1.7. Topologically protected edge states in classical systems

1.7.1. Review: Effects of topological bands

* Quantized Hall response:

SP properties

$$\sigma_{xy} \overset{\uparrow}{=} -\frac{e^2 \nu}{2\pi \hbar} \quad \text{with} \quad \nu = \sum_{n:\, \varepsilon_n < E_F}{}^{'} C^{[n]} \in \mathbb{Z}$$

MB quantum effect

→ Requires filled bands

→ Many-body phenomenon (Fermi statistics)

→ Genuine quantum effect

* Robust edge states:

Top. system 1D (OBC) $\longrightarrow$ Zero-energy boundary modes

Top. system 2D (OBC) $\longrightarrow$ Gapless edge modes

→ Bulk-boundary correspondence

→ Single-particle phenomena

→ Not a quantum effect !

$$\boxed{\text{Topological features of the band structure (= single-particle feature) are not quantum effects !}}$$

Question: Can we translate edge modes to classical systems?

1.7.5. Example: Topological mechanics and helical edge modes

1. <u>Goal:</u> Realization of the helical edge modes of the QSHE in a classical mechanics setup governed by Newton's equation.

2. <u>Quantum system (QS):</u> $\nexists$ SP Schrödinger equation

$$\underbrace{i\hbar\,\dot{\psi}_i^{\alpha}}_{\text{Quantum dynamics}} = H_{ij}^{\alpha\beta}\,\psi_j^{\alpha}$$

i,j: site indices

α,β: spin indices

H: SP-Hamiltonian
↳ Hermitian matrix

3. <u>Classical systems (CS):</u>

pendula coupled by springs
↳

$\nexists$ Newton's equation for N coupled oscillators

$$\underbrace{\ddot{X}_i}_{\text{Classical dynamics}} = -D_{ij}X_j$$

X_i: states of oscillators

D: Dynamical coupling matrix (real, symmetric, positive semidefinite)

Remember:

$m\ddot{x} = -DX$

$\omega^2 = D/m$

4. Observation:

* QS characterized by eigenstates of H
* CS characterized by eigenmodes of D

But: Edge modes are simply special eigenstates of H

Idea: Use H with edge modes to construct D with edge modes.

5. Model construction:

a) * Two independent copies of the Hofstadter model with spin-dependent flux $\alpha\hat{\phi} = \pm \frac{p}{q} = \pm \frac{1}{3}$

Landau gauge

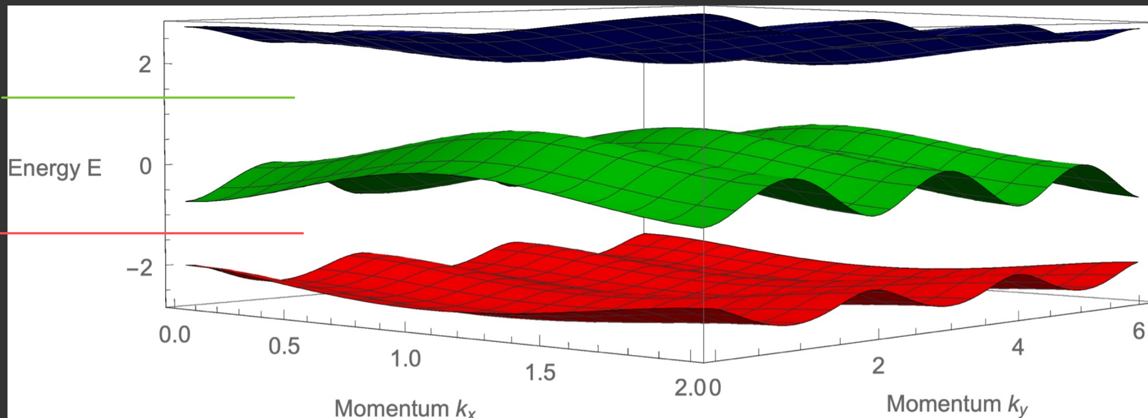
$$H = \begin{pmatrix} H_\uparrow & 0 \\ 0 & H_\downarrow \end{pmatrix} \quad \text{with} \quad H_\alpha = \sum_{x,y} \left\{ |x,y+1,\alpha\rangle\langle x,y,\alpha| + e^{-2\pi i \alpha\hat{\phi} y} |x+1,y,\alpha\rangle\langle x,y,\alpha| \right\} + h.c.$$

b) $q = 3 \rightarrow$ Three gapped, spin-degenerate bands

Chern numbers

$\vartheta = C_1 + C_2 = -1$

$\vartheta = C_1 = 1$



$\rightarrow C_3 = 1$

$\rightarrow C_2 = -2$

$\rightarrow C_1 = 1$

* Finite sample with open boundaries:

$\rightarrow$ Two helical edge modes in each gap

c) Symmetries:

* Time-reversal symmetry: $T = i\sigma^y K$ with $T^2 = -\mathbb{1}$

$$T H T^{-1} = H$$

since $H_\uparrow = H^*_\downarrow$

$\rightarrow$ AII with $\mathbb{Z}_2$ Pfaffian index

* Spin conservation:

$$\sigma^z H \sigma^z = H$$

$$I = \frac{C_\uparrow - C_\downarrow}{2} \bmod 2$$

d) Problem: H is complex

Solution:

$H = H_\uparrow$

$$\underbrace{U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}}_{u} \otimes \mathbb{1}_{\text{lattice}} \Rightarrow D := U^+ H U = \begin{pmatrix} \mathrm{Re}\, H & \mathrm{Im}\, H \\ \mathrm{Im}\, H^T & \mathrm{Re}\, H \end{pmatrix}$$

$\mathrm{Re}\, H = \mathrm{Re}\, H_\uparrow = \mathrm{Re}\, H_\downarrow$, $\mathrm{Im}\, H_\downarrow = \mathrm{Im}\, H^*_\uparrow = \mathrm{Im}\, H^T_\uparrow = \mathrm{Im}\, H^T$

$\rightarrow$ $\mathcal{D}$ real and symmetric (and can be made positive semidefinite by a constant shift $\mathcal{D} \mapsto \mathcal{D} + \text{const}$)

e) Transformed basis:

$$\begin{pmatrix} X_{xy} \\ Y_{xy} \end{pmatrix} = u^{+} \begin{pmatrix} \psi_{xy\uparrow} \\ \psi_{xy\downarrow} \end{pmatrix}$$

(X_{xy}, Y_{xy}): Position of 2D harmonic oscillator on site (x,y)

& Spin-up mode on site (x,y):

$$\Psi_{xy\uparrow}(t) = \psi_0 e^{-i\omega t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{xy}$$ "Spin basis"

$$\Rightarrow \quad u^{+} \Psi_{xy\uparrow}(t) = \psi_0 e^{-i\omega t} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}_{xy}}_{\text{Jones vectors}}$$ "Oscillator basis"

$$\Rightarrow \quad \mathrm{Re}\left[u^{+} \Psi_{xy\uparrow}(t) \right] \propto \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}$$ left-circular polarization

Spin UP $\leftrightarrow$ Left circular polarization

Spin DOWN $\leftrightarrow$ Right circular polarization

f) Requirements on the setup:

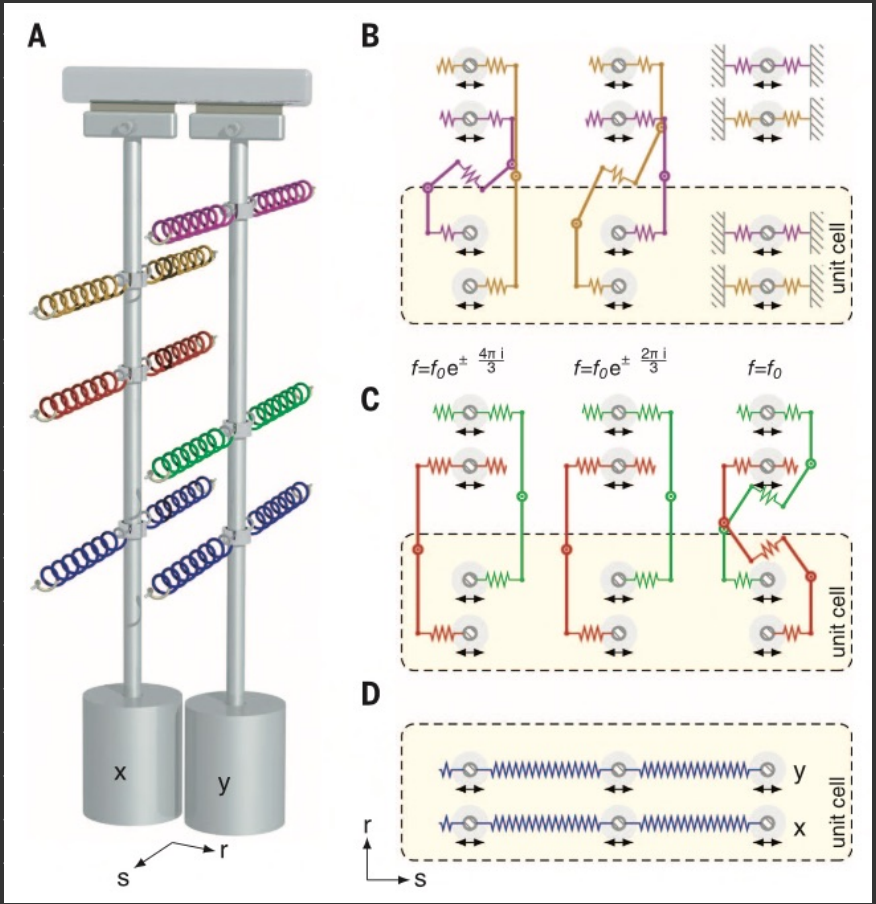
* Hofstadter model with $q=3$ $\rightarrow$ 3 sites per magnetic unit cell
* Two-fold spin/polarization degeneracy per site
  $\rightarrow$ 2 harmonic oscillators per site

$\rightarrow$ 6 x 1D pendula per unit cell coupled by springs according to D:

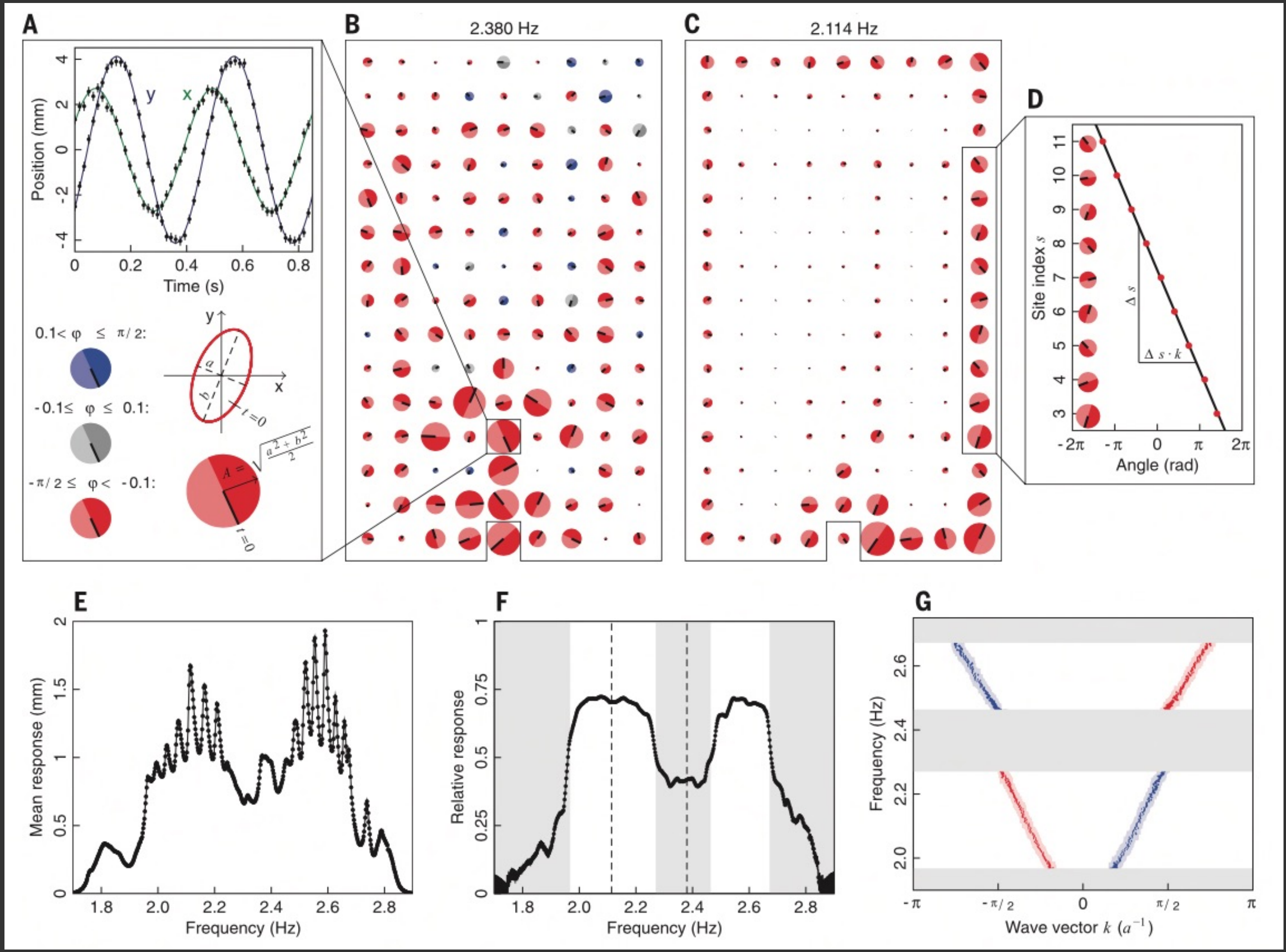
* Re H $\rightarrow$ XX- and YY-couplings
* Im H $\rightarrow$ XY-couplings

6. Experiment & Results:

a) Construction:

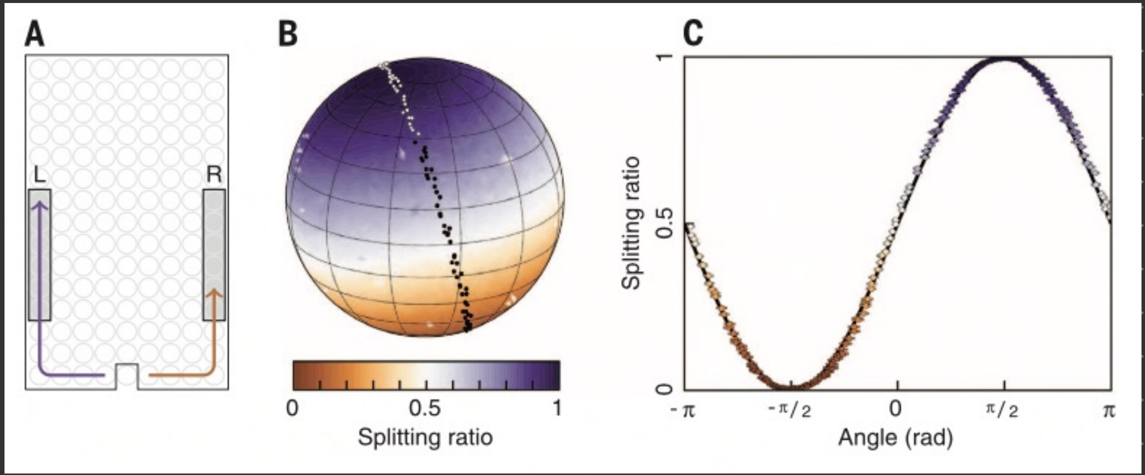


b) Edge modes:

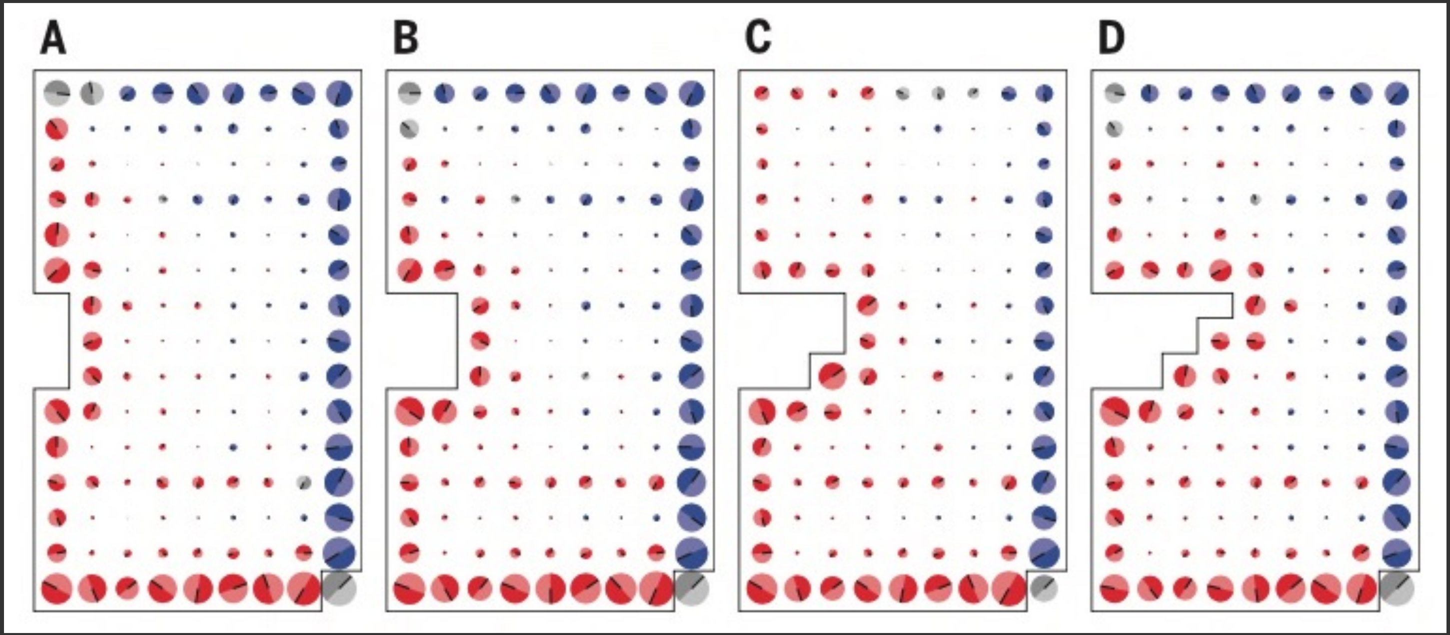


A
B
2.380 Hz
C
2.114 Hz
D
Position (mm)
Time (s)
y
x
0.1< φ ≤ π/2:
-0.1≤ φ ≤ 0.1:
-π/2 ≤ φ< -0.1:
t=0
Site index s
Angle (rad)
Δ s
Δ s · k
E
Mean response (mm)
Frequency (Hz)
F
Relative response
Frequency (Hz)
G
Frequency (Hz)
Wave vector k (a⁻¹)

c) Application: Beam splitter



d) Robustness



Symmetry protection:

$$H \overset{!}{=} T H T^{-1} = U_T H^* \overset{i\sigma_y}{U_T^\dagger} = U_T \left(U U^\dagger H U U^\dagger \right)^* U_T^\dagger$$

$$\Rightarrow \quad D = U^\dagger H U \overset{!}{=} \left(U^\dagger U_T U^* \right) D \underbrace{\left(U^{*\dagger} U_T^\dagger U \right)}_{S} = S^\dagger D S$$

with unitary symmetry

$$S = U^{*\dagger} U_T^\dagger U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\rightarrow \quad \text{with} \quad D = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{pmatrix} \quad \Rightarrow \quad \boxed{\begin{aligned} D_{xx} &\overset{!}{=} D_{yy} \\ D_{xy} &\overset{!}{=} -D_{yx} \end{aligned}}$$

Satisfied by $D_{xx} = D_{yy} = \mathrm{Re}\, H$

$D_{xy} = \mathrm{Im}\, H$

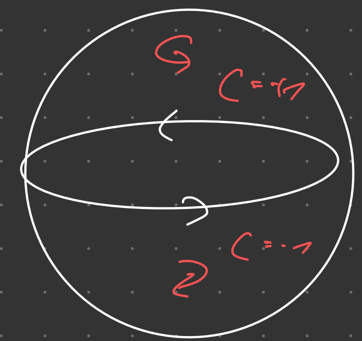
$D_{yx} = \mathrm{Im}\, H^T = \mathrm{Im}\, H^* = -\mathrm{Im}\, H = -D_{xy}$

→ Perturbation

$$D' = D + \begin{pmatrix} \delta D & 0 \\ 0 & \delta D \end{pmatrix}$$

does not destroy the edge modes !?

→ Local symmetry constraint



1.7.3. More classical systems

* Topological mechanics
* Topological acoustics
* Topological photonics (→ Topological insulator lasers)
* Topoelectrical circuits
* Topological fluid dynamics