0 The Big Picture

0.1. Quantum phases and quantum phase transitions

* Phases at $T=0$ $\longrightarrow$ Groundstate(s) of Hamiltonians
* Quantum phase transitions driven by quantum fluctuations
* Phase diagrams wrt. parameters of the Hamiltonian (e.g. magnetic field)

Examples:

* Superconductors
* Superfluids (e.g. sf. Helium ...)
* Bose-Einstein condensates
* Fermi liquids
* quantum Hall states

Paradigmatic example: the transverse field Ising model

$\oint$ Periodic 1D chain of L spin-$\frac{1}{2}$:

$$\underbrace{H = -J\sum_{i=1}^{L}\sigma_i^z\sigma_{i+1}^z - h\sum_{i=1}^{L}\sigma_i^x}_{\text{Transverse field Ising model}}$$

J: ferromag. coupling > 0

h: transverse field > 0

<u>Observation:</u> $[\sigma_i^z\sigma_{i+1}^z, \sigma_i^x] \neq 0 \rightarrow$ quantum fluctuations

$\rightarrow$ Ground state(s) $=$ (entangeld) superpositions of eigenstates of $\sigma_i^z\sigma_{i+1}^z$ for $h \neq 0$

$\rightarrow$ Two qualitatively different parameter regimes:

* $J \ll h$ $(J \approx 0)$: gapped phase with unique ground state

$$|G_+\rangle = |++ + \dots +\rangle \Rightarrow \langle G_+|\sigma_i^z\sigma_j^z|G_+\rangle \xrightarrow{|i-j|\to\infty} 0$$

$\Rightarrow$ Paramagnetic phase (disordered phase)

* $J \gg h$ $(h \approx 0)$: gapped phase with 2-fold degenerate ground state:

$$|G_\uparrow\rangle = |\uparrow\uparrow\ldots\uparrow\rangle \quad \text{and} \quad |G_\downarrow\rangle = |\downarrow\downarrow\downarrow\ldots\downarrow\rangle$$

$$\Rightarrow \quad \langle G|\,\sigma_i^z\sigma_j^z\,|G\rangle \xrightarrow{|i-j|\to\infty} 1$$

$$|G\rangle = \alpha\,|G_\uparrow\rangle + \beta\,|G_\downarrow\rangle$$

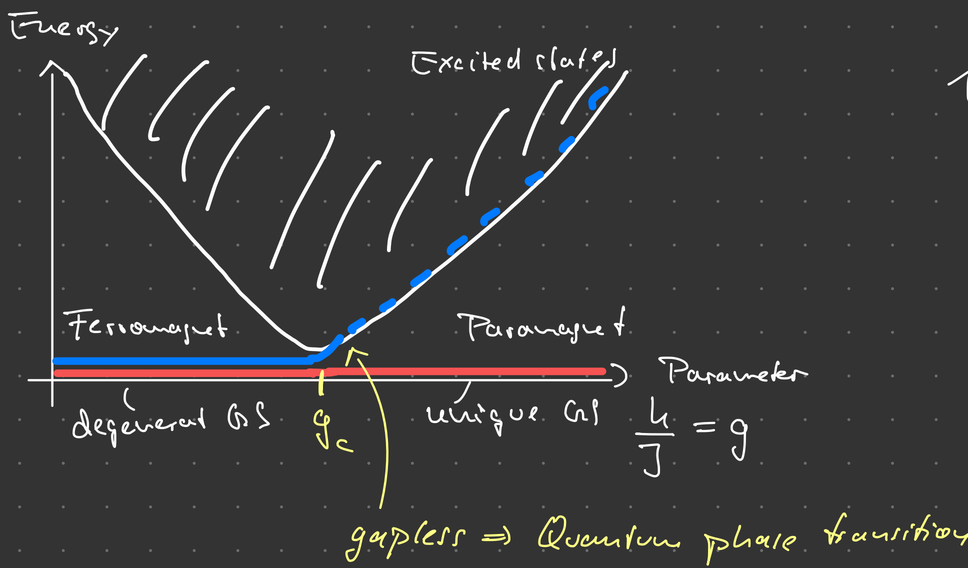
$\Rightarrow$ Ferromagnetic phase (ordered phase)

- σ_i^z is a local order parameter for the ferromagnetic phase since...
  * $\lim_{|i-j|\to\infty} \langle\sigma_i^z\sigma_j^z\rangle = 0$ in paramagnetic phase
  * $\lim_{|i-j|\to\infty} \langle\sigma_i^z\sigma_j^z\rangle \neq 0$ in ferromagnetic phase

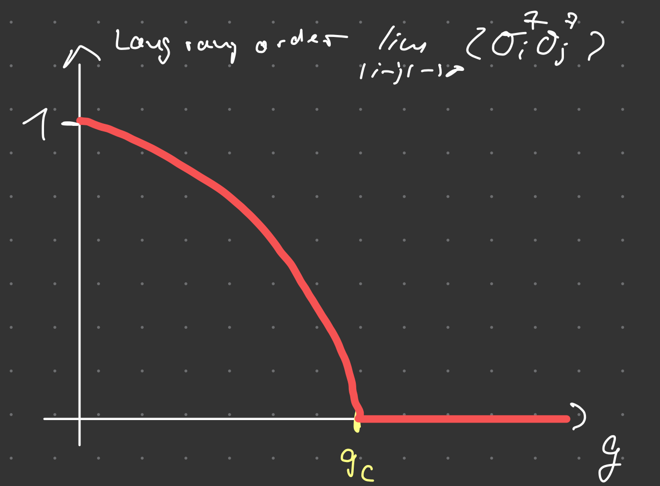
0.2. Spontaneous symmetry breaking

What happens between $J \ll h$ and $J \gg h$?

Schematic spectrum:



Long range order



→ Order parameter continuous at phase transition

→ Continuous (2nd order) phase transition:

* gap closes at phase transition
* long-range fluctuations and self-similarity (= quantum fluctuations on all length scales)
* effective conformal field theory (CFT) description
* algebraic decay of correlations

What happens at the phase transition?

Lev Landau: Spontaneous Symmetry Breaking ! (SSB)

* Symmetry group of TIM Hamiltonian:

$$G_S = \mathbb{Z}_2 = \{ \mathbb{1}, X \} \quad \text{with} \quad X = \prod_i \sigma_i^x \qquad [H, X] = 0$$

δ Symmetry group of the TIM <u>ground states</u>:

$|+ + \dots +\rangle$

* Paramagnetic phase: $G_E^{(P)} = \{\mathbb{1}, X\} = G_S$ since $X\,|G_+\rangle = |G_+\rangle$

→ symmetric phase (= disordered phase)

$|\uparrow\uparrow\uparrow\dots\rangle$ $|\downarrow\downarrow\downarrow\dots\rangle$

* Ferromagnetic phase: $G_E^{(f)} = \{\mathbb{1}\} \subsetneq G_S$ since $X\,|G_\uparrow\rangle = |G_\downarrow\rangle$

→ Symmetry-broken phase (= ordered phase)

$$\boxed{\begin{array}{l}\text{In the ferromagnetic phase, the ground states } |G_{\uparrow,\downarrow}\rangle \text{ spontaneously} \\ \text{break the symmetry } G_S \text{ of the Hamiltonian} \\ \to \text{Spontaneous Symmetry Breaking}\end{array}}$$

Landau's paradigm (SSB)

$G_E = G_S$	$G_E^{(1)}$	$G_E^{(2)}$	$G_E^{(3)}$	....

Labels of phases = Subgroups $G_E^{(i)}$ of symmetry groups G_S

$\Gamma \quad |G_\uparrow\rangle, \quad |G_\downarrow\rangle$

$|G_\uparrow\rangle + |G_\downarrow\rangle = |\uparrow\uparrow\uparrow \ldots\rangle + |\downarrow\downarrow \ldots\rangle$

$L \Rightarrow \mathcal{H}_\uparrow \cup \mathcal{H}_\downarrow$

0.3 Extending Landau's paradigm: Topological phases

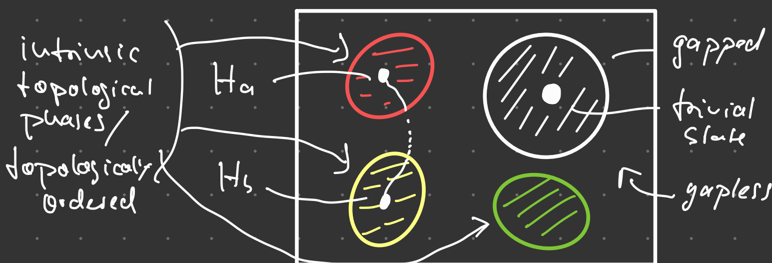
Formal definition of quantum phases:

Def. Quantum Phases

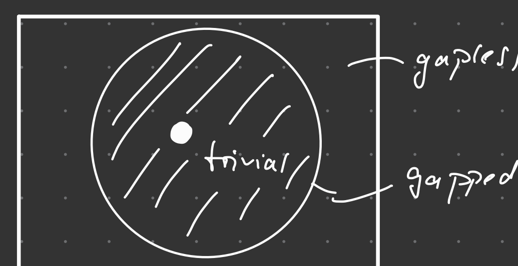
$\exists$ Gapped, local Hamiltonians H_a and H_b with unique ground states $|\psi_a\rangle$ and $|\psi_b\rangle$

They belong to the same quantum phase iff there is a family of gapped and local Hamiltonians $\hat{H}(\alpha)$ that depends continuously on $\alpha \in [0,1]$ such that $H_a = \hat{H}(0)$ and $H_b = \hat{H}(1)$.

$\exists$ Parameter space of local Hamiltonians (without SSB, $G_E = G_S$):



D ≥ 2

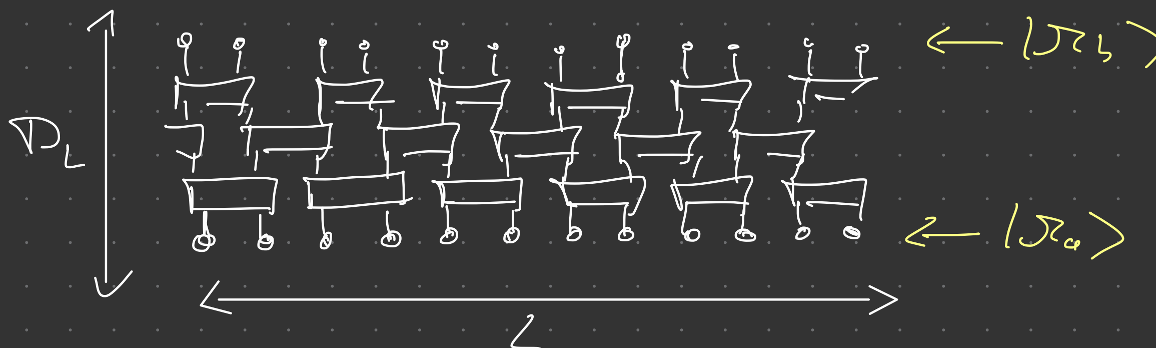


D = 1

$\Rightarrow$ no topolog. order in 1D spin systems / bosonic

- Trivial phase: Ground state = disentangled product state
- Topological phase: Ground state = long-range entangled state

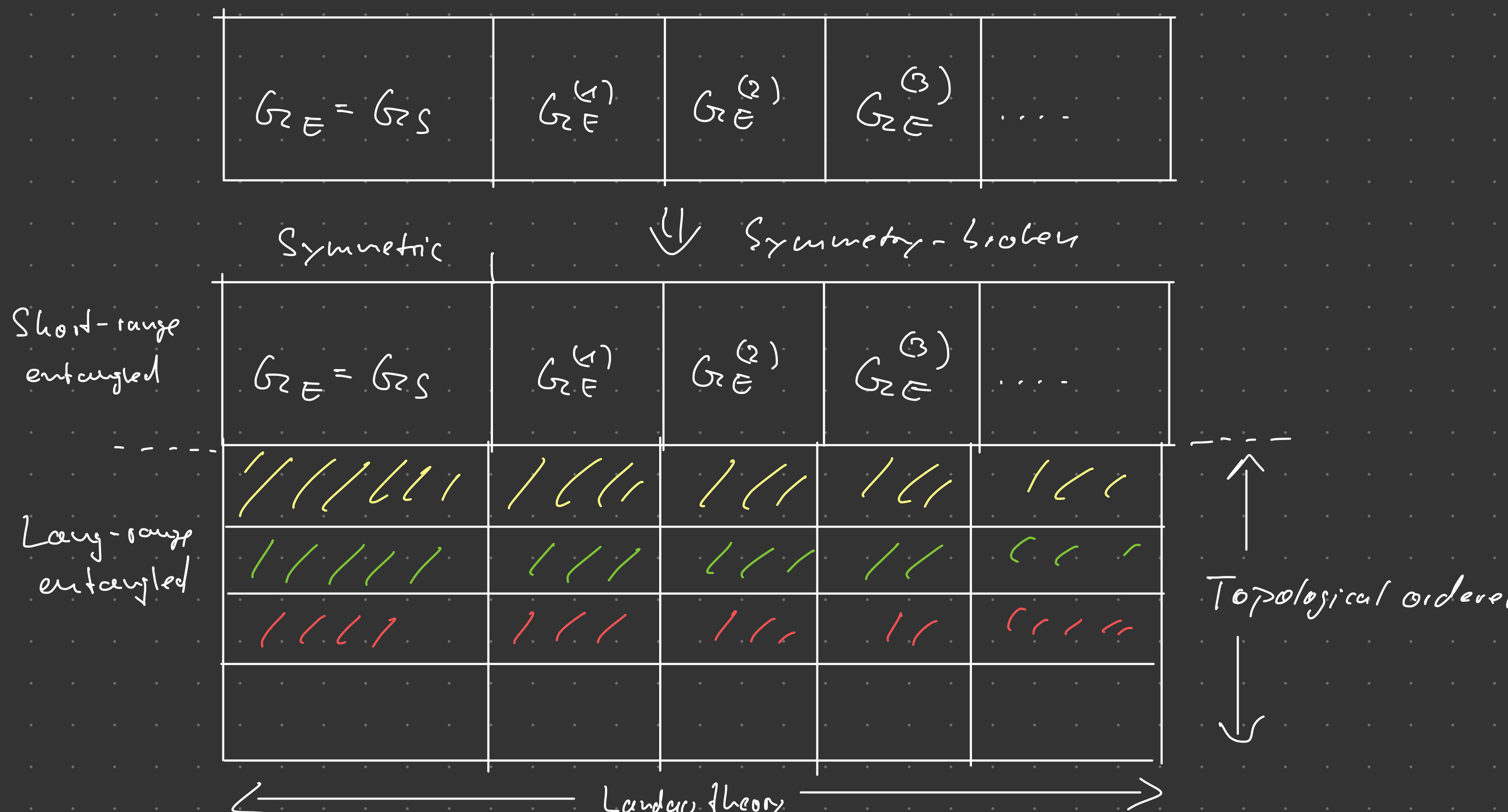
Note: Equivalent Definition



same phase

$D_L \to$ const. $\quad L \to \infty \quad \Rightarrow \quad |\Omega_b\rangle = U_D |\Omega_a\rangle \Rightarrow |\Omega_a\rangle \sim |\Omega_b\rangle$

First extension of Landau's paradigm: (Intrinsic) Topological order

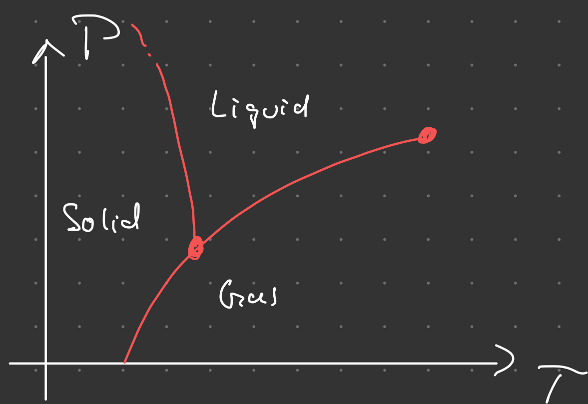


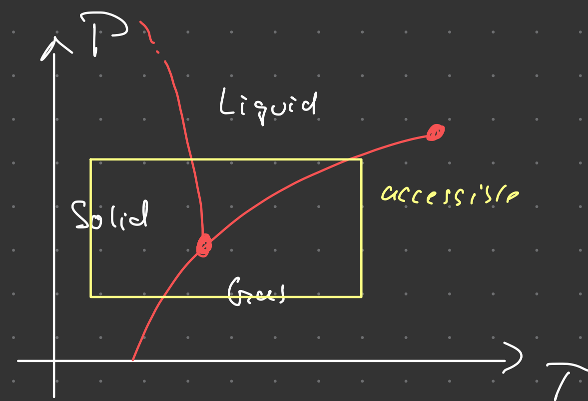
$$\boxed{\text{Topological order} = \text{Patterns of long-range entanglement}}$$

Examples: Fractional quantum Hall states

Interlude: Restrictions on perturbations

* Phase diagram of water:





⇒ Restrictions on allowed paths can be useful

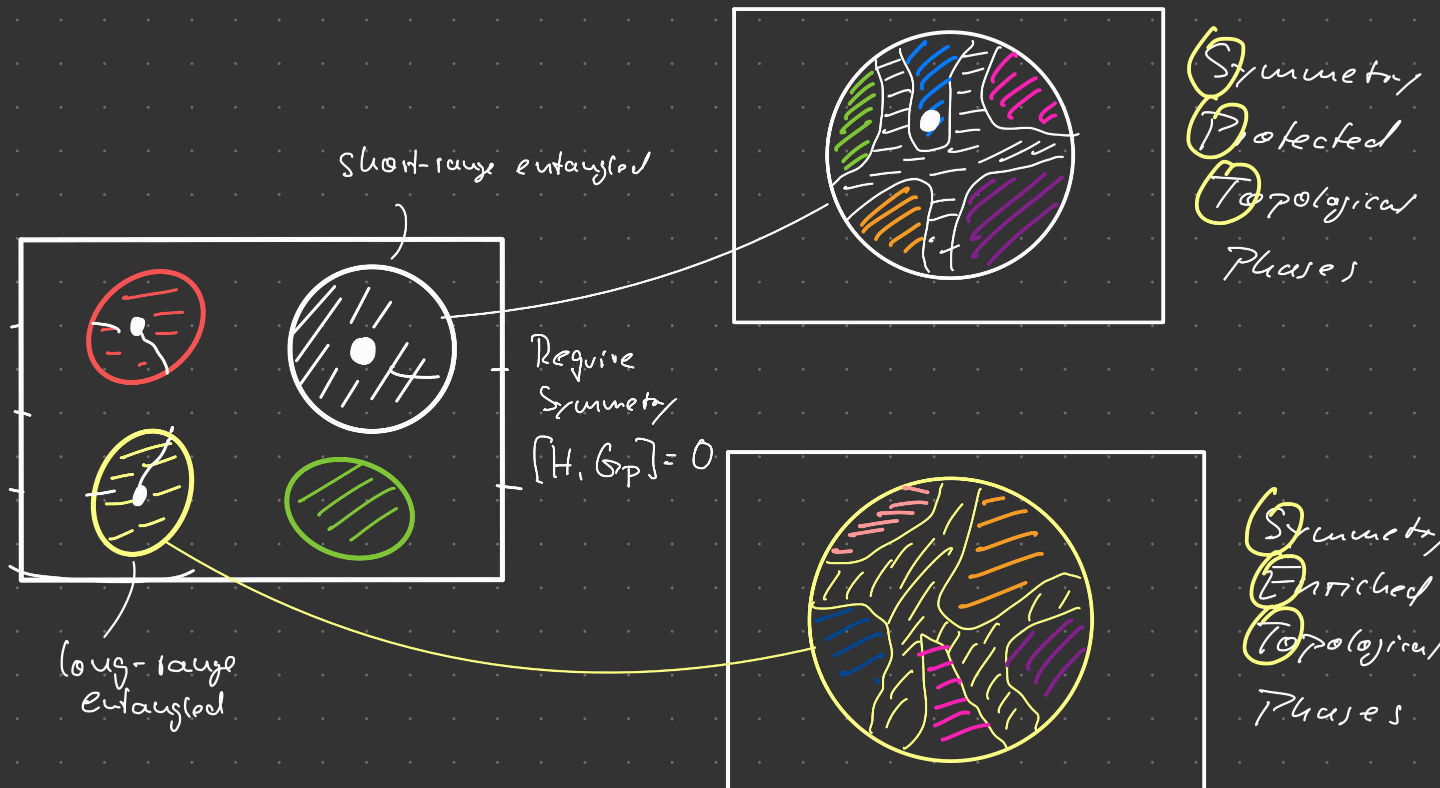
Here we will restrict Hamiltonians by (protecting) symmetries $G_P \subseteq G_S$:

Def. Symmetry-protected Quantum Phases

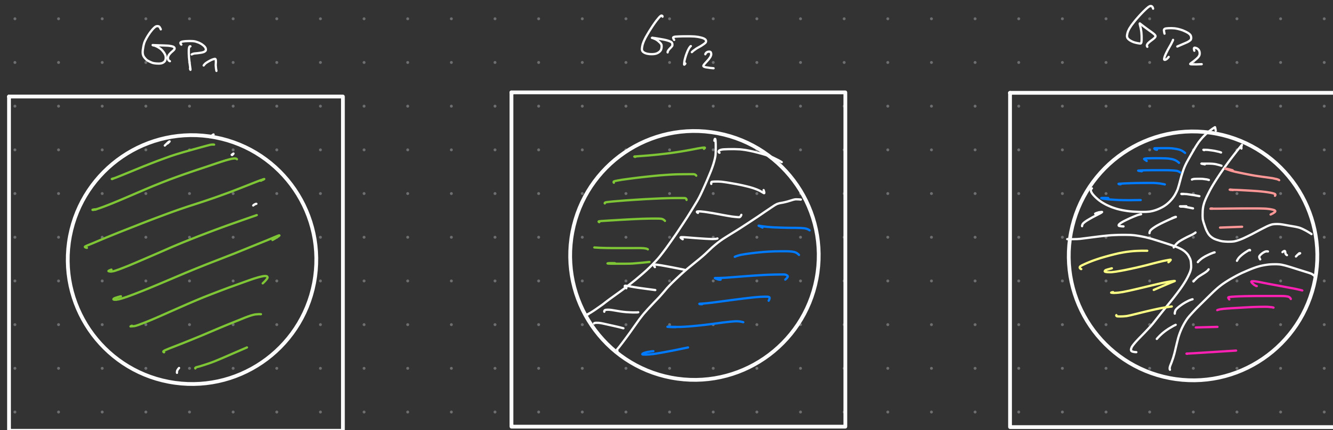
$\forall$ Gapped, local Hamiltonians H_a and H_b with unique ground states $|\Psi_a\rangle$ and $|\Psi_b\rangle$ and symmetry group G_P with $[H_{a,b}, g] = 0$ for all $g \in G_P$.

They belong to the same symmetry-protected topological phase (SPT) iff there is a family of gapped and local Hamiltonians $\hat{H}(\alpha)$ that depends continuously on $\alpha \in [0,1]$ such that $H_a = \hat{H}(0)$ and $H_b = \hat{H}(1)$ and $[\hat{H}(\alpha), g] = 0$ for all $g \in G_P$ and $\alpha \in [0,1]$.

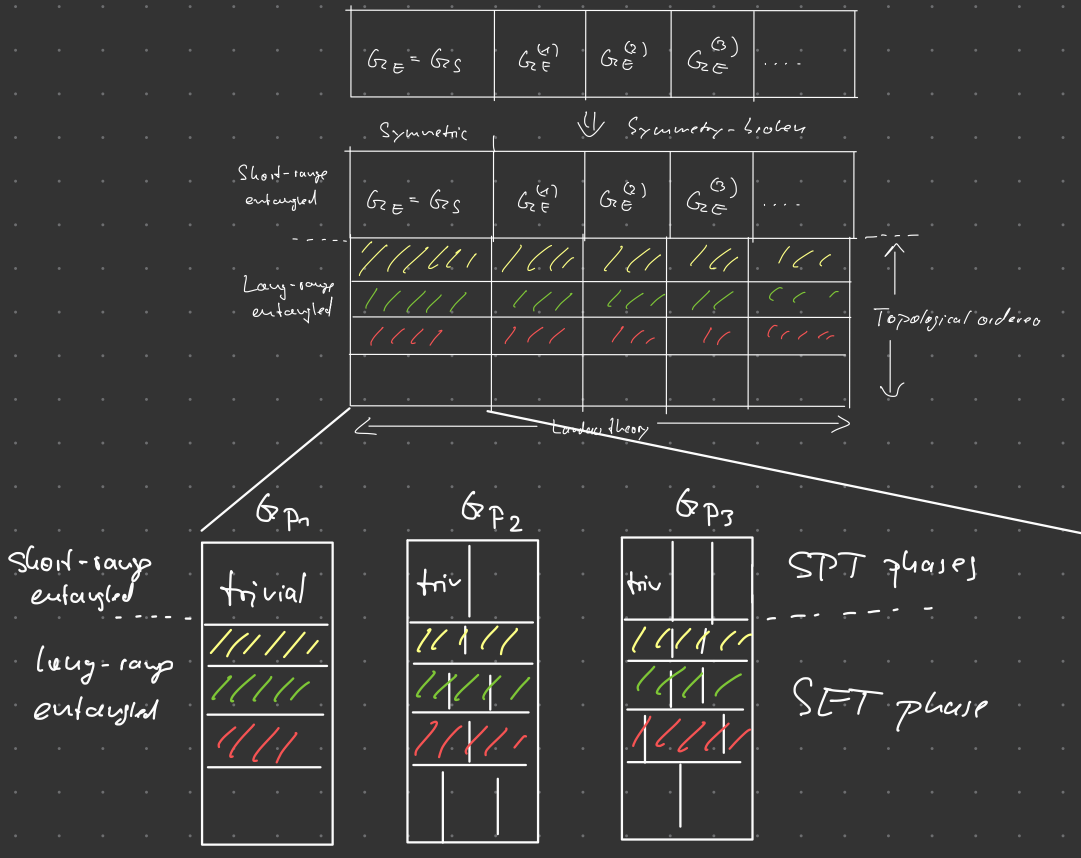
$\rightarrow$ Phases with the same entanglement pattern can split further:


short-range entangled
Require Symmetry
$[H, G_p] = 0$
long-range entangled
Symmetry Protected Topological Phases
Symmetry Enriched Topological Phases

Possible SPTs depend on the protecting symmetry G_P:

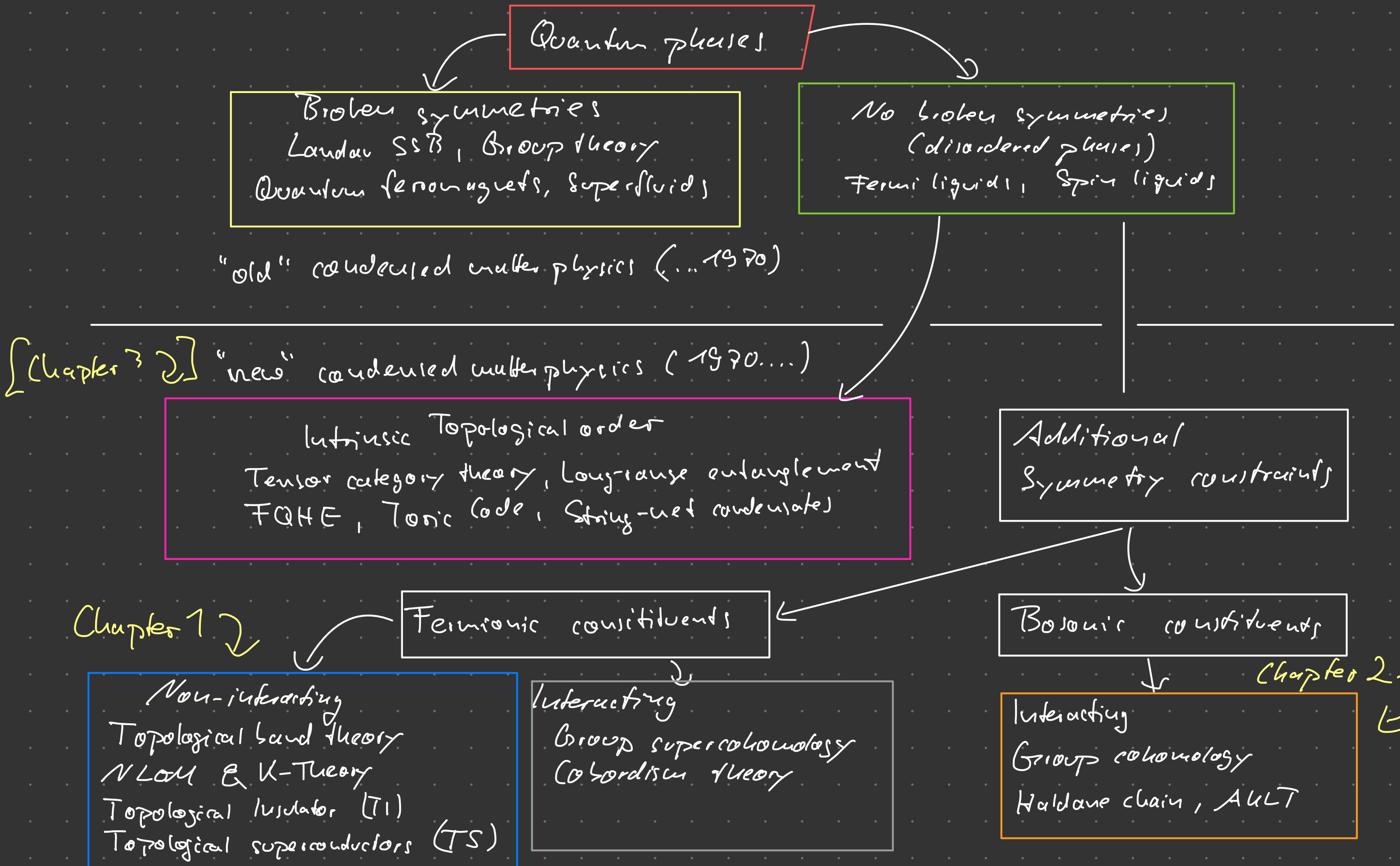


Second extension of Landau's paradigm: Symmetry-protected topological order



How to characterize SPT phases?

* Non-interacting fermions → Chapter I
* Interacting bosons (1D) → Chapter II


Quantum phases
Broken symmetries
Landau SSB, Group theory
Quantum ferromagnets, superfluids
No broken symmetries
(disordered phases)
Fermi liquids, Spin liquids
"old" condensed matter physics (...1970)
[Chapter 3]
"new" condensed matter physics (1970....)
Intrinsic Topological order
Tensor category theory, Long-range entanglement
FQHE, Toric Code, String-net condensates
Additional
Symmetry constraints
Fermionic constituents
Bosonic constituents
Chapter 1
Non-interacting
Topological band theory
NLσM & K-Theory
Topological Insulator (TI)
Topological superconductors (TS)
Interacting
Group supercohomology
Cobordism theory
Chapter 2
Interacting
Group cohomology
Haldane chain, AKLT

Def. Nomenclature:

In this course, quantum phases that (1) do not break symmetries and are either (2a) topologically ordered (TO) (with and without protecting symmetries) or (2b) symmetry-protected (SPT) will jointly be refered to as topological phases (TP).

Hallmarks of topological phases:

* TPs cannot be characterized by a local order parameter
* Some TPs can have topology-dependent, robust ground state degeneracies
* Some TPs can have localized excitations that obey anyonic statistics
* These excitations can carry fractionalized charges
* In some TPs, (lattice) defects can exhibit anyonic statistics

* Some TPs feature robust, gapless edge states on manifolds with boundaries (=) scattering free transport)

* The linear response of TPs can be quantized (=) topological origin)

* Some TPs have an efficient low-energy description in terms of topological quantum field theories (TQFT)