T}^{+T} d^4x\,\mathcal{L}(\phi)\right]} \tag{8.5}$$

↑ Notes

The denominator ensures independence of the boundaries at $T \to \pm\infty$, ϕ_a and ϕ_b.

8.3 Application: Quantization of the Electromagnetic Field

Goal: Apply PI formalism to derive the photon propagator $\frac{-ig_{\mu\nu}}{k^2+i\varepsilon}$

1. Action:

$$S[A] = \int d^4x \left[-\frac{1}{4}(F_{\mu\nu})^2\right]$$

Partial integration with $A \xrightarrow{|x^\mu|\to\infty} 0$; use $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\hat{=} \frac{1}{2}\int d^4x\, A_\mu(x)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right) A_\nu(x)$$

Fourier transform

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \underbrace{\tilde{A}_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\tilde{A}_\nu(-k)}_{\Box}$$

2. Set $\tilde{A}_\mu(k) = k_\mu\,\alpha(k) \to \Box = 0 \to S[A] = 0 \to \int \mathcal{D}A\, e^0 = \infty$
   That's bad!

3. *Problem:* Gauge invariance $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha$
   Integration over continuity of *gauge-equivalent* configurations $A_\mu \sim 0 \Leftrightarrow A_\mu \propto \partial_\mu\alpha$ leads to divergence!

4. *Solution:* Count each physical configuration once (➔ Faddeev & Popov)

   a) Gauge fixing: $G(A) \overset{!}{=} 0$ (e.g. Lorenz gauge: $G(A) = \partial_\mu A^\mu$)

   b) Let $A^\alpha_\mu := A_\mu + \frac{1}{e}\partial_\mu\alpha$, then

$$1 = \int \mathcal{D}\alpha\, \delta(G(A^\alpha))\, \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

> **Note 8.1**
>
> $$\left[\prod_i \int da_i\right]\delta^{(n)}(\vec{g}(\vec{a}))\,\det\left(\frac{\partial\vec{g}}{\partial\vec{a}}\right) = \left[\prod_i \int dg_i\right]\delta^{(n)}(\vec{g}) = 1$$
>
> Here, $\delta^{(n)}(\vec{g}(\vec{a})) = \delta(g_1(\vec{a}))\cdots\delta(g_n(\vec{a}))$ and $\det\left(\frac{\partial\vec{g}}{\partial\vec{a}}\right)$ is the Jacobian of the vector-valued map $\vec{g} = \vec{g}(\vec{a})$.

   c) Assume that $\frac{\delta G(A^\alpha)}{\delta\alpha}$ is *independent* of A and α (true for the Lorenz gauge)
   This cannot be satisfied for non-abelian gauge theories → Ghost fields

↑ Notes

d)

$$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]}\, \delta(G(A^\alpha))$$

Substitute $\tilde{A} = A^\alpha = A + \frac{1}{e}\partial\alpha \to \mathcal{D}\tilde{A} = \mathcal{D}A$

Use gauge invariance: $S[A] = S[\tilde{A}]$

$$= \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \underbrace{\int \mathcal{D}\alpha}_{=\infty} \underbrace{\int \mathcal{D}\tilde{A}\, e^{iS[\tilde{A}]}\, \delta(G(\tilde{A}))}_{\text{only physically distinct configurations}} \tag{8.6}$$

e) Choose $G(A) = \partial^\mu A_\mu - \omega(x) \to \det\left(\frac{\delta G}{\delta\alpha}\right) = \det\left(\frac{1}{e}\partial^2\right)$

∂^2 is a linear operator on a function space; since the latter is infinite dimensional, think of ∂^2 as an "infinite-dimensional matrix".

$$(8.6) = \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x)) \tag{8.7}$$

(We renamed $\tilde{A}$ as A.)

f) True for any $\omega \to$ True for *normalized linear combinations*:

$$(8.7) = \underbrace{\mathcal{N}(\xi)}_{\text{Normalization}} \underbrace{\int \mathcal{D}\omega\, e^{-i\int d^4x\, \frac{\omega^2}{2\xi}}}_{\text{Linear combination}}$$

$$\times \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x))$$

So far, $\xi \in \mathbb{R}$ is arbitrary

$$= \mathcal{N}(\xi) \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right)$$

$$\times \int \mathcal{D}A\, e^{iS[A]} \underbrace{\exp\left[-i\int d^4x\, \frac{(\partial^\mu A_\mu)^2}{2\xi}\right]}_{\text{New term (breaks gauge symmetry)}}$$

Note that breaking gauge invariance in the new effective Lagrangian does not alter expectation values of physical (and therefore gauge-invariant) operators. Different Lagrangians can describe the same physics!

g) $\sphericalangle$ $O(\hat{A})$ *gauge invariant* operator: $O(\hat{A}^\alpha) = O(\hat{A})$, then

$$\langle\Omega|\mathcal{T}\, O(\hat{A})|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathcal{D}A\, O(A)\, \exp\left\{i\int_{-T}^{+T} d^4x \left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]\right\}}{\int \mathcal{D}A\, \exp\left\{i\int_{-T}^{+T} d^4x \left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]\right\}}$$

This follows along the same lines as (8.5). The gauge-invariance of the operator is needed in step (8.6) where we substitute A by $\tilde{A}$.

Important: The unkown and diverging prefactors have canceled!

↑ Notes

5. *New* action (same calculation as in step 1):

$$\tilde{S}[A] = \int d^4x \left[-\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \right]$$

Partial integration with $A \xrightarrow{|x^\mu|\to\infty} 0$; use $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\hat{=} \frac{1}{2} \int d^4x\, A_\mu(x)[\partial^2 g^{\mu\nu} - (1-\xi^{-1})\partial^\mu\partial^\nu]A_\nu(x)$$

Fourier transform

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k)[-k^2 g^{\mu\nu} + (1 - \underbrace{\xi^{-1}}_{\text{New!}})k^\mu k^\nu]\tilde{A}_\nu(-k)$$

Skip first and second step.

→ Argument of Step 2 no longer applies!

6. *Propagator:*

$$D_F^{\mu\nu}(x-y) = \langle\Omega|\mathcal{T} A^\mu(x)A^\nu(y)|\Omega\rangle$$

→ $\langle\Omega|\tilde{A}^\mu(k)\tilde{A}^\nu(q)|\Omega\rangle = 0$ for $k \neq -q$ (due to translation invariance)

Therefore

$$\tilde{D}_F^{\mu\nu}(q) = \langle\Omega|\tilde{A}^\mu(q)\tilde{A}^\nu(-q)|\Omega\rangle$$

Use $\tilde{A}^\nu(-q) = (\tilde{A}^\nu(q))^*$ since A^ν is real

Add $+i\varepsilon$ for regularization to the action

$$= \frac{\int \mathcal{D}A\, \tilde{A}^\mu(q)\tilde{A}^\nu(-q)\, \exp\{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)\overbrace{[-k^2g^{\mu\nu} + (1-\xi^{-1})k^\mu k^\nu]}^{\equiv M^{\mu\nu}(k)\text{ (symmetric)}}\tilde{A}_\nu(-k)\}}{\int \mathcal{D}A\, \exp\left\{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)[-k^2g^{\mu\nu} + (1-\xi^{-1})k^\mu k^\nu]\tilde{A}_\nu(-k)\right\}}$$

PI measure: $\mathcal{D}A = \prod_{\mu;k,k^0>0} d(\mathrm{Re}\,\tilde{A}^\mu(k))\; d(\mathrm{Im}\,\tilde{A}^\mu(k))$

Diagonalize $M^{\mu\nu}$, complete the square, and evaluate Gaussian integrals

Details ➔ Problemset 12

$$\hat{=}\, i(M^{-1}(q))^{\mu\nu}$$

Finally

$$\boxed{\tilde{D}_F^{\mu\nu}(q) \,\hat{=}\, \frac{-i}{q^2+i\varepsilon}\left[g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right]}$$

Check that this is the inverse of $M^{\mu\nu}(q)$!

7. *Gauges:*

- Set $\xi = 0$:

$$\tilde{D}_F^{\mu\nu}(q) = \frac{-i}{q^2+i\varepsilon}\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right] \quad \text{(Landau gauge)}$$

↑ Notes

- Set $\xi = 1$:

$$\tilde{D}_F^{\mu\nu}(q) = \frac{-i g^{\mu\nu}}{q^2 + i\varepsilon} \quad \text{(Feynman gauge)}$$

This form is Lorentz invariant since $g^{\mu\nu}$ is.

Note 8.2

- Correlators of gauge invariant operators are independent of ξ.
- For $\xi \to \infty$ we have $-k^2 g^{\mu\nu} + k^\mu k^\nu = M^{\mu\nu}(k)$.
  Since $(-k^2 g^{\mu\nu} + k^\mu k^\nu)k_\nu = 0$, the inverse $M^{-1}(k)$ does not exist!
- $-k^{-2} \lim_{\xi\to\infty} M^{\mu\nu} = g^{\mu\nu} - k^\mu k^\nu / k^2 = T^{\mu\nu}$ is a projector on transversal fields:
  $T^{\mu\nu} k_\nu = 0$ and $T^{\mu\sigma} T_\sigma{}^\nu = T^{\mu\nu}$
  → The (original) divergence is due to *longitudinal* gauge fields.

Problem Set 12

(due **10.07.2020**)

1. Calculation of the propagator of a generic, quadratic field theory in the path integral formalism
2. Path integrals and Weyl order: Quantization and non-commuting quantities

9 Non-Abelian Gauge Theories

Motivation:

- *So far:* Only few interactions considered: ϕ^4 and $\overline{\Psi}\gamma^\mu\Psi A_\mu$
- *Goal:* Construct interacting theories of vector particles: A^4 or $(\partial A)A^2$?
- *Problem:* Quantization is complicated by negative-norm states of the time component A^0:

$$\left[A_\mu(\vec{x}), \Pi_\nu(\vec{y})\right] = i g_{\mu\nu}\delta^{(3)}\left(\vec{x}-\vec{y}\right) \quad \text{but} \quad g_{00} = -g_{ii}$$

  (➔ Gupta-Bleuler quantization of Maxwell theory).
- *Observation:* In Maxwell theory, negative-norm states are cancelled by longitudinal polarization states. This is rooted in the gauge symmetry (Ward identity).
- *Idea:* Generalize Maxwell theory (or QED, if matter is involved) to gauge theories with other symmetry groups.
- *Spoiler:* This type of theory turns out to describe *all* fundamental forces of nature (except gravity); it is the foundation of the standard model!

We start with a thorough analysis of the gauge symmetry of QED. In a second step, we generalize our findings to non-abelian gauge groups. This yields the famous *Yang-Mills theories*.

9.1 The Geometry of Gauge Invariance

1. $\sphericalangle$ *Local* $U(1)$ symmetry G of Dirac field

$$\tilde{\Psi}(x) = e^{i\alpha(x)}\Psi(x)$$

   for arbitrary $\alpha(x) : \mathbb{R}^{1,3} \to \mathbb{R}$
   (In the following, a tilde always denotes symmetry-transformed quantities.)
2. *Goal:* Construct invariant Lagrangian
3. No problem without derivatives:
   All terms invariant under *global* $U(1)$ transformations allowed (e.g. $\overline{\Psi}(x)\Psi(x)$)
4. $\sphericalangle$ Directional derivative along $n \in \mathbb{R}^{1,3}$:

$$n^\mu\partial_\mu\Psi := \lim_{\varepsilon\to 0}\frac{\Psi(x+\varepsilon n) - \Psi(x)}{\varepsilon}$$

   $\Psi(x+\varepsilon n)$ and $\Psi(x)$ transform *differently* under G
   $\to n^\mu\partial_\mu\Psi$ has no simple transformation law
   ($\to$ not a useful building block for symmetric Lagrangians)
   To see why, calculate $n^\mu\partial_\mu\tilde{\Psi}$; the result is *not* just $e^{i\alpha(x)}n^\mu\partial_\mu\Psi$. This makes the construction of invariant terms for a symmetric Lagrangian very complicated.

↑ Notes

5. → We need a sensible way to compare fields at different points.
→ Postulate the existence of a "*comparator*" $U : \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \to \mathbb{C}$ with transformation

$$\tilde{U}(y,x) = e^{i\alpha(y)} U(y,x) e^{-i\alpha(x)} \quad \text{and} \quad U(y,y) = 1 \tag{9.1}$$

[we require $U(y,x) = e^{i\phi(y,x)}$]

→ $\Psi(y)$ and $U(y,x)\Psi(x)$ have same transformation law
(and therefore can be meaningfully compared)

Note that we do neither prove the existence of U nor provide its construction; we simply take such a function for granted. For more details, ➔ *fiber bundles* in *differential geometry*. In particular, the "comparator" relates to the concept of *parallel transport* between fibers of *principal bundles*.

6. *Covariant derivative:*

$$\boxed{n^\mu D_\mu \Psi := \lim_{\varepsilon\to 0} \frac{\Psi(x+\varepsilon n) - U(x+\varepsilon n, x)\Psi(x)}{\varepsilon}} \tag{9.2}$$

7. Assume $U(y,x)$ continuous →

$$U(x+\varepsilon n, x) = 1 - ie\,\varepsilon n^\mu A_\mu(x) + \mathcal{O}(\varepsilon^2) \tag{9.3}$$

e: arbitrary constant (rescales A_μ)
A_μ: new vector field = *(gauge) connection*

8. (9.3) in (9.2) $\overset{\circ}{\to}$

$$\boxed{D_\mu \Psi(x) = \partial_\mu \Psi(x) + ieA_\mu \Psi(x)}$$

9. (9.3) in (9.1) $\overset{\circ}{\to}$

$$\boxed{\tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)} \tag{9.4}$$

10. $\overset{\circ}{\to}$

$$\tilde{D}_\mu \tilde{\Psi}(x) = e^{i\alpha(x)} D_\mu \Psi(x)$$

→ $D\Psi$ transforms like Ψ (this makes it easy to construct invariant terms!)

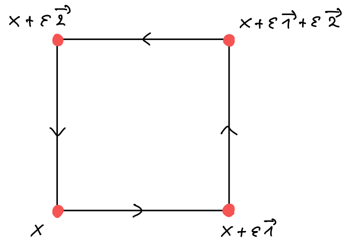
→ All terms invariant under *global* $U(1)$ transformations allowed if ∂ is replaced by D [e.g. $\overline{\Psi}(x)(i\not{D})\Psi(x)$]

11. *Conclusion:*
Local symmetry → Gauge field A_μ needed for covariant derivatives
Note that we did *not* put in the gauge field by hand! It automatically emerges as a necessary ingredient for terms that are locally symmetric and involve derivatives.

12. Last but not least: Kinetic energy term for A_μ?
(= Locally invariant term that depends only on A_μ and its derivatives.)

a) ⊲ Locally invariant loop (= local limit of a *Wilson loop*):
($\vec{1}$ and $\vec{2}$ are two arbirary orthogonal unit vectors.)

$$\begin{aligned}\mathbb{U}(x) := \quad & U(x, x + \varepsilon\vec{2}) \\ & \times U(x + \varepsilon\vec{2}, x + \varepsilon\vec{1} + \varepsilon\vec{2}) \\ & \times U(x + \varepsilon\vec{1} + \varepsilon\vec{2}, x + \varepsilon\vec{1}) \\ & \times U(x + \varepsilon\vec{1}, x)\end{aligned}$$



$\rightarrow \tilde{\mathbb{U}} = \mathbb{U}$ by construction

b) Use

$$U(x + \varepsilon n, x) = \exp\left[-ie\,\varepsilon n^\mu A_\mu\left(x + \frac{\varepsilon}{2}n\right) + \mathcal{O}(\varepsilon^3)\right]$$

To derive this form, recall $U(y,x) = e^{i\phi(y,x)}$ and $U(x,x) = 1$. Without changing our definition of A_μ in (9.3), we can restrict U to the form

$$U(y,x) = \exp\left[-ie\,\varepsilon n^\mu A_\mu(x + \varepsilon C) + \mathcal{O}(\varepsilon^3)\right]$$

where C is arbitrary. The additional constraint $U^\dagger(x,y) = U(y,x)$ then determines $C = \frac{1}{2}n$. Relaxing this assumption introduces additional vector fields (for the higher orders) that render the theory more complicated than necessary.

$\rightarrow$

$$\mathbb{U}(x) \overset{\circ}{=} 1 - i\varepsilon^2 e \underbrace{\left[\partial_1 A_2(x) - \partial_2 A_1(x)\right]}_{=:F_{12}} + \mathcal{O}(\varepsilon^3)$$

$\rightarrow$

$$\boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{(Field-strength tensor)}}$$

is locally invariant by construction

Note that a similar construction (that is, parallel transport along a small closed loop) gives rise to the notion of *curvature* of Riemannian manifolds. Consequently, the Riemann curvature tensor $R_{\mu\nu}$ of general relativity plays a similar role than the field-strength tensor $F_{\mu\nu}$ of Maxwell theory.

13. Most general gauge (and Lorentz-) invariant Lagrangian in $D = 3 + 1$:

- Gauge invariant →
  Constructed from Ψ, $D\Psi$, $F_{\mu\nu}$, $\partial F_{\mu\nu}$ etc. and *globally* U(1)-invariant
- Relativistic → Lorentz scalar
- Renormalizable → Terms of mass dimension at most 4
  (Otherwise the coupling constants of such terms have *negative* mass dimension and render the theory non-renormalizable.)

↑ Notes

→

$$\mathcal{L} = \overline{\Psi}(i\not{D})\Psi - m\overline{\Psi}\Psi - \frac{1}{4}(F_{\mu\nu})^2 - \underbrace{c_1 \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}}_{\text{breaks } P \text{ and } T} + \underbrace{c_2(\overline{\Psi}\Psi)^2 + \dots}_{\text{non-renormalizable}}$$

→ Most general P/T-symmetric Lagrangian: Maxwell-Dirac → QED

Note that $\varepsilon^{\alpha\beta\mu\nu}$ is a *pseudo tensor*, in contrast to $g^{\alpha\mu}g^{\beta\nu}$; therefore the c_1-term is a *pseudo scalar*.

9.2 The Yang-Mills Lagrangian

Goal: Replace local symmetry group U(1) by non-abelian Lie group G

Examples: O(3), SU(2), SU(3), ...

Details: ➔ Problemset 13

1. $\sphericalangle$ Lie group G represented by $n \times n$ unitary matrices V
   Typically, we consider the fundamental (or *defining*) representation of matrix Lie groups, e.g., $V = \exp\left(i\omega_j \frac{\sigma^j}{2}\right)$ for $G = \mathrm{SU}(2)$ with $n = 2$ and σ^j Pauli matrices.

2. Fields $\Psi = (\Psi_1, \dots, \Psi_n)^T$ are n-plets of Dirac fields Ψ_i:
   $\Psi : \mathbb{R}^{1,3} \to \mathbb{C}^4 \otimes \mathbb{C}^n \simeq \mathbb{C}^{4n}$ and transform as

$$\tilde{\Psi}(x) = V(x)\Psi(x) = V_{ij}(x)\Psi_j(x)$$

   with $V : \mathbb{R}^{1,3} \to G$ arbitrary
   Note that i, j are *not* spinor- but G-indices; each Ψ_j is a Dirac bispinor.

3. G *Lie group* → *Lie algebra* $\mathfrak{g}$ with N Hermitian generators t^a ($n \times n$-matrices, $a = 1, \dots, N$) that obey

$$\left[t^a, t^b\right] = if^{abc}t^c \qquad \text{Einstein notation!}$$

   with *structure constants* $f^{abc} \in \mathbb{C}$.

   The structure constants define the Lie algebra. One can always choose a basis $\{t^a\}$ such that they are completely antisymmetric in the three indices. Note that $a = 1, \dots, N$ is finite since G is assumed to be compact and the matrix representations of t^a are Hermitian because the matrix representations V are assumed to be unitary. ➔ P&S pp. 495–502 for details.

   →

$$V(x) = \exp\left[i\alpha^a(x)t^a\right] = 1 + i\alpha^a(x)\,t^a + \mathcal{O}(\alpha^2)$$

↑ Notes

4. The "*comparator*" is now a $n \times n$ unitary matrix with transformation

$$\tilde{U}(y,x) = V(y)U(y,x)V^\dagger(x) \quad \text{and} \quad U(y,y) = \mathbb{1} \tag{9.5}$$

→

$$U(x+\varepsilon n, x) = 1 + ig\,\varepsilon n^\mu A_\mu^a t^a + \mathcal{O}(\varepsilon^2) \tag{9.6}$$

g: arbitrary constant (rescales A_μ^a)
A_μ^a: N vector fields (= gauge connections, one for each generator t^a)
The "comparator" acts on Ψ, i.e., on the representation of G given by V; thus its infinitesimal action must be generated by the representation of the corresponding Lie algebra $\{t^a\}$.

5. (9.2) → *Covariant derivative:*

$$D_\mu \overset{\circ}{=} \partial_\mu - igA_\mu^a t^a \tag{9.7}$$

In the literature, it is often written $A_\mu \equiv A_\mu^a t^a$, i.e., A_μ is a Lie-algebra valued field.

6. Transformation of A_μ^a:
   a) (9.6) in (9.5) →

$$1 + ig\,\varepsilon n^\mu \tilde{A}_\mu^a t^a = V(x+\varepsilon n)\left(1 + ig\,\varepsilon n^\mu A_\mu^a t^a\right)V^\dagger(x)$$

   b) Use

$$V(x+\varepsilon n)V^\dagger(x) \overset{\circ}{=} 1 + \varepsilon n^\mu V(x)\left[-\partial_\mu V^\dagger(x)\right] + \mathcal{O}(\varepsilon^2)$$

   [Recall that $0 = \partial_\mu \mathbb{1} = \partial_\mu(VV^\dagger) = (\partial_\mu V)V^\dagger + V(\partial_\mu V^\dagger)$.]

   to show

$$\tilde{A}_\mu^a t^a = V(x)\left[A_\mu^a t^a + \frac{i}{g}\partial_\mu\right]V^\dagger(x) \tag{9.8}$$

   This transformation law is exact, i.e., true for any V. Note that ∂_μ acts only on $V^\dagger$ and not on what comes after $\tilde{A}_\mu^a t^a$!

   c) $\partial_\mu V^\dagger(x)$ is not easy to evaluate (non-commuting operators in the exponent!)→ $\sphericalangle$ Infinitesimal transformation $V^\dagger(x) \approx \mathbb{1}$:

$$V(x) = \mathbb{1} + i\alpha^a(x)t^a + \mathcal{O}(\alpha^2)$$
$$\text{and} \quad \partial_\mu V^\dagger(x) = -i\partial_\mu\alpha^a(x)t^a + \mathcal{O}(\alpha^2)$$

   With $f^{cba} = -f^{abc}$ →

$$\tilde{A}_\mu^a \overset{\circ}{\approx} A_\mu^a + \frac{1}{g}\partial_\mu\alpha^a + \underbrace{f^{abc}A_\mu^b\alpha^c}_{\text{New!}}$$

↑ Notes

This transformation law is only true for *infinitesimal transformations* $V \approx \mathbb{1}$ (hence the "$\approx$"). For an abelian Lie group [such as U(1)], it is $f^{abc} \equiv 0$ and this expression is exact.

7. (9.8) in (9.7) → Transformation of $D_\mu \Psi$:

$$\tilde{D}_\mu \tilde{\Psi} \stackrel{\circ}{=} V\, D_\mu \Psi \tag{9.9}$$

Use again $(\partial_\mu V^\dagger)V = -V^\dagger(\partial_\mu V)$ to show this.

→ $D_\mu \Psi$ transforms like Ψ
→ $\overline{\Psi} D_\mu \Psi$ is gauge-invariant and $\overline{\Psi} \not{D} \Psi$ is both gauge- and Lorentz invariant

8. Last but not least: Kinetic energy term for A_μ^a?
Here, we follow an alternative approach to find such terms (without using the infinitesimal loop construction from above):
   a) Iteration of (9.9) implies $\tilde{D}_\mu \tilde{D}_\nu \tilde{\Psi} = V\, D_\mu D_\nu \Psi$

$$\Rightarrow \quad [\tilde{D}_\mu, \tilde{D}_\nu]\,\tilde{\Psi} = V\,[D_\mu, D_\nu]\,\Psi = V\,[D_\mu, D_\nu]\,V^\dagger\,\tilde{\Psi}$$
$$\Rightarrow \quad [\tilde{D}_\mu, \tilde{D}_\nu] \quad = V\,[D_\mu, D_\nu]\,V^\dagger \tag{9.10}$$

   b) On the other hand: (9.7) →

$$-ig\, \underbrace{F_{\mu\nu}^a t^a}_{=:F_{\mu\nu}} := [D_\mu, D_\nu]$$
$$\text{with} \quad F_{\mu\nu}^a \stackrel{\circ}{=} \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

   $F_{\mu\nu}^a$: N field-strength tensors
   (Note that $F_{\mu\nu} \equiv F_{\mu\nu}^a t^a$ is a $n \times n$-matrix, not a derivative.)

   c) (9.10) →

$$\tilde{F}_{\mu\nu} = \tilde{F}_{\mu\nu}^a t^a = V F_{\mu\nu} V^\dagger$$

   → $F_{\mu\nu}$ is no longer gauge invariant
   (cf. Maxwell theory where V and t^a are 1×1-matrices so that $\tilde{F}_{\mu\nu} = F_{\mu\nu}$)

   d) Simplest invariant term:

$$\mathcal{L}_{YM} = -\frac{1}{2}\,\mathrm{Tr}\left[F^2\right] \equiv -\frac{1}{2}\,\mathrm{Tr}\left[(F_{\mu\nu}^a t^a)(F^{\mu\nu b} t^b)\right]$$

Use $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$

$$= -\frac{1}{4}(F_{\mu\nu}^a)^2 \qquad \text{(Yang-Mills theory)}$$

One can always choose a basis $\{t^a\}$ of $\mathfrak{g}$ where $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$, ➔ P&S p. 498ff.

↑ Notes

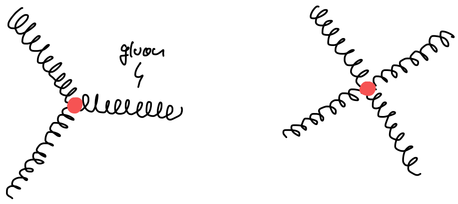
Note 9.1

Important:

$$F^2 \sim (\partial A)^2 + \underbrace{f\,(\partial A)AA + f^2\,AAAA}_{\text{Interactions}}$$

→ *Interacting* QFT for $f \neq 0$ (= non-abelian)!
→ Gauge bosons scatter off each other

Example:
Quantum Chromodynamics [$G = \mathrm{SU}(3)$] (➔ last lecture)
Gauge bosons = *Gluons* → Pure gluon vertices in Feynman diagrams:



→ Bound states of gluons: *Glueballs* (not yet observed!)
That the mass of glueballs cannot be arbitrarily small is (part of) one of the *Millennium Prize Problems* of the *Clay Mathematics Institute*: the "Yang-Mills Existence and Mass Gap" problem, ➔ `https://www.claymath.org/millennium-problems`.

9. Couple Dirac fermions to Yang-Mills gauge field:

$$\mathcal{L}_{YM+D} = \overline{\Psi}\left(i\not{D} - m\right)\Psi - \frac{1}{4}(F^a_{\mu\nu})^2$$

Two parameters:
m: Fermion mass
g: Coupling constant (hidden in D and F^2)

This is the most general Lagrangian that is ...

- gauge invariant
- Lorentz invariant
- renormalizable
- P- and T-symmetric

→ Yang-Mills theories describe *all* fundamental forces of the standard model! (➔ last lecture)

Note: Let us be precise about the symbols:

$$\not{D} = \gamma^\mu D_\mu = \partial_\mu\,\gamma^\mu \mathbb{1}_n - igA^a_\mu\,\gamma^\mu t^a$$

where

$$\gamma^\mu t^a \equiv \gamma^\mu \otimes t^a = \gamma^\mu_{\alpha\beta}\cdot t^a_{mn} = (\gamma^\mu t^a)_{(\alpha,m)(\beta,n)}$$

↑ Notes

so that

$$\Psi : \mathbb{R}^{1,3} \to \mathbb{C}^4 \otimes \mathbb{C}^n \simeq \mathbb{C}^{4n}$$

carries a four-dimensional bispinor representation of the Lorentz group, $\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$ with $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, and the representation V of the gauge group G, $V = \exp(i\alpha^a t^a)$. Then it follows in particular

$$\overline{\tilde{\Psi}} = (V\Psi)^\dagger \gamma^0 = \Psi^\dagger V^\dagger \otimes \gamma^0 = \Psi^\dagger \gamma^0 V^\dagger = \overline{\Psi} V^\dagger$$

so that $\overline{\Psi}\Psi$ is gauge invariant.

Note 9.2

The mass term A^2 is *not* allowed as it is not gauge invariant!
Recall that $\frac{1}{2}m\phi^2$ was responsible for the mass gap of ϕ^4-theory and $m\overline{\Psi}\Psi$ for the mass of Dirac fermions.

→ Gauge bosons of Yang-Mills theories are massless.
For QED, this is fine: The photon *is* massless.

Problem:
The weak interaction is short-ranged, i.e., its gauge bosons $W^\pm$ and Z *have* mass!

Solution:
Higgs mechanism (➔ next lecture)

10 Excursions

10.1 The Higgs Mechanism

Motivation:

- *Problem 1*: Recall that we cannot add a mass term A^2 to the Yang-Mills Lagrangian as it would break gauge invariance (➔ note at the end of the last lecture).

  How do the $W^\pm$ and Z bosons that mediate the short-ranged weak interaction obtain their observed masses?

- *Problem 2*: Although we have shown that a Dirac mass term $\overline{\Psi}\Psi$ is allowed in general Yang-Mills theories, in the particular case of the standard model, it is *forbidden* (➔ next lecture).

  How do quarks and leptons gain their observed masses?

Solution to *both* problems: *Higgs mechanism*

(For simplicity, will consider only *classical* field theories and skip their quantization as the crucial mechanisms are already present at this level.)
For the quantization of gauge theories with Higgs field, ➔ Chapter 21 of P&S (p. 731ff.)

10.1.1 Abelian Example: The Standard Approach

This approach follows loosely the essay ➔ http://philsci-archive.pitt.edu/9295/1/Spontaneous_symmetry_breaking_in_the_Higgs_mechanism.pdf (see also references therein); this is also roughly the approach of P&S, ➔ pp. 690–692.

Goal: $\triangleleft$ Abelian gauge theory to understand the Higgs mechanism

1. $\triangleleft$ Maxwell theory coupled to a complex scalar field:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu\phi|^2 - V(\phi)$$
$$\text{with potential} \quad V(\phi) = \mu^2|\phi|^2 + \lambda|\phi|^4 \tag{10.1}$$

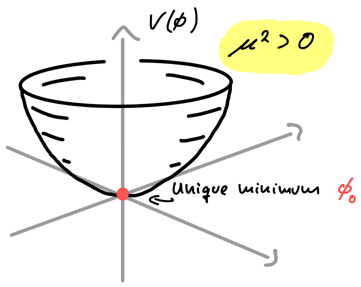
   and $D_\mu = \partial_\mu + ieA_\mu$

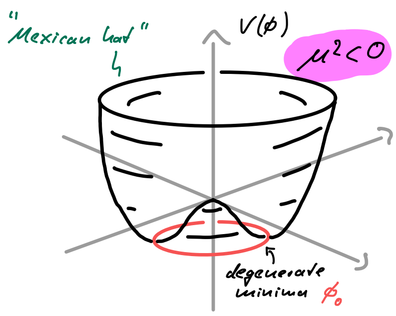
2. $\mathcal{L}$ is Invariant under the $\mathrm{U}(1)$ *gauge transformation*

$$\tilde{\phi}(x) = e^{i\alpha(x)}\phi(x) \quad \text{and} \quad \tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)$$

3. $\triangleleft$ $V(\phi)$ in the complex plane $\phi \in \mathbb{C}$:

↑ Notes





- $\mu^2 > 0$: Unique minimum with $\langle\phi\rangle = 0$
- $\mu^2 < 0 \to$ *Mexican hat potential*:
  Degenerate minima with non-zero vacuum expectation value (VEV)

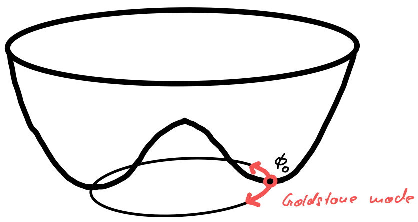
$$\phi_0 := \langle\phi\rangle \quad \text{and} \quad v := |\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}} \neq 0$$

  → Ground states are *not* symmetric under global phase rotations
  → *Spontaneous symmetry breaking (SSB)* of the *global* U(1) symmetry

4. Aside: *The Goldstone theorem:*

> If a *global, continuous* symmetry is *spontaneously broken*, there is one *massless scalar* (= Spin-0) particle for each broken symmetry generator; these particles are known as *(Nambu-)Goldstone bosons*.

"Proof by picture:"



Long wavelength deformations of the field with the broken symmetry generator (red arrows) cost arbitrary low energy → Gapless Goldstone mode

Examples:

- Breaking of translation and rotation invariance in crystals
  → Transversal and longitudinal phonons
  This is a subtle example. There are in total 6 generators that are broken: 3 translations P_x, P_y, P_z and 3 rotations L_x, L_y, L_z—but there are only 3 (not 6!) Goldstone modes, namely two transversal and one longitudinal phonon. The reason is that the Euclidean group of translations and rotations is $\mathrm{E}(3) = \mathrm{O}(3) \ltimes \mathrm{T}(3)$ with rotations O(3) and translations $\mathrm{T}(3) = \mathbb{R}^3$ and *not* $\mathrm{E}(3) = \mathrm{O}(3) \times \mathrm{T}(3)$; in particular, the generators of rotations L_i (= angular momentum operators) and translations P_i (= momentum operators) do *not* commute. Thus for nonrelativistic field theories, the above statement is only true if the different generators commute; ➔ https://doi.org/10.1103/PhysRevLett.108.251602 for details on counting the Goldstone modes correctly in such theories.

↑ Notes

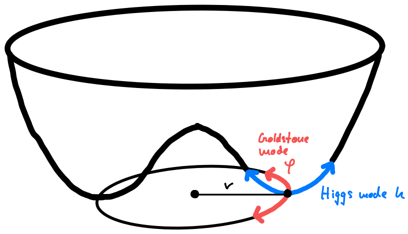
- Breaking of rotation symmetry in a ferromagnet
  → Magnons (= Spin waves)

But there is one notable *exception:*
In conventional superconductors the U(1) symmetry (generated by particle number conservation) is broken spontaneously (➔ Ginzburg-Landau theory)—but there is *no* massless Goldstone boson! (Recall that the photon in superconductors is short-ranged and therefore massive; it is also not a scalar.)

→ How can the Goldstone theorem fail?
→ Answer: Gauge symmetry & Higgs mechanism! (➔ below)

5. Assume that $\langle\phi\rangle = \phi_0 = v$ breaks the global U(1) symmetry
   → Expand ϕ in small fluctuations around $\langle\phi\rangle$:

$$\phi(x) = [v + h(x)] \cdot e^{i\varphi(x)}$$



with two *real* fields:

$h(x)$: Higgs field
$\varphi(x)$: Goldstone boson
(The terms "field", "mode" and "boson" are often used interchangeably.)

→

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \left[(\partial_\mu + ieA_\mu)(v+h)e^{i\varphi}\right]\left[(\partial^\mu - ieA^\mu)(v+h)e^{-i\varphi}\right] - \mu^2(v+h)^2 - \lambda(v+h)^4$$

$$\mathrel{\hat=} \underbrace{-\frac{1}{4}(F_{\mu\nu})^2 + e^2v^2 A_\mu^2}_{\text{Massive gauge field (Yay!)}} + \underbrace{(\partial_\mu h)^2 - m_h^2 h^2}_{\text{Higgs field with mass } m_h^2 = 4\lambda v^2} + \underbrace{v^2(\partial_\mu\varphi)^2}_{\text{Massless Goldstone mode}} + \underbrace{2ev^2(\partial_\mu\varphi)A^\mu}_{\text{Quadratic coupling}} + \underbrace{\dots}_{\text{Interactions}}$$

The interactions include terms cubic and quartic in the dynamical fields φ, h and A_μ.

Note that this Lagrangian is still gauge invariant under the gauge transformation

$$\tilde\varphi = \varphi + \alpha \quad\text{and}\quad \tilde A_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha \quad\text{and}\quad \tilde h = h$$

6. Fix the gauge in the *unitary gauge* $\phi \overset{!}{=} \phi^* \;\Leftrightarrow\; \varphi \equiv 0$ with the gauge transformation $\alpha(x) = -\varphi(x)$, i.e.,

$$\tilde\phi = e^{-i\varphi}\phi \quad\text{and}\quad \tilde A_\mu = A_\mu + \frac{1}{e}\partial_\mu\varphi(x)$$

As the gauge is now fixed, the theory has no longer a gauge symmetry! Indeed, $\phi = \phi^*$ is violated by the transformation $\phi \mapsto \phi e^{i\alpha(x)}$. Note that the local gauge symmetry is

↑ Notes

lost *not* because of SSB but because of explicit gauge fixing.

→

$$\tilde{\mathcal{L}} = \underbrace{-\frac{1}{4}(F_{\mu\nu})^2 + e^2v^2 A_\mu^2}_{\text{Massive gauge field}} + \underbrace{(\partial_\mu h)^2 - m_h^2 h^2}_{\text{Massive Higgs field}} + \text{Interactions} \tag{10.2}$$

→ Goldstone mode φ has disappeared!

Reason: φ is a *pure gauge* d.o.f. and therefore *not physical*!

- This is what one means by

  "The Goldstone boson is 'eaten' by the Gauge boson to give it a mass."

  Personally, I do not like this metaphoric description as a didactic "auxiliary structure" for a mathematically subtle mechanism because it explains nothing and makes only sense if you already understood the math.
- This explains how the Goldstone theorem can fail for gauge theories. Conventional superconductivity is therefore a non-relativistic example for the Higgs mechanism where the Goldstone mode vanishes and instead the gauge boson (the photon, now a quasiparticle excitation) obtains a mass m which leads to the Meissner effect; the London penetration depth is then given by $\lambda_L \propto m^{-1}$. Historically, this observation in condensed matter physics motivated the application of the Higgs mechanism to the problem of mass generation in high-energy physics.

7. *Consistency check:* Counting physical degrees of freedom:

$$\begin{aligned}\#(\text{d.o.f.}) \text{ before SSB} &= 2\,(\text{massless vector boson}) + 2\,(\text{complex scalar field}) = 4\\ \#(\text{d.o.f.}) \text{ after SSB} &= 3\,(\text{massive vector boson}) + 1\,(\text{real scalar Higgs field}) = 4\end{aligned}$$

→ We did not loose any d.o.f. but merely "mixed" them differently!

Note that a massless vector boson (like the photon) has only *two* transversal polarizations. By contrast, a massive vector boson has an additional longitudinal polarization.

Note 10.1

- We have seen that the Goldstone theorem is not valid for gauge theories (since the Goldston boson can become "pure gauge").
- The Higgs mechanism also describes conventional superconductivity as spontaneous U(1) symmetry breaking in a charged superfluid (➔ Ginzburg-Landau theory).
  In a superconductor, the photon (then a quasiparticle) acquires a mass and can no longer propagate (➔ Meissner effect).
- There is also an intuitive picture how the Goldstone theorem fails in the presence

↑ Notes

of a gauge field:

The proof of the Godstone theorem relies on the *absence of long-range interactions* (like the Coulomb interaction). Only then, a massless Goldstone boson can be predicted. However, coupling a (yet massless) gauge field to the (yet U(1)-symmetric) complex scalar field adds exactly such long-range interactions between fluctuations of the scalar. Due to these long-range interactions, the long-wavelength fluctuations of the real mode of the scalar field "parallel" to the symmetry—that under normal circumstances give rise to the massless Goldstone mode—develops a mass gap and mixes with the gauge bosons. The result is a massive Spin-1 gauge boson, now a collective "quasiparticle" excitation of the former gauge field and the Goldstone mode of the scalar. The other real mode of the complex scalar that is "orthogonal" to the symmetry gives rise to the Higgs boson.

- The Higgs mechanism is sometimes explained as "spontaneous breaking of a gauge symmetry." This is a misleading statement as gauge symmetries are redundancies of our mathematical description; breaking a gauge symmetry should consequently not lead to observable phenomena. As the mass generation due to the Higgs mechanism is clearly observable, it cannot be rooted in the breaking of a gauge symmetry. In addition, there is *Elitzur's theorem* (➔ `https://journals.aps.org/prd/abstract/10.1103/PhysRevD.12.3978`) that rigorously forbids SSB for local (gauge) symmetries.

  Indeed, there are equivalent descriptions of the Higgs mechanism that circumvent the concept of "gauge symmetry breaking" altogether. Here a few references for the interested student:

  - A gauge-invariant treatment of the Higgs mechanism (for the weak interaction) is given in ➔ `https://doi.org/10.1016/0370-2693(94)90953-9`.
  - A gauge-invariant treatment of the U(1) symmetry breaking in superconductors is discussed in ➔ `https://arxiv.org/abs/cond-mat/0503400`.
  - A few general remarks on the impossibility of spontaneously breaking gauge symmetries can be found here ➔ `http://web.physics.ucsb.edu/~d_else/gauge_rant.pdf`.

  ➔ Gauge-invariant approach below

10.1.2 Bonus: A Gauge-Invariant Approach

This approach is based on Chapter 6.1 (p. 105ff.) of *Rubakov, "Classical theory of gauge fields", Princeton University Press (2002).*

1. ⊲ Again (10.1)

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu\phi|^2 - \mu^2|\phi|^2 - \lambda|\phi|^4$$

2. Let $\mu^2 < 0$ (= symmetry-broken phase) → Classical ground state (= vacuum):

$$\phi_0(x) = e^{i\alpha(x)}\phi_0$$

with $\alpha(x)$ arbitrary (w.l.o.g. $\alpha(x) \equiv 0$) and $|\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}} = v \neq 0$ (w.l.o.g. $\phi_0 = v$)

3. $\lessdot$ Small fluctuations around ϕ_0 and introduce the new *real* fields $h(x)$, $\varphi(x)$ and $B_\mu(x)$:

$$\phi \equiv [v + h(x)]\, e^{i\varphi(x)} \quad \text{and} \quad B_\mu(x) \equiv A_\mu(x) + \frac{1}{e}\partial_\mu \varphi(x)$$

Note that $\beta(x)$ is only well-defined if $\phi(x) \neq 0$ everywhere and we can ignore the ambiguity $\beta = \beta + 2\pi$; this is true for small fluctuations around the vacuum $\phi_0 = v$.

$\to$ Gauge transformations:

$$\begin{aligned} \tilde{\varphi} &= \varphi + \alpha && \to \text{pure gauge} && = \text{only gauge d.o.f.} \\ \tilde{h} &= h && \to \text{gauge invariant} && = \text{only physical d.o.f.} \\ \tilde{B}_\mu &= B_\mu && \to \text{gauge invariant} && = \text{only physical d.o.f.} \end{aligned}$$

$\to$ B_μ is not a gauge field as it is gauge *invariant*.

Indeed,

$$\tilde{B}_\mu = \tilde{A}_\mu + \frac{1}{e}\partial_\mu \tilde{\varphi} = A_\mu + \frac{1}{e}\partial_\mu(\tilde{\varphi} - \alpha) = B_\mu$$

Compare this to

$$\begin{aligned} \tilde{\phi} &= e^{i\alpha}\phi && \to \text{gauge dependent} && = \text{physical and gauge d.o.f.} \\ \tilde{A}_\mu &= A_\mu - \frac{1}{e}\partial_\mu \alpha && \to \text{gauge dependent} && = \text{physical and gauge d.o.f.} \end{aligned}$$

4. Express Lagrangian in new fields:

$$\mathcal{L} \mathrel{\hat{=}} -\frac{1}{4}(B_{\mu\nu})^2 + e^2 v^2\, B_\mu^2 + (\partial_\mu h)^2 - m_h^2\, h^2 + \ldots \overset{B_\mu \overset{!}{=} A_\mu}{=} \tag{10.2}$$

$$\text{with} \quad B_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu \mathrel{\hat{=}} \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

$\to$

- Gauge d.o.f. φ drops out and is unconstrained by the Lagrangian
- $\mathcal{L}$ is manifestly gauge-invariant (Note that $\mathcal{L} = \tilde{\mathcal{L}}$ only if we fix the unitary gauge $B_\mu \overset{!}{=} A_\mu$ [this is a constraint on A_μ].)
- B_μ is a massive vector boson
- h is a massive Higgs mode

5. *Take-home-message:*
The crucial ingredient of the Higgs mechanism is the *non-zero vacuum expectation value* of

↑ Notes

the Higgs field $\phi_0 = v$ which *can* be explained by the spontaneous breaking of a *global*, *continuous* symmetry. However:

> There is *no* spontaneous breaking of *local* gauge symmetries in the Higgs mechanism.
>
> As a matter of fact, local gauge symmetries can *never* break spontaneously (→ Elitzur's theorem), they are a consequence of redundancies in our mathematical description. In particular, they do *not* give rise to conserved charges (→ Noether's theorem).

Note 10.2

The Higgs mechanism can be straightforwardly generalized to non-abelian gauge symmetries.

→ Next lecture for the electroweak interaction with $SU(2) \times U(1)$ gauge symmetry.

Problem Set 13

(due **17.07.2020**)

1. Non-abelian gauge theories and the Yang-Mills Lagrangian: The basic concept

10.2 The Standard Model

This section does *not* follow P&S but is a collage of various sources.

Preliminaries:

- Define the chiral projectors

$$P_R := \frac{1}{2}(\mathbb{1}_4 + \gamma^5) \overset{\text{Weyl}}{=} \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \quad \text{and} \quad P_L := \frac{1}{2}(\mathbb{1}_4 - \gamma^5) \overset{\text{Weyl}}{=} \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix}$$

  and the chiral fermion fields

$$\Psi_R := P_R \Psi \quad \text{and} \quad \Psi_L := P_L \Psi$$

- With $\overline{\Psi} P_R = \overline{\Psi}_L$ and $\overline{\Psi} P_L = \overline{\Psi}_R$ show that

$$\overline{\Psi}(i\partial\!\!\!/ - m)\Psi \overset{\circ}{=} \overline{\Psi}_R(i\partial\!\!\!/)\Psi_R + \overline{\Psi}_L(i\partial\!\!\!/)\Psi_L - m\overline{\Psi}_L\Psi_R - m\overline{\Psi}_R\Psi_L \tag{10.3}$$

  Only the mass term mixes right- and left-handed fermions. We did not use this notation so far, because there was no reason to (and the left-hand side is shorter).

- The Dirac representation is *reducible*, a fact that is manifest in the Weyl basis, recall (3.7). Alternatively, it is easy to check that

$$\left[P_{R/L}, \Lambda_{\frac{1}{2}}\right] \overset{\circ}{=} 0$$

  so that the decomposition (10.3) is irreducible for Lorentz transformations.

  → Terms like $\overline{\Psi}_R(i\partial\!\!\!/)\Psi_R$ and $\overline{\Psi}_L\Psi_R$ are Lorentz invariant on their own and do not mix with their counterparts $\overline{\Psi}_L(i\partial\!\!\!/)\Psi_L$ and $\overline{\Psi}_R\Psi_L$ under Lorentz transformations.

  → Under additional (gauge) symmetries, the left- and right-handed fields $\Psi_{L/R}$ (then multiplets) can transform under *different* representations of these new symmetry groups!

10.2.1 Overview

1. *Field content:*
   - *Fermions (= Spin-$\frac{1}{2}$):*

Generation n	I		II		III	
Leptons	e_L	e_R	μ_L	μ_R	τ_L	τ_R
	ν_{eL}	(ν_{eR})	$\nu_{\mu L}$	$(\nu_{\mu R})$	$\nu_{\tau L}$	$(\nu_{\tau R})$
Quarks	u_L	u_R	c_L	c_R	t_L	t_R
	d_L	d_R	s_L	s_R	b_L	b_R

↑ Notes

- Here, each symbol $x_{L/R}$ denotes a four-component, chiral bispinor field which describes both a fermion and its corresponding antifermion (recall the QED Lagrangian). Note that the chirality is reversed for the antiparticles: e_L describes left-handed electrons and *right*-handed positrons.
- The right-handed neutrinos (in parantheses) have not been observed. In the standard model, these fields are completely uncharged (mathematically speaking, they transform under the trivial representation of all gauge groups) and decouple completely from the observable sector; thus these fields are typically omitted in the Lagrangian (➔ below).
- The three generations of fermions are not necessary for the symmetry considerations that follow. We will simply sum over the generation index n. It is unclear why there are *three* generations; however, so far there is no evidence for a fourth generation. All stable baryonic matter in the observable universe is made from first generation fermions as the other generations are much heavier and decay quickly into first generation particles.
- While the total number of generations is not determined, the fact that each generation contains three chiral leptons (e.g. e_L, e_R, ν_{eL}) and four chiral quarks (e.g. u_L, d_L, u_R, d_R) is crucial to cancel the so called *chiral anomaly* when quantizing the theory. The number of quarks and leptons is therefore not independent! ➔ P&S pp. 705–707

- *Vector bosons (= Spin-1):*

Force	Electroweak	Strong
Gauge group	$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$\mathrm{SU}(3)_C$
# Generators	$3+1=4$	8
Gauge fields	$\underbrace{W^i_\mu\ (i=1,2,3),\ B_\mu}_{\text{Before Higgs SSB}}$	$G^a_\mu\ (a=1,\dots,8)$
Gauge bosons	$\underbrace{\gamma, W^+, W^-, Z}_{\text{After Higgs SSB}}$	$8\times$ Gluons

Warning: The gauge field B_μ of the $U(1)_Y$ symmetry does *not* correspond to the photon γ of QED (➔ Higgs mechanism in the GWS theory below).

- *Scalar bosons (= Spin-0):*

$$\underbrace{2\times \textit{Complex}\text{ Higgs fields } \phi^+\ \phi^0}_{\text{Before Higgs SSB}} \xrightarrow{3\times\text{SSB}} \underbrace{1\times \textit{Real}\text{ Higgs field } h}_{\text{After Higgs SSB}}$$

The three missing d.o.f. after SSB give the three vector bosons $W^\pm$ and Z their mass ($\leftrightarrow$ longitudinal component).

Why is nature like this? That doesn't look very pretty!
Well, we don't know! The most probable answer is that at very high energies (=

↑ Notes

Planck scale) the picture becomes more symmetric with fewer distinct fields. The mess we observe may be caused by spontaneous symmetry breaking at our "low" energies. Finding a "prettier" construction is the quest for a GUT, a *Grand Unified Theory*.

2. Question: How to put this "chaos" into a consistent (= relativistic, renormalizable) QFT?

3. Answer:

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{EWS}} + \mathcal{L}_{\text{QCD}} \qquad \text{(Standard model)}$$

4. Two parts:
   - Electroweak Standard Model $\mathcal{L}_{\text{EWS}}$
     = Glashow-Weinberg-Salam (GWS) Theory
     = Unification of weak & electromagnetic force
     (+ mass generation through Higgs mechanism)
   - Quantum Chromodynamics $\mathcal{L}_{\text{QCD}}$ = Strong force

10.2.2 The Glashow-Weinberg-Salam Theory

GWS theory = Unification of the *electromagnetic* and *weak* interaction of the standard model; explains the masses of $W^\pm$ and Z bosons and all the fermions (including quarks) with the Higgs mechanism.

Goal: Generalize the Higgs mechanism to the Standard model

1. *Lagrangian:*

$$\mathcal{L}_{\text{EWS}} = \mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{Yang–Mills}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}$$

   We will discuss each term separately in the following.

2. *Gauge symmetry* (pre-Higgs, i.e., without SSB of the vacuum):

$$\underbrace{\text{SU}(2)_L}_{\text{Weak isospin}} \times \underbrace{\text{U}(1)_Y}_{\text{Weak Hypercharge}}$$

   - $\text{SU}(2)_L \to$ 3 generators T^i, $i = 1, 2, 3$ with

$$\left[T^i, T^j\right] = i\varepsilon^{ijk} T^k$$

     → *Irreducible representations:* (hats denote representation matrices)
     - 1D: Trivial representation $\hat{T}^i = 0$ (= *Singlet* representation)
     - 2D: Pauli matrices $\hat{T}^i = \frac{\sigma^i}{2}$ (= *Doublet* representation)
       (In the following, $\hat{T}^i$ always denotes the doublet representation.)

     → Eigenvalue of $\hat{T}^3$ = *The* weak isospin T^3
     ($T^3 = \pm\frac{1}{2}$ for doublet and $T^3 = 0$ for singlet)
     (For eigenvalues, we do not write hats as these are not matrices.)

↑ Notes

- $\mathrm{U}(1)_Y \to 1$ generator Y

$$\left[Y, T^i\right] = 0$$

(Since the gauge group is direct product of $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$.)

Schur's lemma $\to \hat{Y} = \text{Number} \times \mathbb{1} = \text{Hypercharge } Y \times \mathbb{1}$

3. $\mathrm{SU}(2)_L$ *Representations:*
We focus here on the first generation fermions. The values (= representations) of the weak hypercharge cannot be infered at this point; we will discuss them after the Higgs mechanism.

- *Left-handed fields = Isospin doublets:*

$$\Psi_L = \underbrace{\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}}_{\text{Gen. I}}, \underbrace{\begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}}_{\text{Gen. II}}, \underbrace{\begin{pmatrix} t_L \\ b_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}}_{\text{Gen. III}} \tag{10.4}$$

$\to$ Weak isospin: $T^3(\nu_{eL}) = +\frac{1}{2}$ and $T^3(e_L) = -\frac{1}{2}$ ...

The notation used here is conventional but a bit confusing: With

$$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$$

we mean that the chiral bispinor field $\nu_{eL}(x)$ is of the form

$$\nu_{eL}(x) = \psi_L(x) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in L^2(\mathbb{R}^{1,3}) \otimes \mathbb{C}^4 \otimes \mathbb{C}^2_L$$

with some left-chiral bispinor field $\psi_L(x)$. The last factor $\mathbb{C}^2_L$ is the spin-$\frac{1}{2}$ representation space of $\mathrm{SU}(2)_L$ and $\mathbb{C}^4$ is the representation space of the Dirac bispinor. It is then

$$T^3(\nu_{eL}) = +\frac{1}{2} \quad :\Leftrightarrow \quad \hat{T}^3 \nu_{eL}(x) = \psi_L(x) \otimes \hat{T}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2} \nu_{eL}(x) \,.$$

The basis vectors that span $\mathbb{C}^2_L$, say $\nu_{eL} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_L \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are refered to as *flavours* (of first-generation leptons). Similarly, u_L and d_L are the flavours of first-generation quarks.

- *Right-handed fields = Isospin singlets:*

$$\psi_R = \underbrace{u_R, d_R, e_R}_{\text{Gen. I}}, \underbrace{c_R, s_R, \mu_R}_{\text{Gen. II}}, \underbrace{t_R, b_R, \tau_R}_{\text{Gen. III}} \tag{10.5}$$

In the following, Ψ denotes a doublet and ψ a singlet. If we write Ψ_L, we refer to a doublet of left-handed components (as above).

Note that we omit right-handed neutrinos $\nu_{eR}, \dots$ as these would completely decouple from the observable sector, ➔ below.

$\to$ Weak isospin: $T^3(e_R) = 0$ ...

↑ Notes

- *Higgs fields = Isospin doublet:*

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

→ Weak isospin: $T^3(\phi^+) = \frac{1}{2}$ and $T^3(\phi^0) = -\frac{1}{2}$

Both Φ and its scalar components ϕ^+ and ϕ^0 are often refered to as "Higgs field". Note that despite the vectorial notation, the fields ϕ^+ and ϕ^0 are complex (*Lorentz*) scalars. That is, "scalar" refers to their trivial transformation under *Lorentz transformations*. The Higgs field Φ does *not* transform trivially under $\mathrm{SU}(2)_L$ gauge transformations, as its doublet structure reveals. In a nutshell: The Higgs field is a Lorentz *scalar* (= Spin-0 irrep of $\mathrm{SO}^+(1,3)$) but a $\mathrm{SU}(2)_L$ gauge *doublet* (= Spin-$\frac{1}{2}$ irrep of $\mathrm{SU}(2)$).

→ Gauge transformations on fields:

$$\text{Left-handed doublet:} \quad \tilde{\Psi}_L = e^{i\hat{Y}_L\alpha(x)} \underbrace{e^{i\hat{T}^i\beta^i(x)}}_{\equiv V_L(x)} \Psi_L$$

$$\text{Right-handed singlet:} \quad \tilde{\psi}_R = e^{i\hat{Y}_R\alpha(x)} \psi_R$$

$$\text{Higgs doublet:} \quad \tilde{\Phi} = e^{i\hat{Y}_H\alpha(x)} e^{i\hat{T}^i\beta^i(x)} \Phi$$

where $\hat{Y}_L = Y \cdot \mathbb{1}_2$, $\hat{Y}_R = Y \cdot 1$ and $\hat{Y}_H = Y \cdot \mathbb{1}_2$

Note that here also the hypercharge is an *operator*. As we consider a direct sum of possibly unitary equivalent but different copies of irreps, Y can take *different* values on these irreps.

Note:
The *weak hypercharge* Y is a fixed number for each irrep, e.g., $Y(u_L) = Y(d_L)$, but can differ for different irreps: $Y(u_L) \neq Y(e_L)$ (➔ Higgs mechanism below)

4. *Kinetic energy for fermions & Minimal coupling:*

$$\mathcal{L}_{\text{Fermion}} = \sum_{\Psi_L} \overline{\Psi}_L(i\not{D}_L)\Psi_L + \sum_{\psi_R} \overline{\psi}_R(i\not{D}_R)\psi_R \tag{10.6}$$

The sums go here over the doublets in (10.4) and the singlets in (10.5), respectively.

with covariant derivatives

$$D_{L\mu} = \partial_\mu - igW_\mu^i\hat{T}^i - ig'B_\mu\hat{Y}_L$$
$$D_{R\mu} = \partial_\mu \qquad\qquad - ig'B_\mu\hat{Y}_R$$

g / W_μ^i: coupling constant / gauge field for weak isospin
g' / B_μ: coupling constant / gauge field for weak hypercharge

Note that $[\hat{T}^i, \hat{Y}_L] = 0$ for all i so that the fields W_μ^i and B_μ do not mix under gauge transformations and thus can have different coupling constants g and g', respectively.

↑ Notes

The Lagrangian (10.6) violates the symmetries C (swaps left-handed fermions with left-handed antifermions) and P (swaps left-handed and right-handed fermions) as much as possible since left(right)-handed (anti)fermions couple weakly but right(left)-handed (anti)fermions do not. Note that CP swaps a left-handed fermion with a right-handed antifermion so that (10.6) is CP-symmetric.

→ Transformation of the gauge fields:

$$\tilde{B}_\mu = B_\mu + \frac{1}{g'}\partial_\mu \alpha \quad \text{and} \quad \tilde{W}_\mu = V_L \left[W_\mu + \frac{i}{g}\partial_\mu \right] V_L^\dagger$$

Recall (9.8) and (9.4). Here we use the shorthand notation $W_\mu \equiv W_\mu^i \hat{T}^i$.

5. *Dirac mass terms?* Should be of the form

$$m(\overline{\Psi}_L \psi_R + \overline{\psi}_R \Psi_L) \quad \to \quad \text{Undefined!}$$

⊲ Elementary terms of the form $\overline{x}_L y_R$ with x, y Dirac spinors
→ *Not* $\mathrm{SU}(2)_L$ gauge invariant since

- x_L is *component* of a SU(2) doublet
- but y_R transforms as a SU(2) singlet

The argument here is the same as, e.g., for an expression like Ep^2 that is not a Lorentz scalar since $E = p^0$ is component of a four vector.

→ $\overline{x}_L y_R$ is *not* a SU(2) singlet (i.e., not gauge invariant)
→ We cannot add Dirac mass terms to the Lagrangian!

Solution: Yukawa coupling and Higgs mechanism, ➔ below.

6. *Kinnetic energy for gauge bosons:*

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{4}(B^{\mu\nu})^2 - \frac{1}{4}(W^i_{\mu\nu})^2$$

$$\text{with} \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$
$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu \underbrace{+ g\varepsilon^{ijk} W^j_\mu W^k_\nu}_{\text{Interactions between gauge bosons}}$$

Here, $\varepsilon^{ijk} = f^{ijk}$ are the structure constants of SU(2), ➔ Problemset 13.

7. *Higgs field:*

$$\mathcal{L}_{\text{Higgs}} = (D^\mu_H \Phi)^\dagger (D_{H\,\mu} \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2 \tag{10.7}$$

with covariant derivative

$$D_{H\,\mu} = \partial_\mu - igW^i_\mu \hat{T}^i - ig' B_\mu \hat{Y}_H$$

↑ Notes

Note that $(\Phi^\dagger\Phi)^2 \neq |\phi^+|^4 + |\phi^0|^4$; the latter term is Lorentz but *not* gauge invariant so that only the former is an allowed interaction. The form of the Higgs potential is then given by the condition of renormalizability. To make the vacuum stable, $\lambda > 0$ is required.

8. *Higgs mechanism Part I:* Masses for the gauge bosons

a) Let $\mu^2 < 0 \to$ Non-zero VEV of Higgs field:

$$\text{W.l.o.g.}\quad \langle\Phi\rangle = \Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} \quad \text{with} \quad v = \sqrt{\frac{-\mu^2}{\lambda}}$$

b) Define the *electric charge (operator)*

$$\boxed{Q = T^3 + Y \;\in \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_Y} \tag{10.8}$$

$\to$ Choose $Y(\Phi) = +\frac{1}{2}$ so that

$$\hat{Q}\Phi_0 = \left(-\frac{1}{2} + \frac{1}{2}\right)\Phi_0 = 0 \quad \Rightarrow \quad e^{i\hat{Q}\alpha(x)}\Phi_0 = \Phi_0$$

This is why the lower Higgs field is called ϕ^0: it is uncharged, $Q(\phi^0) = 0$. By contrast, the upper field ϕ^+ has charge $Q(\phi^+) = +1$.

$\to$ Gauge symmetry $\mathrm{U}(1)_Q$ generated by Q is *unbroken*:

$$\boxed{\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \xrightarrow{3\times\text{SSB}} \underbrace{\mathrm{U}(1)_Q}_{\substack{\text{Unbroken gauge}\\ \text{group of QED}}}}$$

3 generators of the global symmetry group are spontaneously broken while 1 generator (Q) remains unbroken. This is what we want, as we *know* that there should be one massless gauge boson: the photon.

Conclusion: The generator of $\mathrm{U}(1)_Y$ (the weak hypercharge Y) and the generator of $\mathrm{U}(1)_Q$ (the electric charge Q) are not the same!

c) $\sphericalangle$ Fluctuations of Φ around Φ_0 in the *unitary gauge*:

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v + h(x)\end{pmatrix}$$

$h(x)$: *real* scalar Higgs field
The excitations of this field are the famous Higgs bosons.

d) $\Phi(x)$ in (10.7) (we focus here on the terms that generate the gauge boson masses)

$$(D_H^\mu\Phi)^\dagger(D_{H\,\mu}\Phi) \stackrel{\circ}{=} \frac{v^2}{8}\left\{g^2\left[(W_\mu^1)^2 + (W_\mu^2)^2\right] + (-gW_\mu^3 + g'B_\mu)^2\right\} + \dots \tag{10.9}$$

↑ Notes

e) Define the new fields

$$W_\mu^\pm := \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2)$$

$$Z_\mu := \frac{1}{\sqrt{g^2+g'^2}}(g W_\mu^3 - g' B_\mu)$$

$$A_\mu := \frac{1}{\sqrt{g^2+g'^2}}(g' W_\mu^3 + g B_\mu)$$

The ratio of g and g' defines the so called *Weinberg angle* θ_W: $\cos\theta_W = g/\sqrt{g^2+g'^2}$. This parameter is not predicted by the SM but one of the many input parameters that have to be determined experimentally. It is also called *weak mixing angle* as it describes the mixing of W_μ^3 and B_μ that yields A_μ.

→

$$(10.9) \stackrel{\circ}{=} \underbrace{\left(\frac{gv}{2}\right)^2}_{m_W^2} W_\mu^+ W^{-\mu} + \frac{1}{2}\underbrace{\left(\frac{v}{2}\right)^2 (g^2+g'^2)}_{m_Z^2} (Z_\mu)^2 + \dots$$

and (express the covariant derivative in the new fields)

$$D_{H\,\mu} = \partial_\mu - (\dots) - i \underbrace{\frac{gg'}{\sqrt{g^2+g'^2}}}_{\text{Electron charge } e} A_\mu \hat{Q}$$

We conclude:

- A_μ: massless, neutral ($Q=0$) gauge field of QED
- $W_\mu^\pm$: *massive*, charged ($Q=\pm 1$) gauge bosons of weak interaction
- Z_μ: *massive*, neutral ($Q=0$) gauge boson of weak interaction

9. *Interlude:* With (10.8) we can fix the hypercharge Y by the (observed) electric charge Q
Examples:

$$Y(e_L) = Q(e_L) - T^3(e_L) = -1 - \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

$$Y(e_R) = Q(e_R) - T^3(e_R) = -1 - 0 = -1$$

10. *Higgs mechanism Part II:* Masses for the fermions
    a) How to form a gauge invariant term including left- and right-handed fermions?
    Must be a $SU(2)_L$ singlet and hypercharge-neutral ($Y=0$)!
    → Couple left-handed fermion doublet, Higgs doublet, and right-handed fermion

↑ Notes

singlet via a Yukawa term:
[Compare: $\phi\overline{\Psi}\Psi$ (Yukawa) vs. $A_\mu\overline{\Psi}\gamma^\mu\Psi$ (Maxwell)]

$$-\gamma_e\,(\overline{\Psi}_L\cdot\Phi)\,e_R+\text{h.c.}\quad\text{with}\quad\Psi_L=\begin{pmatrix}\nu_{eL}\\ e_L\end{pmatrix}\tag{10.10}$$

γ_e: coupling constant
Note that

$$Y(\Phi)+Y(e_R)-Y(\Psi_L)=\frac{1}{2}-1-\left(-\frac{1}{2}\right)=0$$

and

$$(\overline{\Psi}_L\cdot\Phi)\,e_R=\underbrace{(\overline{\nu}_{eL}\quad\overline{e}_L)\cdot\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}}_{\text{SU}(2)_L\text{ singlet}}\,e_R=\underbrace{\phi^+\cdot\overline{\nu}_{eL}e_R+\phi^0\cdot\overline{e}_Le_R}_{\text{Scalars}\,\times\,\text{Dirac inner products}}$$

so that (10.10) is both $\text{SU}(2)_L$ and $\text{U}(1)_Y$ invariant. The last expression reveals the Yukawa-form of the interaction clearly.
Higgs mechanism: $\phi^+\mapsto 0$ and $\phi^0\mapsto v/\sqrt{2}\rightarrow$

$$(10.10)=-\frac{\gamma_e v}{\sqrt{2}}\,(\overline{e}_Le_R+\overline{e}_Re_L)+\ldots$$

with fermion mass $m_e=\gamma_e v/\sqrt{2}$
The same works for the other charged leptons and the quarks (but not the neutrinos if their right-handed counterparts are excluded).

b) In general, we can couple different fermion generations:
This is possible since fermions of the same type (charged lepton l, neutrino ν, up-type u and down-type d quark) but different generations have the same hypercharge and isospin.

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}}=&-\Gamma^u_{mn}\,\overline{Q}^m_L\,\hat{\Phi}\,u^n_R-\Gamma^d_{mn}\,\overline{Q}^m_L\,\Phi\,d^n_R\\ &-\Gamma^l_{mn}\,\overline{L}^m_L\,\Phi\,l^n_R-\Gamma^\nu_{mn}\,\overline{L}^m_L\,\hat{\Phi}\,\nu^n_R+\text{h.c.}\end{aligned}\tag{10.11}$$

There are implicit sums over the fermion generations m and n. All other symbols are fixed labels.

- $m,n\in\{\text{I,II,III}\}$: fermion generations
- $x\in\{u,d,l,\nu\}$: fermion types
  Examples: $l^{\text{I}}_R=e_R$, $l^{\text{II}}_R=\mu_R$, $u^{\text{I}}_R=u_R$, $u^{\text{II}}_R=c_R$, ...
- Γ^x_{mn}: coupling constants
  Example: $\Gamma^l_{\text{I,I}}=\gamma_e$ from above
- Q^m_L, L^m_L: left-handed quark- resp. lepton doublets of generation m
  Examples: $\overline{Q}^{\text{I}}_L=(\overline{u}_L\quad\overline{d}_L)$ and $\overline{L}^{\text{II}}_L=(\nu_{\mu L}\quad\mu_L)$, ...

↑ Notes

- $\hat{\Phi}_i \equiv \varepsilon^{ij}\Phi_j^*$: Higgs doublet with opposite hypercharge: $Y(\hat{\Phi}) = -\frac{1}{2}$
  This representation is required to make the terms hypercharge-neutral [to see this, use $Y(u_R^n) = \frac{2}{3}$, $Y(d_R^n) = -\frac{1}{3}$ and $Y(\overline{Q}_L^m) = -\frac{1}{6}$]. Note that $\hat{\Phi}$ transforms in the same isospin irrep as Φ.

c) The Yukawa couplings (10.11) …

- …generate mass terms for quarks and charged fermions
- …cannot generate mass terms for neutrinos if there are no right-handed neutrinos (→ Neutrinos are *massless* in the standard model)

  Adding right-handed neutrinos can be used to explain the experimentally observed masses of neutrinos and might even contribute to dark matter (➔ *Seesaw mechanism*). Such neutrinos are called *sterile neutrinos* as they have vanishing weak isospin $T^3 = 0$ and electric charge $Q = 0$ (and therefore hypercharge $Y = 0$). Hence they do not take part in any interaction described by the standard model.
- …leads to generation-changing transitions of quarks
  (➔ *CKM matrix* and P&S pp. 721–724)

  The generation mixing in (10.11) implies that the quark states that take part in weak interactions (= interaction eigenstates) are *not* the eigenstates of the mass operator (= mass eigenstates) that describe freely propagating particles. Then one can show that a (mass eigenstate) s-quark that propagates freely can decay into a (mass eigenstate) u-quark by coupling to a (virtual) W^--boson. If there are no right-handed neutrinos, such transitions are forbidden for leptons (which matches experimental observations).

10.2.3 Quantum Chromodynamics

We discuss QCD here only superficially to connect with concepts that we learned previously.

1. *Gauge symmetry:*

$$\boxed{\underbrace{\mathrm{SU}(3)_C}_{\text{Color charge}}}$$

→ 8 generators K^a, $a = 1, \dots, 8$ with (in general, $\mathrm{SU}(N)$ has $N^2 - 1$ generators)

$$\left[K^a, K^b\right] = i f^{abc} K^c$$

Here we use the unconventional label K^a to distinguish the generators from the $\mathrm{SU}(2)_L$ generators T^i of the weak force.

→ *Irreducible representations:*

- 1D: Trivial representation $\hat{K}^a = 0$ (= *Singlet* representation)
- 3D: Defining representation (physicist parlance: *fundamental representation*):
  $\hat{K}^a = \frac{\lambda_a}{2}$ with 3×3 Hermitian *Gell-Mann matrices* λ_a (= *Triplet* representation)

  Gell-Mann matrices are the analog of Pauli matrices for SU(3).

2. *Field representations:*

- Quarks = $SU(3)_C$ *triplets*

$$q = \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix} \quad \text{for } q \in \{u, d, c, s, t, b\}$$

with *colors* r (red), g (green), b (blue)
Note that each color field q_c is a Dirac spinor, i.e., we extended the number of quark fields threefold!

The notion of "colors" is not gauge invariant: For instance, a "red" quark q_r can be transformed into a mixture of red, green, and blue quarks by a gauge tranformation $U_C(x)$:

$$\begin{pmatrix} \tilde{q}_r \\ \tilde{q}_g \\ \tilde{q}_b \end{pmatrix} = U_C \begin{pmatrix} q_r \\ 0 \\ 0 \end{pmatrix}$$

- Leptons & Higgs fields = $SU(3)_C$ *singlets* → Ignore them for QCD

Since the leptons $(e, \nu_e, \dots)$ do *not* interact via the strong force, they carry the trivial (singlet-) representations of $SU(3)_C$, i.e., their fields are not extended into triplets and it is $\hat{K}^a = 0$ for actions of $SU(3)_C$ transformations on their fields.

→ Gauge transformation of fields:

$$\text{Quark triplet:} \quad \tilde{q} = \underbrace{e^{i\hat{K}^a\beta^a(x)}}_{\equiv U_C(x)} q$$

3. *Lagrangian:*

$$\mathcal{L}_{\text{QCD}} = \sum_q \bar{q}(i\not{D}_C)q - \frac{1}{4}(G^a_{\mu\nu})^2$$

Note the missing mass terms! As explained above, the masses are generated by the Higgs mechanism and electroweak SSB.

with covariant derivative

$$D_{C\,\mu} = \partial_\mu - i g_s G^a_\mu \hat{K}^a$$

g_s: coupling constant of the strong force
G^a_μ: 8 gauge fields → 8 gauge bosons = 8 *Gluons*
➲ (9.8) for the transformation of $G_\mu \equiv G^a_\mu \hat{K}^a$ under $U_C(x)$.

The Gauge field strength is defined as usual:

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu$$

↑ Notes

Note 10.3

- *No additional* Higgs mechanism:
  - Quarks masses are generated by electroweak SSB
  - Gluons are massless
- Gluons carry color charges and can therefore interact with each other (➔ Note 9.1)

  Mathematically, this means that gluons transform in a non-trivial representation of $SU(3)_C$ (not the three-dimensional defining irrep of quarks but the so called *adjoint representation* which is 8-dimensional for SU(3)). Gluons act then on colored quarks and change their color. That is, if we write $|c\rangle$ for a the three color states of a quark ($c = r, g, b$), gluon states can be generated from matrices of the form $|c\rangle\langle c'|$ (one says that gluons carry a color c and an *anticolor* c'). However, this suggests $3 \times 3 = 9$ gluon states, but there are only 8! The hitch is that the linear combination

  $$\hat{K}^0 \equiv |r\rangle\langle r| + |g\rangle\langle g| + |b\rangle\langle b|$$

  is forbidden (physically, this means that a gluon can never transform in the singlet representation, i.e., a gluon cannot be coloreless). That $\hat{K}^0$ is *not* part of the generating set of $\mathfrak{su}(3)$ can be seen easily since

  $$e^{i\hat{K}^0\cdot\pi} = -\mathbb{1}_3$$

  has determinant -1! That is, a coloreless gluon would imply a gauge group U(3) rathern than SU(3). However, such a gluon would not be constrained by confinement, and therefore contradicts current experimental evidence. Thus the gauge group of QCD is SU(3) with 8 gluons and not U(3) with 9.

  The Gell-Mann matrices are then 8 particular linear combinations of the 9 matrices $|c\rangle\langle c'|$ that are linearly independent of $\hat{K}^0$, e.g., $\lambda_1 = |r\rangle\langle g| + |g\rangle\langle r|$.

- Renormalization: Let $\alpha_s \equiv \frac{g_s^2}{4\pi}$, then

  $$\alpha_s^{\text{eff}}(q^2) \xrightarrow{q^2\to\infty} 0 \quad \rightarrow \quad \text{Asymptotic freedom}$$
  $$\alpha_s^{\text{eff}}(q^2) \xrightarrow{q^2\to 0} \infty^* \quad \rightarrow \quad \text{Confinement}^*$$

  Compare this with the running of $\alpha_{\text{eff}}(q^2)$ in QED, ➔ (6.37).
  For experimental results ➔ P&S Fig. 17.23 on p. 595.

  That is, quarks at very high energies (e.g. in hadrons) behave almost like free particles. By contrast, at low energies, their interaction becomes so strong that free particles that carry a color charge (i.e. are not a color singlet = coloreless) do not exist (thus we observe only mesons and baryons that are coloreless).

  (*) Note that a diverging coupling constant does *not* prove confinement (QED also has a divergence, known as *Landau pole*, at high energies which is believed to be an unphysical artifact). Up to know, the existence of confinement in QCD and the true

↑ Notes

IR behaviour of α_s^{eff} remains unproven resp. unkown (confinement is supported by numerical lattice QCD calculations though) since this regime is not accessible by perturbation theory.

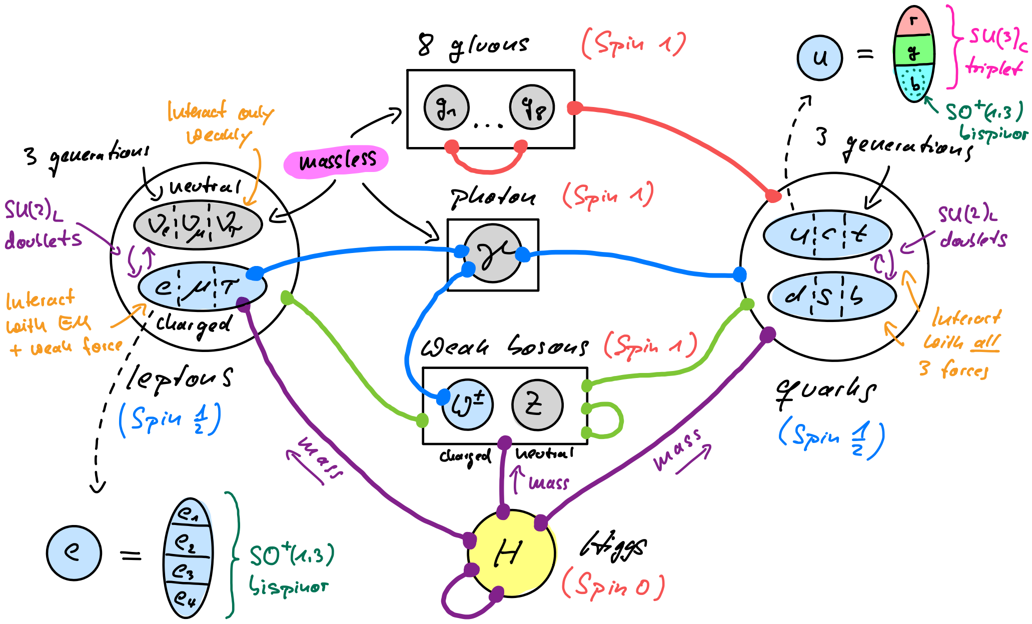
10.2.4 Summary

- Gauge symmetry group of the standard model:

$$\underbrace{\underbrace{\mathrm{SU}(2)_L}_{\text{Weak isospin}} \times \underbrace{\mathrm{U}(1)_Y}_{\text{Weak Hypercharge}}}_{\xrightarrow{\text{Electroweak SSB}} \underbrace{\mathrm{U}(1)_Q}_{\text{Electric charge}}} \times \underbrace{\mathrm{SU}(3)_C}_{\text{Color charge}}$$

Our vacuum has lost the global $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry since the Higgs field developed a VEV (by the way, it is unclear *why* this happened). The "true Lagrangian of the universe" still has this symmetry, only our low-energy vacuum "hides" this symmetry from us. Thus spontaneous symmetry breaking is sometimes refered to as *spontaneous symmetry hiding*.

- *Fermions and their interactions:*



In total there are

$$[2 \text{ Leptons} + 2 \text{ Quarks} \times 3 \text{ Colors}] \times 3 \text{ Generations} = 24 \text{ Dirac bispinors}$$

each consisting of 4 complex fields $\rightarrow$ 96 complex fields for fermions

- The standard model Lagrangian $\mathcal{L}_{\mathrm{M}}$ contains *18 parameters* (can be more if additional extensions to the SM are considered, e.g., neutrino masses) that cannot be derived but must be provided by experiments:

↑ Notes

- 9 × Fermion masses: $m_e, m_u, \dots$ (recall that neutrinos are massless in the SM)
- 1 × Higgs mass $m_h \approx 125\,\text{GeV}$
  (This is the famous result from the observation at LHC in 2012, https://doi.org/10.1016/j.physletb.2012.08.021)
- 1 × Higgs field VEV v
- 3 × Gauge field couplings: g, g', g_s
- 4 × CKM matrix parameters: $\theta_{12}, \dots$ (describe the mixing of quark generations and possible CP-violating terms, hidden in the Yukawa coupling matrices of (10.11)) (P&S p. 721ff.)

In conclusion, the SM does not seem to be good candidate for a truly fundamental theory (which should be a more efficient "compression" of the laws of nature). This is one of the reasons to look for a GUT that allows for the computation of these parameters *ab initio*.

- Have a look at https://arxiv.org/abs/0904.1556 if you want to know more about the representation theory of the standard model (and its possible extensions to GUTs).