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Problem 1: Propagator in the path integral formalism (Written, 3 points)

Learning objective

In this problem, you use path integrals to construct the propagator (or two-point correlation function) of a generic quadratic field theory. You show that the propagator is given by the inverse of the quadratic form associated with the field theory.

Consider a generic quadratic theory in momentum space with action

$$S = \frac{1}{2}\sum_k \phi_{i,k} M_k^{ij} \phi_{j,-k} \tag{1}$$

with a symmetric matrix M, which also satisfies $M_k = M_{-k}$. Since the fields ϕ_i are assumed to be real-valued, their Fourier components fulfill the relation $\phi_{i,-k} = \phi_{i,k}^*$ and we can rewrite the action as

$$S = \sum_{k^0>0} \phi_{i,k} M_k^{ij} \phi_{j,k}^* \,. \tag{2}$$

Note that the sum over the momenta now only runs over one half of the four-momentum space, that is $k^0 > 0$.

In order to calculate the two-point correlation function, it proves useful to introduce the *generating functional*

$$Z[J, J^*] = \int \mathcal{D}\phi \exp\left(i \sum_{k^0>0} \left[\phi_{i,k} M^{ij} \phi_{j,k}^* + (J_k^i)^* \phi_{i,k} + J_k^j \phi_{j,k}^* \right] \right) \,. \tag{3}$$

a) Calculate the path integral in (3) and show that the generating functional is given by

$$Z[J, J^*] = \prod_{k^0>0} \left(\frac{i\pi}{\det M_k} \right) \exp\left(-i J_{i,k} (M_k^{-1})^{ij} J_{j,k}^* \right) \,. \tag{4}$$

Hints:

- Since the matrix M is symmetric, it can be diagonalized by a unitary transformation, that is $M = U^\dagger D U$, where D is diagonal and U is unitary.
- After diagonalizing the matrix and decoupling the fields, split each field into real and imaginary part, $\phi_{i,k} = \phi_{i,k}^{\text{re}} + i\phi_{i,k}^{\text{im}}$.
- The path integral measure is given by $\mathcal{D}\phi = \prod_{j,k^0>0} d\phi_{j,k}^{\text{re}} d\phi_{j,k}^{\text{im}}$.
- The remaining Gaussian integrals can be calculated by completing the square.

b) Use (3) to relate the two-point correlation function in Fourier space to the generating functional $Z[J, J^*]$ and show that

$$\tilde{D}_F^{ij}(k) = \langle \phi_k^i \phi_k^{j*} \rangle = \frac{1}{Z[0,0]} \left(-i \frac{\partial}{\partial J_{i,k}^*} \right) \left(-i \frac{\partial}{\partial J_{j,k}} \right) Z[J, J^*] \bigg|_{J,J^*=0} . \tag{5}$$

c) Finally, use (4) to show that $\tilde{D}_F^{ij}(k) = i \left(M_k^{-1} \right)^{ij}$.

Problem 2: Path integral and Weyl order (Written, 5 points)

Learning objective

This problem deals with the connection between the transition amplitude of a quantum system and the path integral formalism. In doing so, the peculiarity of non-commuting operators arises which can be resolved by employing a special ordering of operators in the Hamiltonian called *Weyl order*. As an example, you will calculate the transition amplitude for a non-relativistic particle in one dimension.

Consider a general quantum system described by a set of coordinates q^i, conjugate momenta p^i and Hamiltonian $H(q,p)$. In the lecture, it was shown that the transition amplitude $U(q_a, q_b; T) = \langle q_b | e^{-iHT} | q_a \rangle$ can be computed by breaking the time interval into N short slices of length ϵ and inserting a complete set of intermediate states between each slice such that

$$U(q_a, q_b; T) = \prod_i \prod_{k,l} \int dq_l^i \, \langle q_{k+1} | e^{-i\epsilon H} | q_k \rangle \, , \tag{6}$$

where $k = 0, \ldots, N-1$, $l = 1, \ldots, N-1$ and $q_a = q_0$ and $q_b = q_N$. Since $\epsilon \to 0$, we may expand the exponential as $e^{-i\epsilon H} = 1 - i\epsilon H + \cdots$. In a first step, consider a Hamiltonian which is a pure function of either q or p, that is $H(q,p) = f(q)$ or $H(q,p) = f(p)$.

a) Show that if H is only a function of the coordinates, the matrix element can be written as

$$\langle q_{k+1} | f(q) | q_k \rangle = f\left(\frac{q_{k+1} + q_k}{2} \right) \left(\prod_i \int \frac{dp_k^i}{2\pi} \right) \exp\left(i \sum_i p_k^i (q_{k+1}^i - q_k^i) \right) . \tag{7}$$

b) Now consider the case when the Hamiltonian only depends on the momenta. Show that

$$\langle q_{k+1} | f(p) | q_k \rangle = \left(\prod_i \int \frac{dp_k^i}{2\pi} \right) f(p_k) \exp\left(i \sum_i p_k^i (q_{k+1}^i - q_k^i) \right) . \tag{8}$$

Show also that if H contains only terms of the form $f(q)$ and $f(p)$, its matrix elements can be written

$$\langle q_{k+1} | H(q,p) | q_k \rangle = \left(\prod_i \int \frac{dp_k^i}{2\pi} \right) H\left(\frac{q_{k+1} + q_k}{2}, p_k \right) \exp\left(i \sum_i p_k^i (q_{k+1}^i - q_k^i) \right) . \tag{9}$$

c) In general, (9) is false when there are products of q's and p's in the Hamiltonian as on the left-hand side the order of the (non-commuting) operators matters while on the right-hand side we only deal with numbers. Show this explicitly for $H = p^2q^2$ and show that putting the Hamiltonian into *Weyl order*, that is

$$H(q,p) = p^2q^2 \mapsto H_W(q,p) = \frac{1}{4}(q^2p^2 + 2qp^2q + p^2q^2)\,, \tag{10}$$

resolves this issue. Note that any Hamiltonian can be put into Weyl order by commuting p's and q's on the cost of some extra terms appearing on the right-hand side of (9).

d) Show that for a Weyl-ordered Hamiltonian, the propagator (6) is given by

$$U(q_N, q_0; T) = \left(\prod_{i,k,l} \int dq_l^i \int \frac{dp_k^i}{2\pi}\right) \exp\left[i \sum_k \left(\sum_i p_k^i (q_{k+1}^i - q_k^i) - \epsilon H\left(\frac{q_{k+1} + q_k}{2}, p_k\right)\right)\right]. \tag{11}$$

This expression is the discretized form of

$$U(q_a, q_b; T) = \int \mathcal{D}q(t)\, \mathcal{D}p(t) \exp\left[i \int_0^T dt \left(\sum_i p^i \dot{q}^i - H(q,p)\right)\right] \tag{12}$$

and defines what we understand as a *path integral.*

e) As a special case, consider the Hamiltonian $H = p^2/2m + V(q)$ of a single, non-relativistic particle in one dimension. Show that the transition amplitude reads

$$U(q_a, q_b; T) = \left(\frac{1}{C(\epsilon)} \prod_k \int \frac{dq_k}{C(\epsilon)}\right) \exp\left[i \sum_k \left(\frac{m}{2} \frac{(q_{k+1} - q_k)^2}{\epsilon} - \epsilon V\left(\frac{q_{k+1} + q_k}{2}\right)\right)\right] \tag{13}$$

and determine $C(\epsilon)$.