

1.6. Classification of non-interacting fermionic topological phases

1.6.1. Generic symmetries and "tenfold way"

1. Goal: Classify TPs of non-interacting fermions

Approach: Use SP Hamiltonian H to describe & classify MO Hamiltonian $\hat{H}$

2. Which symmetries of $\hat{H}$ to use?

Remember: X symmetry of $\hat{H} \Leftrightarrow [\tilde{X}, \hat{H}] = 0$

Wigner's $\rightarrow X$ unitary or antiunitary
theorem

→ Four possibilities on Fock space:

• **Unitary:** $U; U^{-1} = +i$, $U c_i U^{-1} = U_{ij} c_j$

(unitary MB sym.) $[U, \hat{H}] = 0 \iff [U, H] = 0$ (SP unitary sym.)

• **TRS:** $T; T^{-1} = -i$, $T c_i T^{-1} = U_{ij} c_j$

(antiunitary MB sym.) $[T, \hat{H}] = 0 \iff [UK, H] = 0$ (antiunitary SP sym.)

• **PHS:** $C; C^{-1} = +i$, $C c_i C^{-1} = U_{ij} c_j^+$

(unitary MB sym.) $[C, \hat{H}] = 0 \iff \{UK, H\} = 0$ (antiunitary SP pseudosym.)

• **SLS:** $S; S^{-1} = -i$, $S c_i S^{-1} = U_{ij} c_j^+$

(antiunitary MB sym.) $[S, H] = 0 \iff \{U, H\} = 0$ (unitary SP pseudosym.)

Using unitary symmetries of $\hat{H}$ (H) is possible but not universal!

→ $\nexists$ TRS, PHS and SLS

and only SP Hamiltonians without unitary symmetries:

$$H \text{ irreducible} : \Leftrightarrow [U, H] = 0 \Rightarrow U = e^{i\lambda} \mathbb{1}$$

3. For a given irreducible SP Hamiltonian H check

$$\exists U_T? : [U_T U, H] = 0 \quad \text{and if so: } U_T U_T^* \stackrel{?}{=} \pm \mathbb{1}$$

$$\exists U_C? : \{U_C U, H\} = 0 \quad \text{and if so: } U_C U_C^* \stackrel{?}{=} \pm \mathbb{1}$$

$$\exists U_S? : \{U_S, H\} = 0$$

$$T_U = U U$$

$$T = U_T U$$

$$C = U_C U$$

$$S = U_S$$

→ Define:

$$\text{TRS: } T = U_T U$$

$$\text{PHS: } C = U_C U$$

$$\text{SLS: } S = U_S$$

→ Labeling scheme:

$$[T, H] \neq 0 \quad \Rightarrow \quad T = 0$$

$$[T, H] = 0 \text{ with } T^2 = +1 \quad \Rightarrow \quad T = +1$$

$$[T, H] = 0 \text{ with } T^2 = -1 \quad \Rightarrow \quad T = -1$$

$$\{C, H\} \neq 0 \quad \Rightarrow \quad C = 0$$

$$\{C, H\} = 0 \text{ with } C^2 = +1 \quad \Rightarrow \quad C = +1$$

$$\{C, H\} = 0 \text{ with } C^2 = -1 \quad \Rightarrow \quad C = -1$$

$$\{S, H\} \neq 0 \quad \Rightarrow \quad S = 0$$

$$\{S, H\} = 0 \quad \Rightarrow \quad S = 1$$

→ Triple (T, C, S) classifies
the symmetries

4. Important: TRS T_{U_1} , PHS C_{U_1} , SLS S_{U_1} are **unique** (if present):

& T_{U_1} and T_{U_2} different:

$$[T_{U_1}, H] = 0 \quad \text{and} \quad [T_{U_2}, H] = 0$$

Then $\tilde{U} := T_{U_1} T_{U_2} = U_1 U_2^*$ is unitary symmetry of H :

$$[\tilde{U}, H] = 0 \quad \xrightarrow{\text{irred.}} \quad \tilde{U} = e^{i\lambda} \mathbb{1}$$

$$\Rightarrow T_{U_1} = U_1 K = e^{i\lambda} U_2^{* \dagger} K \propto T_{U_2}^{-1}$$

5. Sublattice symmetry:

$$S = T \circ C = U_T U_C^* \quad \text{unitary operator}$$

$$\text{w.l.o.g.} \quad S^2 = +\mathbb{1}$$

$$\Rightarrow \left. \begin{array}{l} \text{TRS: } [T, H] = 0 \\ \text{PHS: } \{C, H\} = 0 \end{array} \right\} \Rightarrow \{C, H\} = 0$$

→ C cannot be eliminated in favour of T since S is not unitary symmetry (but pseudosymmetry)

→ Keep T , C , and S

6. The "Tenfold way":

$$* (T \neq 0 \vee C \neq 0) \Rightarrow S = |TC|$$

$$* (T=0 \wedge C=0) \Rightarrow S = \begin{cases} 0 \\ 1 \end{cases}$$

$\rightarrow 3 \times 3 + 1 = 10$ symmetry classes

1.6.2 The periodic table of topological insulators and superconductors

1. $\otimes$ Gapped Hamiltonians H of X in dimension d

Question: How to label the topological phases that can be realized by these systems?

2. Answer: Periodic table of top. insulators & superconductors:

Class	T	C	S
A	0	0	0
AII	0	0	1
AI	+1	0	0
BDI	+1	+1	1
D	0	+1	0
DIII	-1	+1	1
AII	-1	0	0
CII	-1	-1	1
C	0	-1	0
CI	+1	-1	1

Symmetries Dimensions →

	Class	T	C	S	0D	1D	2D	3D	...	8D	...
Complex (w/o T & C)	A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	2-periodic →	$\mathbb{Z}$	...
	AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$		0	...
	AI	+1	0	0	$\mathbb{Z}$	0	0	0		$\mathbb{Z}$	...
Real (w/ T or C)	BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	"Bott periodicity" ⚡ real K-groups	$\mathbb{Z}_2$	...
	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0		$\mathbb{Z}_2$	...
	DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		0	...
	AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$2\mathbb{Z}$	...
	CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$		0	...
	C	0	-1	0	0	0	$2\mathbb{Z}$	0		0	...
	CI	+1	-1	1	0	0	0	$2\mathbb{Z}$		0	...

This course:

Remember,
remember...

Chern number

- IQHE
- QWZ
- Haldane

Pfaffian Index

- Kane-Mele model

No Kramers pairs?
(Kane-Mele model)

Winding number

- SSH-chain

All paths contractible?
(Majorana chain)

Endpoints = Poles

- Majorana chain

Winding number

- Majorana chain

→ In every dimension, 5 out of 10 symmetry classes support TPs.

3. Recipe:

- Check whether the SP Hamiltonian has features TRS, PHS, and/or SCS and (if so) whether TRS/PHS square to $\pm 1 \rightarrow (T, C, S) \rightarrow \text{Class } X$
- Use the periodic table to check whether $\times$ supports TPs in the spatial dimension d of your system Ref. in Script 0
- Look up the associated topological invariant I for (X, d)
- Compute $I = I[H]$ as a function of the control parameters and check whether it is non-zero in (some of) the phases

1.6.3. Frameworks for classification

- * Anderson localization on the boundary
 - * Quantum anomalies on the boundary
 - * K-Theory:
- } no transl. invariance
boundary-based
- } transl. invariance
bulk-based

1. $\mathbb{Z}$ Gapped system in filled (unempty) bands described by
Block Hamiltonian $H(\vec{u})$

2. Spectral flattening:

$$H(\vec{u}) \xrightarrow{\text{continuously deformed}} L(\vec{u})$$

$$\text{with } \sigma(L(\vec{u})) = \left(\underbrace{-1, \dots, -1}_n, \underbrace{+1, \dots, +1}_n \right)$$

3. Simplest case: Class $A \rightarrow$

$$L(\vec{u}) = U(\vec{u}) \underbrace{\begin{pmatrix} \mathbb{1}_m & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}}_{\times} U^T(\vec{u}) \quad U(\vec{u}) \in U(m+n)$$

4. "Gauge symmetry:"

$$U \sim U' \Leftrightarrow U = U' \cdot \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \quad \text{for} \quad \begin{matrix} U_1 \in U(m) \\ U_2 \in U(n) \end{matrix}$$

since

$$L(\vec{u}) = U(\vec{u}) \times U^T(\vec{u}) = U'(\vec{u}) \times U'^T(\vec{u})$$

$\rightarrow$

$$L: T^d \ni \vec{u} \mapsto L(\vec{u}) \hat{=} [U(\vec{u})]_{\sim} \in \frac{U(m+n)}{U(m) \times U(n)} = G_{m, m+n}(\mathbb{C})$$

$\rightarrow G_{m, m+n}(\mathbb{C})$ is the classifying space C_0 $\left\{ \begin{array}{l} \text{complex Grassmannian} \\ \leftarrow \text{w.r. to subspaces} \end{array} \right.$
for symmetry class A

	Class	T	C	S	Classifying Space	Name
Complex classes $UH U^t = -H$	A	0	0	0	$U(n+m) / U(n) \times U(m)$	C_0
	AIII	0	0	1	$U(n) \times U(n) / U(n)$	C_1
	AI	+1	0	0	$O(n+m) / O(n) \times O(m)$	R_0
Real classes $UH^* U^t = H$	BDI	+1	+1	1	$O(n) \times O(n) / O(n)$	R_1
	D	0	+1	0	$O(2n) / U(n)$	R_2
	DIII	-1	+1	1	$U(2n) / Sp(2n)$	R_3
	AII	-1	0	0	$Sp(n+m) / Sp(n) \times Sp(m)$	R_4
	CII	-1	-1	1	$Sp(n) \times Sp(n) / Sp(n)$	R_5
	C	0	-1	0	$Sp(2n) / U(n)$	R_6
	CI	+1	-1	1	$U(n) / O(n)$	R_7

5. Simplification: $T^d \rightarrow S^d$

Gravimetric for sy. A

$$\{\text{Topological phases}\} = \pi_d(C_0)$$

Example for $d=2$: $\pi_2(C_0) \stackrel{*}{=} \mathbb{Z} \rightarrow$ Chern number

6. Undo simplification $S^d \rightarrow T^d$ & include symmetry constraints:

Kitaev $\rightarrow$

$$\{\text{Topological phases of } (X, d)\} =$$

$$\pi_0(X) = \text{connected components of } X$$

$$\underbrace{\pi(\overline{T}^d, \mathbb{R}_q)}_{K_n^{-q}(\overline{T}^d)} \stackrel{\text{K-theory}}{=} \underbrace{\pi_0(\mathbb{R}_{q-d})}_{\substack{\text{mod } \mathbb{P} \\ \text{(Strong) topological index}}} \oplus \underbrace{\bigoplus_{s=0}^{d-1} \binom{d}{s} \pi_0(\mathbb{R}_{q-s})}_{\text{Weak topological indices}}$$

$\rightarrow$ computing $\pi_0(\mathbb{R}_{q-d})$ for $q=0, \dots, p$ and $d=0, 1, 2, \dots$ yield the periodic table.

Example: $g=4$ ($A\mathbb{Z}$) and $d=2$ (e.g. Hase-Meiss mode):

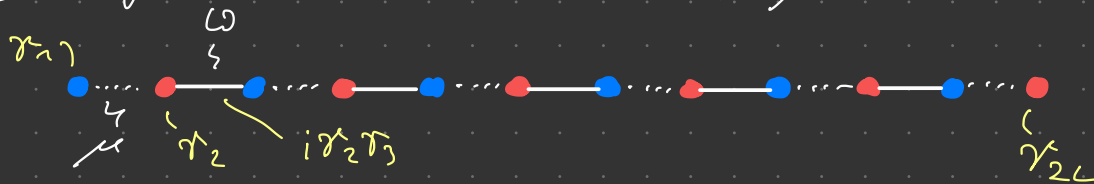
$$\begin{aligned}\pi(\overline{T}^2, R_4) &= \underbrace{\pi_0(R_2)}_{\pi_2} \oplus 1 \times \underbrace{\pi_0(R_4)}_{\pi} \oplus 2 \times \underbrace{\pi_0(R_3)}_{0} \\ &= \underbrace{\pi_2}_{\text{Pfaffian index}} \oplus \underbrace{\pi}_{\substack{\# \text{ Valence} \\ \text{bands}}} \oplus \underbrace{2 \times 0}_{\text{No weak top. indices}}\end{aligned}$$

1.6.4 (consequences of interactions)

Question: What happens to the periodic table if interactions are included?

Quick answer: Unknown for **quartic** interactions; No complete classification known for arbitrary interactions.

1. $\mathbb{Z}$ Majorana chain $\omega = \Delta > 0$ and $\mu = 0$:



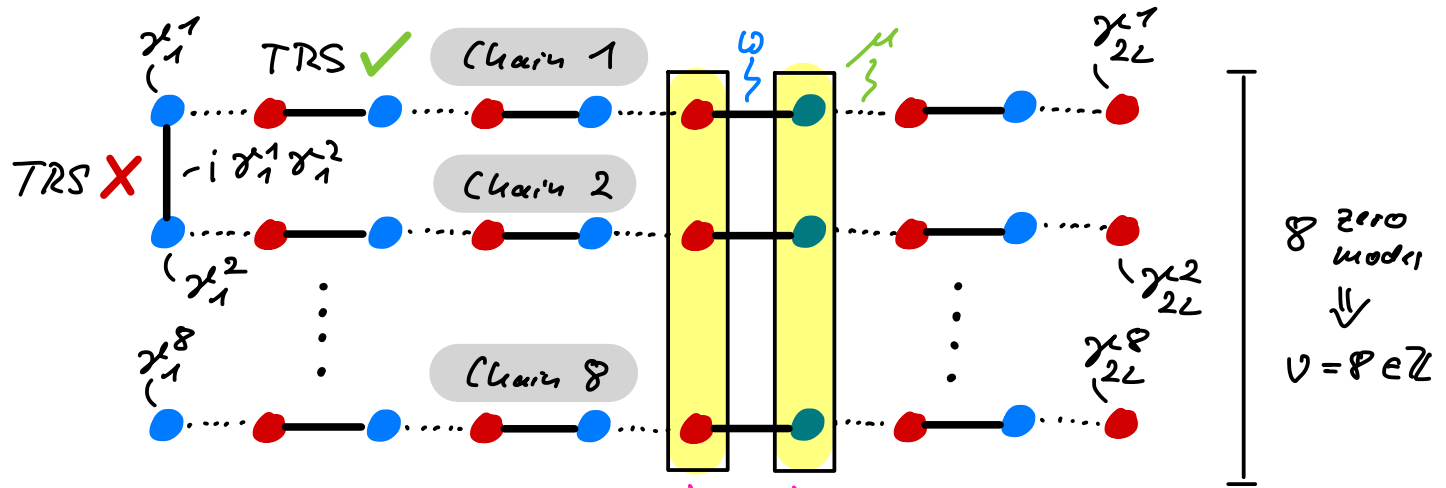
TRS & THP $\rightarrow$ Symmetry class BDI in $d=1 \rightarrow \mathbb{Z}$ index

2. Time-reversal symmetry: $(\tau c_i \tau^{-1} = c_i, \quad \tau^2 = +1)$

$$\tau i \tau^{-1} = -i \quad \text{and} \quad \tau \gamma_{2i-1} \tau^{-1} = +\gamma_{2i-1} \quad \text{and} \quad \tau \gamma_{2i} \tau^{-1} = -\gamma_{2i}$$

$\rightarrow$ Only couplings between even (red) and odd (blue) Majoranas $\triangleright$

3. Stack of Majorana chains;



→ $\mathbb{Z}$ -index = # dangling Majorana zero modes γ_1^α

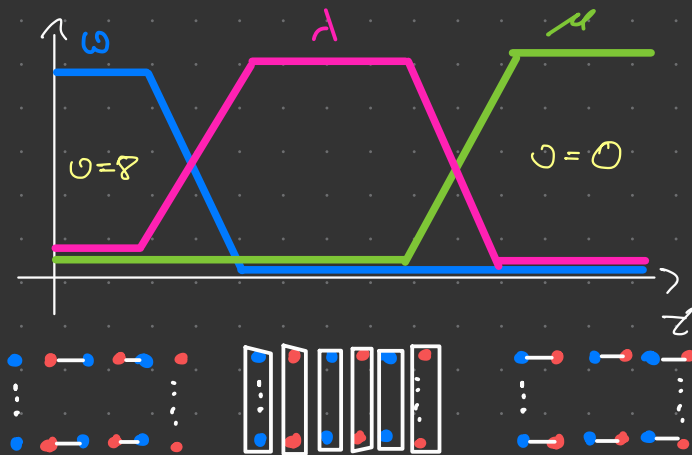
& 8 topological chains → BDI index $v = 8$

4. Connect topological to trivial phase via **quartic interaction**:

$$\hat{H}(\mu, \omega, \lambda) = \hat{H}_{nc}(\mu, \omega) + \lambda \sum_{u=1}^{2L} \mathcal{O}_u$$

with $\mathcal{O}_u = \tau_u^1 \tau_u^2 \tau_u^3 \tau_u^4 + \dots$ (many more quartic terms)

Protocol:



On this path...

- * the bulk gap remains open
- * and TRS is not broken

5. $\omega=0$ and $\omega=8$ are the same phase $\mathcal{D}$

→ $\mathbb{Z}_2$ -index of BDI in $d=1$ reduces to $\mathbb{Z}_8$ -index