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Problem 1: Cross section of two scattering particles (Written, 2 points)

Learning objective

In this problem, we will study the cross section of two scattering particles within the ϕ^4-theory. It serves as an example of the use of Feynman diagrams to calculate scattering cross sections.

From the lecture, you know the relation between S-matrix elements and cross sections which is given by

$$d\sigma = \frac{1}{4|E_{\text{in}}p^z_{\text{in}} - E'_{\text{in}}p'^z_{\text{in}}|}\left(\prod \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}(p_{\text{in}}, p'_{\text{in}} \to \{p_f\})|^2(2\pi)^4\delta^{(4)}(p_{\text{in}}+p'_{\text{in}}-\sum_f p_f)\,, \quad (1)$$

where we chose the reference frame such that the particles collide along the z-axis.

Specialize now to the case of two particles, with the same mass m, interacting with the Hamiltonian

$$H_I = \frac{\lambda}{4!}\int d^3x\, \phi_I^4(x)\,. \quad (2)$$

a) Transform into the center-of-mass frame and integrate over the final momenta. Show that to lowest order the differential cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\lambda^2}{64\pi^2E^2_{\text{cm}}}\,, \quad (3)$$

where E_{cm} is the total initial energy.

Hint: In general, the invariant matrix element $\mathcal{M}$ is defined by

$$\langle \mathbf{p}_1\mathbf{p}_2\cdots|\, iT\, |\mathbf{p}_{\text{in}}\mathbf{p}'_{\text{in}}\rangle = (2\pi)^4\delta^{(4)}(p_{\text{in}} + p'_{\text{in}} - \sum_f p_f)i\mathcal{M}(p_{\text{in}}, p'_{\text{in}} \to \{p_f\}) \quad (4)$$

and

$$\langle \mathbf{p}_1\mathbf{p}_2\cdots|\, iT\, |\mathbf{p}_{\text{in}}\mathbf{p}'_{\text{in}}\rangle = \lim_{T\to\infty(1-i\varepsilon)}\left({}_0\langle \mathbf{p}_1\mathbf{p}_2\cdots|\, \mathcal{T}\exp\left(-i\int_{-T}^{T} dt\, H_I(t)\right)|\mathbf{p}_{\text{in}}\mathbf{p}'_{\text{in}}\rangle_0\right)_{\substack{\text{fully connected,}\\ \text{amputated}}}\,, \quad (5)$$

with the asymptotic incoming and outgoing states $|\mathbf{p}_{\text{in}}\mathbf{p}'_{\text{in}}\rangle_0$ and $|\mathbf{p}_1\mathbf{p}_2\cdots\rangle_0$, respectively.

b) Calculate the total cross section of the scattering process.

Problem 2: Important relations for gamma matrices (Written, 4 points)

> **Learning objective**
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> The evaluation of scattering amplitudes in quantum electrodynamics becomes much easier with an appropriate compendium of relations for gamma matrices. Here you derive the most important ones from their algebraic properties.

Consider a set of four matrices γ^μ satisfying the *Dirac algebra*

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,. \tag{6}$$

In addition, we introduce a fifth matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{7}$$

If not stated otherwise, all results below should be derived *algebraically*, i.e., without reference to a specific representation of the Dirac algebra.

a) As warm-up, show that

$$\left(\gamma^5\right)^2 = 1\,, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0\,. \tag{8}$$

b) Show that the *Weyl (chiral) representation* satisfies the additional *Hermiticity condition*

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \qquad \text{and that this implies} \qquad \left(\gamma^5\right)^\dagger = \gamma^5\,. \tag{9}$$

Is this true for arbitrary representations?

c) Prove the following *trace identities*:

$$\begin{aligned}
\text{tr}\left[\gamma^\mu\right] &= 0 && \text{(10a)}\\
\text{tr}\left[\gamma^\mu\gamma^\nu\right] &= 4g^{\mu\nu} && \text{(10b)}\\
\text{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) && \text{(10c)}\\
\text{tr}\left[\gamma^5\right] &= 0 && \text{(10d)}\\
\text{tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] &= 0 && \text{(10e)}\\
\text{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right] &= -4i\varepsilon^{\mu\nu\rho\sigma} && \text{(10f)}\\
\text{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\dots\right] &= \text{tr}\left[\dots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu\right] && \text{(10g)}
\end{aligned}$$

Hint: Only for the last relation (10g) the Hermiticity condition $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ is required.

d) Finally, prove the following *contraction identities*:

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= 4 && \text{(11a)}\\
\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu && \text{(11b)}\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} && \text{(11c)}\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu && \text{(11d)}
\end{aligned}$$