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Problem 1: Rutherford scattering (Written, 3 points)

> **Learning objective**
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> Here, you will derive the famous Rutherford formula for the differential cross section for the elastic scattering of non-relativistic charged particles interacting via the Coulomb interaction.

The cross section for the scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = \int d^3x \, e\bar{\psi}\gamma^\mu\psi A_\mu, \tag{1}$$

where $\psi(x)$ is the usual quantized Dirac field.

a) Show that the T-matrix element for electron scattering off a localized classical potential is, to lowest order,

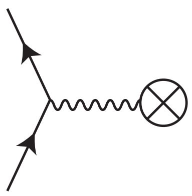
$$\langle p'| \, iT \, |p\rangle = -ie\bar{u}(p')\gamma^\mu u(p)\widetilde{A}_\mu(p'-p), \tag{2}$$

where $\widetilde{A}_\mu(q) = \int d^4x \, e^{iqx} A_\mu(x)$.

b) If $A_\mu(x)$ is time independent, its Fourier transform contains a δ-function of energy. It is then natural to define

$$\langle p'| \, iT \, |p\rangle \equiv -i\mathcal{M} \cdot (2\pi)\delta(E_f - E_i), \tag{3}$$

where E_i and E_f are the initial and final energies of the particle, and to adopt a new Feynman rule for computing $\mathcal{M}$:



$= -ie\gamma^\mu \widetilde{A}_\mu(\mathbf{q})$

where $\widetilde{A}(\mathbf{q})$ is the three-dimensional Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross section for scattering off a time-independent localized potential is

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\mathcal{M}(p_i - p_f)|^2(2\pi)\delta(E_f - E_i), \tag{4}$$

where v_i is the particle's initial velocity. Integrate over $|\mathbf{p}_f|$ to find a simple expression for $d\sigma/d\Omega$.

Hints:

- Start by considering a wave packet

$$|\phi\rangle = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} e^{-i\mathbf{b}\cdot\mathbf{p}}\phi(\mathbf{p})\,|\mathbf{p}\rangle\,, \tag{5}$$

  where **b** is the impact parameter accounting for the transverse displacement of the incoming wave packet and assume $\phi(\mathbf{p})$ to be narrowly peaked around $\mathbf{p} = (0, 0, p)$.

- Calculate the probability to scatter the incoming state into a final state whose momentum lies in a small region d^3p'. Take care of the proper normalization.
- In order to calculate the differential cross section, integrate this probability over the impact parameter **b**.

c) Specialize to the case of electron scattering from a Coulomb potential ($A^0 = Ze/4\pi r$). Working in the non-relativistic limit, derive the *Rutherford formula*,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2v^4\sin^4(\theta/2)}\,, \tag{6}$$

where θ is the angle between initial and final momentum.

Problem 2: Scattering cross section for $e^+e^- \to \mu^+\mu^-$ in QED (Written, 5 points)

Learning objective

The scattering of an electron e^- and a positron e^+ into a muon/antimuon pair $\mu^+\mu^-$ is the simplest and one of the most important scattering processes described by QED. In the lecture you already evaluated the cross section for vanishing electron mass. Here you will retrace this calculation step by step with non-vanishing electron mass to practice the evaluation of cross sections.

We want to calculate the *unpolarized* scattering cross section of the elementary QED process

$$e^+e^- \to \mu^+\mu^- \tag{7}$$

to lowest order. Here "unpolarized" means that the spins of incoming particles are distributed uniformly and the spins of the scattering products cannot be resolved by the experiment.

a) Draw and label the lowest-order Feynman diagram for this process and calculate the amplitude

$$i\mathcal{M}[e^-(p,s)e^+(p',s') \to \mu^-(k,r)\mu^+(k',r')] \tag{8}$$

with momenta p, p', k, k' and spins s, s', r, r'.

b) Since we want to calculate the *unpolarized* scattering cross section, we have to average uniformly over all incoming spins and sum over the outgoing ones. I.e., calculate

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{2}\sum_{s}\frac{1}{2}\sum_{s'}\sum_{r,r'}|\mathcal{M}|^2 \tag{9}$$

and evaluate the traces using the identities from Problem 1.

Hint: You should end up with

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{(p+p')^4}\left[(p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k) + (p\cdot p')m_\mu^2 + (k\cdot k')m_e^2 + 2m_\mu^2 m_e^2\right],$$

where m_μ and m_e are the masses of the muon and electron, respectively.

c) In the center-of-mass frame, calculate the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ and the total cross section σ in terms of the energy E of the incoming particles and the scattering angle θ of the outgoing particles.

Hint: As there are only two particles in the final state, you can use the simplified expression for the differential scattering cross section

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CM}} = \frac{1}{2E_A 2E_B|v_A - v_B|}\frac{|\mathbf{k}|}{(2\pi)^2\, 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \tag{10}$$

where $E_A = E_B = E$, $E_{\text{cm}} = 2E$ and $v_A = \sqrt{E^2 - m_e^2}/E = -v_B$ since the two incoming particles have the same mass.

d) In reality, the muon is much heavier than the electron: $m_e/m_\mu \approx 1/200$. Show that in the limit of small electron mass, $m_\mu \gg m_e$, the differential and total cross section take the form derived in the lecture.

e) Starting from your result for finite m_e, calculate the differential and total cross section in the high-energy limit $E \gg m_\mu, m_e$ to the lowest order where the results depend on both masses m_e and m_μ.