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Problem 1: Infrared divergence of the electron vertex function (Written, 4 points)

Learning objective

The calculation of the one-loop correction of the electron vertex function is riddled with both an ultraviolet and an infrared divergence—caused by the momentum integration of the loop. While the ultraviolet divergence is controlled by Pauli-Villars regularization, the infrared divergence can be parametrized by introducing a small, artificial photon mass $\mu > 0$. It is important to extract the asymptotic behaviour of this divergence for $\mu \to 0$ to prepare its cancellation with a similar term found for soft bremsstrahlung. Here you work out the details of this asymptotic behaviour.

As shown in the lecture, the regularized form factor F_1 of the electron vertex in QED up to one-loop order reads

$$F_1(q^2) = 1 + \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right)\right. \tag{1}$$

$$\left. + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \mu^2 z} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \mu^2 z}\right] + \mathcal{O}(\alpha^2)\,.$$

Here, x, y, z are Feynman parameters, m is the electron mass, $q = p' - p$ the momentum transfer and μ the artificial photon mass to regularize the integral; α is the fine structure constant.

We are interested in the (physical) limit of vanishing photon mass ($\mu \to 0$) where Eq. (1) diverges.

a) Show that the dominant terms of Eq. (1) in this limit read

$$F_1^{(1)}(q^2) := \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \tag{2}$$

$$\times\left[\frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \mu^2 z} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \mu^2 z}\right].$$

Hint: Show that the virtual photon is spacelike, i.e., show that $q^2 < 0$; then show that the argument of the logarithm is bounded in the relevant region.

b) Using the previous result, show that the asymptotic behaviour of F_1 is captured by the simpler expression

$$F_1^{(2)}(q^2) := \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z} dy\left[\frac{-2m^2 + q^2}{m^2(1-z)^2 - q^2(1-z-y)y + \mu^2} - \frac{-2m^2}{m^2(1-z)^2 + \mu^2}\right]. \tag{3}$$

c) Use the substitution $y = (1-z)\xi$ and $w = 1-z$ to show that

$$F_1^{(3)}(q^2) := \frac{\alpha}{4\pi}\int_0^1 d\xi \left[\frac{-2m^2+q^2}{m^2-q^2\xi(1-\xi)}\log\left(\frac{m^2-q^2\xi(1-\xi)}{\mu^2}\right)+2\log\left(\frac{m^2}{\mu^2}\right)\right]. \tag{4}$$

d) Finally, show that the asymptotics of F_1 is given by $F_1(q^2) \approx 1 + F_1^{(4)}(q^2) + \mathcal{O}(\alpha^2)$ with

$$F_1^{(4)}(q^2) := -\frac{\alpha}{2\pi} f_{\text{IR}}(q^2)\log\left(\frac{A}{\mu^2}\right) \tag{5}$$

where the function $f_{\text{IR}}(q^2)$ has to be determined and both choices $A \in \{-q^2, m^2\}$ give rise to valid expressions.

Hint: Use that adding constants (with respect to μ) to $F_1^{(3)}$ does not change its asymptotic behaviour for $\mu \to 0$.

What is the sign of $f_{\text{IR}}(q^2)$?

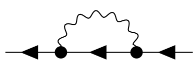
This expression can now be used to cancel the infrared divergence of the electron vertex function with the corresponding divergence found for soft *bremsstrahlung* to obtain a finite result independent of μ (see lecture).

Problem 2: The electron self-energy (Written, 3 points)

Learning objective

The mass-energy equivalence inherent to any relativistic theory implies for quantum field theories that fluctuations of fields around particles with “bare” mass m_0 shift the latter to a larger, observable mass m. In QED, virtual photons that couple to the charged electron make up for its *self-energy* which, in turn, contributes to its mass m; we say that the mass is *renormalized.* Here you work out the details of the one-loop correction discussed in the lecture. As a result, we find that m_0 and m differ by an infininity.

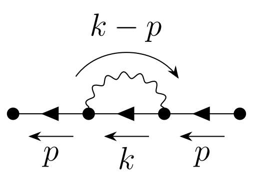
The electron two-point function is given by the sum of diagrams

$\langle\Omega|\, \mathcal{T}\Psi(x)\bar{\Psi}(y)\,|\Omega\rangle =$ x  y + x  y + ... (6)

where the first diagram is just the free-field propagator,

 $= \dfrac{i(\not{p}+m_0)}{p^2-m_0^2+i\epsilon}$, (7)

and the second diagram (the *electron self-energy*) yields the expression



$= \dfrac{i(\not{p}+m_0)}{p^2-m_0^2}[-i\Sigma_2(p)]\dfrac{i(\not{p}+m_0)}{p^2-m_0^2}$ (8)

according to the Feynman rules of QED (for the sake of simplicity, we omit the term $e^{-ip(x-y)}$ and the integral $\int d^4p/(2\pi)^4$ for the external points). Here

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 - m_0^2 + i\epsilon} \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\epsilon} \tag{9}$$

contains the two loop propators with their two vertices. m_0 is the bare mass of the electron and $\mu > 0$ is a small photon mass to regulate the infrared divergence of the integral.

a) Using Feynman parameters, show that the second-order self-energy $-i\Sigma_2(p)$ takes the form

$$-i\Sigma_2(p) = -e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{-2x\not{p} + 4m_0}{[\ell^2 - \Delta_\mu + i\epsilon]^2}\,, \tag{10}$$

where Δ_μ has to be determined.

b) To control the ultraviolet divergence of the integral (10), use the *Pauli-Villars regularization*

$$\frac{1}{(p-k)^2 - \mu^2 + i\epsilon} \to \frac{1}{(p-k)^2 - \mu^2 + i\epsilon} - \frac{1}{(p-k)^2 - \Lambda^2 + i\epsilon} \tag{11}$$

for $\Lambda \to \infty$ and show that

$$\Sigma_2(p) = \frac{\alpha}{2\pi} \int_0^1 dx\,(2m_0 - x\not{p})\,\log\left[\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right] \tag{12}$$

in this limit.

c) Using the expression for the second-order self-energy obtained in b), calculate the mass shift

$$\delta m = m - m_0 = \Sigma_2(\not{p} = m) \approx \Sigma_2(\not{p} = m_0) \tag{13}$$

in first order of α.

Show that the bare mass m_0 and the measurable mass m differ by a diverging quantity.