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April 16th, 2021
SS 2021

You can find detailed information about the lecture, tutorials and the general organization of this course on the website

`https://www.itp3.uni-stuttgart.de/teaching/qft21/`

or on ILIAS:

`https://ilias3.uni-stuttgart.de/goto_Uni_Stuttgart_crs_2346773.html` (Lecture)
`https://ilias3.uni-stuttgart.de/goto_Uni_Stuttgart_crs_2347972.html` (Tutorials).

Please keep yourself up to date concerning this course.

The solutions to all problems have to be handed in via ILIAS (see link above) and will be corrected by the tutors.

This first problem set serves as a repetition for some important concepts from classical field theory:

Problem 1: Functional derivative (Written, 2 points)

Learning objective

In this problem, you should familiarize yourself with functional derivatives which are an important tool in (quantum) field theory, for example in the context of calculating correlation functions of operators.

For a given manifold $\mathcal{M}$ of functions ϕ and a functional F with $F: \mathcal{M} \to \mathbb{R}$ or $\mathbb{C}$, the functional derivative $\frac{\delta F[\phi]}{\delta \phi}$ is defined as

$$\int \mathrm{d}x' \, \frac{\delta F[\phi]}{\delta \phi(x')} f(x') = \lim_{\varepsilon \to 0} \frac{F[\phi(x) + \varepsilon f(x)] - F[\phi(x)]}{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F[\phi + \varepsilon f]\bigg|_{\varepsilon=0} \tag{1}$$

for all test functions $f \in \mathcal{M}$.

a) Show that for two functionals F and G

$$\frac{\delta(F + \lambda G)[\phi]}{\delta \phi(x)} = \frac{\delta F[\phi]}{\delta \phi(x)} + \lambda \frac{\delta G[\phi]}{\delta \phi(x)} \quad \text{for } \lambda \in \mathbb{R} \tag{2}$$

$$\frac{\delta(FG)[\phi]}{\delta \phi(x)} = \frac{\delta F[\phi]}{\delta \phi(x)} G[\phi] + F[\phi] \frac{\delta G[\phi]}{\delta \phi(x)}, \text{ with } (FG)[\phi] = F[\phi] G[\phi]. \tag{3}$$

If $G[\phi]$ is a function-like functional, i.e. can be treated as a function itself

$$\frac{\delta F[G[\phi]]}{\delta \phi(y)} = \int \mathrm{d}x \, \frac{\delta F[G]}{\delta G(x)} \frac{\delta G[\phi](x)}{\delta \phi(y)}. \tag{4}$$

b) Calculate the functional derivative $\frac{\delta F[\phi]}{\delta\phi(x)}$ for the following functionals:

$$F[\phi] = \int \mathrm{d}x'\, K(y,x')\phi(x'), \quad \text{where } K(y,x') \text{ is a so called } \textit{integral kernel} \tag{5}$$
$$F[\phi] = \phi(y) \tag{6}$$
$$F[\phi] = \phi'(y) \tag{7}$$
$$F[\phi] = \int \mathrm{d}y\, f(\phi(y),\phi'(y)) \text{ for a differentiable function } f:\ \mathbb{R}\times\mathbb{R}\to\mathbb{R} \tag{8}$$

Problem 2: Lorentz covariance (Written, 5 points)

> **Learning objective**
>
> This exercise serves as a **brief** revision of tensor calculus and the covariant formulation of classical electromagnetism.

In the following, we will work in units where $c = 1$. Further, we will make use of *Einstein notation* where summation over indices appearing twice is assumed.

We first introduce the four-vector

$$x^\mu = (t, \mathbf{r}), \quad \mu = 0, 1, 2, 3, \tag{9}$$

which we will call a **contravariant** vector or tensor of first order. The vector x_μ is called **covariant** vector. In special relativity, the metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{10}$$

and the relationship between co- and contravariant vectors is given by

$$x_\mu = g_{\mu\nu}x^\nu. \tag{11}$$

A **Lorentz vector** is an object that under a Lorentz transformation $\Lambda^\mu{}_\nu$ transforms like

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{12}$$

In tensors of higher order, each index transforms as a Lorentz vector, e.g.

$$\tilde{A}^{\alpha\beta\gamma}{}_{\delta\varepsilon} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \Lambda^\gamma{}_\xi \Lambda_\delta{}^\rho \Lambda_\varepsilon{}^\sigma A^{\mu\nu\xi}{}_{\rho\sigma}. \tag{13}$$

A **Lorentz scalar** is a quantity that is invariant under Lorentz transformations.

a) Show that $x^\mu x_\mu$ is a Lorentz scalar, i.e. show that $x^\mu x_\mu = \tilde{x}^\sigma \tilde{x}_\sigma$.

Another important object is the four-gradient

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = (\partial_t, \nabla). \tag{14}$$

b) Compute the d'Alembert operator $\partial^\mu \partial_\mu$. Is this quantity a Lorentz scalar?

In a covariant formulation of electromagnetism, the electric and magnetic field **E** and **B**, respectively, are given by the anti-symmetric field tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{15}$$

The fields can also be described by the four-potential $A_\mu = (\Phi, -\mathbf{A})$, where Φ is a scalar potential and $\mathbf{A}$ is a vector potential.

c) Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ reproduces the fields **E** and **B**.

d) Show that $F_{\mu\nu}$ is invariant under the gauge transformation $A_\mu \to A_\mu - \partial_\mu f$, where f is an arbitrary function.

e) Show that in Lorenz gauge, $\partial_\nu A^\nu = 0$, and for no external sources, the Maxwell equations $\partial_\mu F^{\mu\nu} = 0$ reduce to $\partial^\mu \partial_\mu A^\nu = 0$.

Problem 3: Maxwell equations (Written, 4 points)

> **Learning objective**
>
> In this problem, you use the methods from problem 1 and 2 to derive the famous Maxwell equations from a covariant formulation of classical electrodynamics. You calculate the energy-momentum tensor of the electromagnetic field which contains information about the flow of energy and momentum.

The Maxwell equations for classical electromagnetism in vacuum can be derived from the action

$$S = \int \mathrm{d}^4x \mathcal{L} = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{16}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

a) Derive the Maxwell equations from the action (16). Use the Euler-Lagrange equations and treat the components $A_\mu(x)$ as the dynamic variables. Write the two "inhomogeneous" Maxwell equations in their standard form by using $E^i = -F^{0i}$ and $\varepsilon^{ijk} B^k = -F^{ij}$, $i = x, y, z$.

   What happens with the two homogeneous equations?

b) Calculate the energy-momentum tensor $T^{\mu\nu}$ for the electromagnetic field

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L}. \tag{17}$$

c) This tensor, however, is not symmetric which can be fixed by adding a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices.

Calculate the symmetric energy-momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{18}$$

with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu. \tag{19}$$

d) Show that the symmetrized tensor yields the standard form of the electromagnetic energy density and the momentum density (Poynting vector)

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2) \quad \text{and} \quad \mathbf{S} = \mathbf{E} \times \mathbf{B}. \tag{20}$$