Prof. Dr. Hans Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

May 25, 2020
SS 2021

Problem 1: Feynman diagrams for ϕ^4-theory (Written, 4+1 points)

> **Learning objective**
>
> The purpose of this problem is to become familiar with Feynman diagrams and their corresponding perturbative expressions. To this end, we use the interacting ϕ^4-theory and focus on its four-point correlator to apply the machinery of real- and momentum-space Feynman diagrams.

We consider the ϕ^4-theory

$$H = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left[\pi^2(\mathbf{x}) + (\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x}) + 2\,\frac{\lambda}{4!}\phi^4(\mathbf{x})\right] \tag{1}$$

with interacting fields $\phi(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt}$ and vacuum $|\Omega\rangle$.

a) Draw all *relevant* Feynman diagrams (i.e., without vacuum bubbles) for the perturbative expansion of the four-point function

$$\langle\Omega|\,\mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,|\Omega\rangle \tag{2}$$

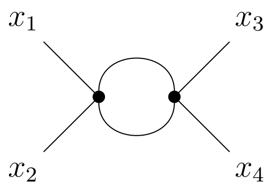
up to second order (λ^2).

Draw two relevant diagrams of third order (λ^3): one connected and one disconnected.

Hint: Ignore symmetry factors and permutations of external points. Use that four-point diagrams are either fully connected or decompose into products of disjoint two-point diagrams. Up to permutations, there are 3 connected diagrams and 6 additional disconnected diagrams up to second order.

b) **Optional (+1 point):** Draw all diagrams of third order. How many are connected and disconnected, respectively (again up to permutations)?

c) Using the *real-space Feynman rules*, write down the term described by the Feynman diagram



d) Label the Feynman diagram above with directed momenta and write down the corresponding expression as prescribed by the *momentum-space Feynman rules.*

e) Use the Fourier expansion of the Feynman propagator

$$D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{i\,e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{3}$$

to show that the expressions of c) and d) are equivalent.

Problem 2: Feynman rules for the interacting complex Klein-Gordon field (Written, 4 points)

Learning objective

Here you derive the Feynman rules for the complex Klein-Gordon field with an arbitrary interaction potential. Generically, this interaction violates causality and the resulting theory is no longer a relativistic quantum field theory. However, in condensed matter physics such theories can be used to describe the low-energy physics of interacting models that are otherwise hard to tackle analytically. This demonstrates that diagrammatic methods for perturbation theory are not restricted to relativistic high-energy physics.

Recall the (free) complex Klein-Gordon field (Problem Set 2) with Hamiltonian

$$H_0 = \int \mathrm{d}^3\mathbf{x}\, \left(\pi^\dagger\pi + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi\right) \tag{4}$$

and fields that satisfy the canonical commutation relations $[\phi(\mathbf{x}),\pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$.

Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a symmetric $[V(\mathbf{r}) = V(-\mathbf{r})]$ but otherwise arbitrary (well-behaved) potential. Here we consider the interacting theory

$$H = H_0 + \frac{\lambda}{2}\int \mathrm{d}^3\mathbf{x} \int \mathrm{d}^3\mathbf{y}\, V(\mathbf{x}-\mathbf{y})\,\phi^\dagger(\mathbf{x})\phi^\dagger(\mathbf{y})\phi(\mathbf{x})\phi(\mathbf{y}) \tag{5}$$

with small parameter λ.

At an arbitrary time t_0, we can expand the interacting field $\phi(t_0,\mathbf{x})$ into modes,

$$\phi(t_0,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{i\mathbf{px}} + b_\mathbf{p}^\dagger e^{-i\mathbf{px}}\right)\,, \tag{6}$$

with the mode algebra

$$\left[a_\mathbf{p},a_\mathbf{q}^\dagger\right] = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{q}) \quad \text{and} \quad \left[b_\mathbf{p},b_\mathbf{q}^\dagger\right] = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{7}$$

(all other commutators vanish). In the interaction picture, we then have

$$\phi_I(x) = e^{iH_0(t-t_0)}\phi(t_0,\mathbf{x})e^{-iH_0(t-t_0)} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{-ipx} + b_\mathbf{p}^\dagger e^{ipx}\right) \tag{8}$$

with $x^0 = t - t_0$. Note that this is just the the time evolution of the free theory H_0 that you derived in Problem 2 b) of Problem Set 2.

a) Let the contraction be defined as difference between time ordering and normal ordering:

$$\overbrace{AB} \equiv \mathcal{T}\{AB\} - {:}AB{:} \tag{9}$$

where $A,B \in \{\phi_I,\phi_I^\dagger\}$.

Use the decomposition $\phi_I = \phi_a^+ + \phi_b^-$ and $\phi_I^\dagger = \phi_a^- + \phi_b^+$ into positive- and negative-frequency parts (and your knowledge from the real Klein-Gordon field) to show that

$$\overbrace{\phi_I(x)\phi_I(y)} = \overbrace{\phi_I^\dagger(x)\phi_I^\dagger(y)} = 0 \tag{10a}$$

$$\overbrace{\phi_I^\dagger(x)\phi_I(y)} = \overbrace{\phi_I(x)\phi_I^\dagger(y)} = D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\,. \tag{10b}$$

b) Prove Wick's theorem for the free complex scalar field. That is, show that

$$\mathcal{T}\{ABC\dots\} = :ABC\dots: + :\{\text{all contractions between pairs of } \phi \text{ and } \phi^\dagger\}: \tag{11}$$

for $A, B, C, \dots \in \{\phi_I, \phi_I^\dagger\}$.

Hint: Use induction (as in Peskin & Schroeder) with the decomposition of ϕ and $\phi^\dagger$ from above.

As shown in the lecture (or in Problem 1 of Problem Set 5), time-ordered correlation functions can be rewritten in terms of interaction picture fields via

$$\langle\Omega|\,\mathcal{T}\{ABC\dots\}\,|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\,\mathcal{T}\{A_I B_I C_I \dots \exp\left(-i\int_{-T}^{T} dt\, H_I(t)\right)\}\,|0\rangle}{\langle 0|\,\mathcal{T}\exp\left(-i\int_{-T}^{T} dt\, H_I(t)\right)\,|0\rangle} \tag{12}$$

for $A, B, C, \dots \in \{\phi, \phi^\dagger\}$. Here $|\Omega\rangle$ is the interacting vacuum and the interaction picture Hamiltonian is given by

$$H_I(t) = \frac{\lambda}{2}\int \mathrm{d}^3\mathbf{x} \int \mathrm{d}^3\mathbf{y}\, V(\mathbf{x}-\mathbf{y})\, \phi_I^\dagger(x)\phi_I^\dagger(y)\phi_I(x)\phi_I(y)\,. \tag{13}$$

c) Use this prescription in combination with Wick's theorem to evaluate the two-point correlator

$$\langle\Omega|\,\mathcal{T}\phi(x)\phi^\dagger(y)\,|\Omega\rangle \tag{14}$$

up to first order in λ.

Compare your result to the ϕ^4-theory.

d) Use the dictionary

$$y \longrightarrow x \quad = \quad \overline{\phi_I(x)\phi_I^\dagger}(y) = D_F(x-y) \tag{15a}$$

$$u \text{ - - - - - - - } w \quad = \quad V(\mathbf{u}-\mathbf{w})\,\delta(u^0-w^0) \tag{15b}$$

to recast the summands found in c) as Feynman diagrams.

Generalize your result to the Feynman rules of the interacting theory of a complex scalar field with interaction potential V.