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Problem 1: Fock states and coherent states (Written, 5 points)

> **Learning objective**
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> In this problem, we construct the Hilbert space of a (bosonic) quantum field theory on the basis of linear superpositions of non-normalizable single-particle states and discuss how the concept of coherent states translates to this setting.

The single-particle states $|\mathbf{p}\rangle = \sqrt{2E_\mathbf{p}} a^\dagger_\mathbf{p} |0\rangle$ are not well suited to build up the Hilbert space as they are not normalizable due their diverging commutation relations, $[a_\mathbf{p}, a^\dagger_\mathbf{q}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$. However, it is possible to build a wave packet by linear superposition of momentum eigenstates,

$$a^\dagger(f) |0\rangle = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} f(\mathbf{p}) a^\dagger_\mathbf{p} |0\rangle \ . \tag{1}$$

a) Calculate the commutator $[a(f), a^\dagger(f)]$ and derive a condition for $f(\mathbf{p})$ such that the states are normalizable.

b) Consider now the generalization to n particles. We define the unnormalized *Fock state* as

$$|n\rangle = \int \frac{d^3p_1}{(2\pi)^3 \sqrt{2E_{\mathbf{p}_1}}} \cdots \int \frac{d^3p_n}{(2\pi)^3 \sqrt{2E_{\mathbf{p}_n}}} F(\mathbf{p}_1, \ldots, \mathbf{p}_n) a^\dagger_{\mathbf{p}_1} \cdots a^\dagger_{\mathbf{p}_n} |0\rangle \ , \tag{2}$$

where F is symmetric under the exchange of two of its arguments. Calculate the norm of (2). Show that (2) is an eigenstate of the number operator

$$N = \int \frac{d^3p}{(2\pi)^3} a^\dagger_\mathbf{p} a_\mathbf{p} \tag{3}$$

and calculate its eigenvalue.

c) Calculate the expectation value of the (normal ordered) Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} E_\mathbf{p} a^\dagger_\mathbf{p} a_\mathbf{p} \tag{4}$$

of the real Klein-Gordon field with respect to the state (2).

d) Consider now a coherent superposition of n-particle states, that is a *coherent state*,

$$|\alpha\rangle = \exp\left(\int \frac{d^3p}{(2\pi)^3 \sqrt{2E_\mathbf{p}}} \alpha(\mathbf{p}) a^\dagger_\mathbf{p} \right) |0\rangle \ . \tag{5}$$

Calculate its norm as well as the overlap of two (normalized) coherents states, $\langle \alpha | \beta \rangle$. Interpret the result.

e) Show that (5) is an eigenstate of the annihilation operator $a_{\mathbf{p}}$ and calculate its eigenvalue. Show that the coherent state remains a coherent state under time evolution with the Hamiltonian (4), that is, show that $e^{-iHt}|\alpha\rangle = |\beta\rangle$, where $|\beta\rangle$ has to be determined.

Problem 2: Free-particle solutions of the Dirac equation (Written, 4 points)

Learning objective

In this problem, you will fill in the missing calculations of the discussion of the free-particle solutions of the Dirac equation in the lecture. In addition, you will calculate two important completeness relations which will prove useful in the evaluation of Feynman diagrams.

In the lecture, you discussed that the general solution of the Dirac equation can be written as a superposition of plane waves whose positive-frequency solutions are given by

$$\psi(x) = u(p)e^{-ipx}\,, \qquad p^2 = m^2\,, \qquad p^0 > 0\,. \tag{6}$$

The two independent solutions for $u(p)$ read

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}\,, \qquad s = 1,2 \tag{7}$$

with the normalization $(\xi^s)^\dagger\xi^r = \delta^{rs}$ of the two-component spinor ξ^s. Here, $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$ with the Pauli matrices σ^i.

a) Prove the identity $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2$. Also show that $(\gamma^\mu p_\mu + m)(\gamma^\mu p_\mu - m) = 0$.

b) Show that $u^{r\dagger}(p)u^s(p)$ is not Lorentz invariant. Show that instead $u(p)$ can be normalized in a Lorentz invariant way according to

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs} \tag{8}$$

with $\bar{u} = u^\dagger\gamma^0$.

Similarly to above, the negative-frequency solutions

$$\psi(x) = v(p)e^{ipx}\,, \qquad p^2 = m^2\,, \qquad p^0 > 0 \tag{9}$$

can be obtained with two linearly independent solutions

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}\,, \qquad s = 1,2\,, \tag{10}$$

where η^s is another basis of two-component spinors. These solutions are normalized according to $\bar{v}^r(p)v^s(p) = -2m\delta^{rs}$.

c) Show that $\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0$ and $u^{r\dagger}(p)v^s(p) \neq 0$ as well as $v^{r\dagger}(p)u^s(p) \neq 0$. However, show that reversing the sign of the 3-momentum in one factor of each spinor product leads to $u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = v^{r\dagger}(\mathbf{p})u^s(-\mathbf{p}) = 0$.

d) Finally, we consider the sum over the polarization states of a fermion which will be important when evaluating Feynman diagrams. Calculate the completeness relations

$$\sum_s u^s(p)\bar{u}^s(p) \quad = \not{p} + m\,, \tag{11}$$

$$\sum_s v^s(p)\bar{v}^s(p) \quad = \not{p} - m \tag{12}$$

with the Feynman slash notation $\not{p} \equiv \gamma^\mu p_\mu$.