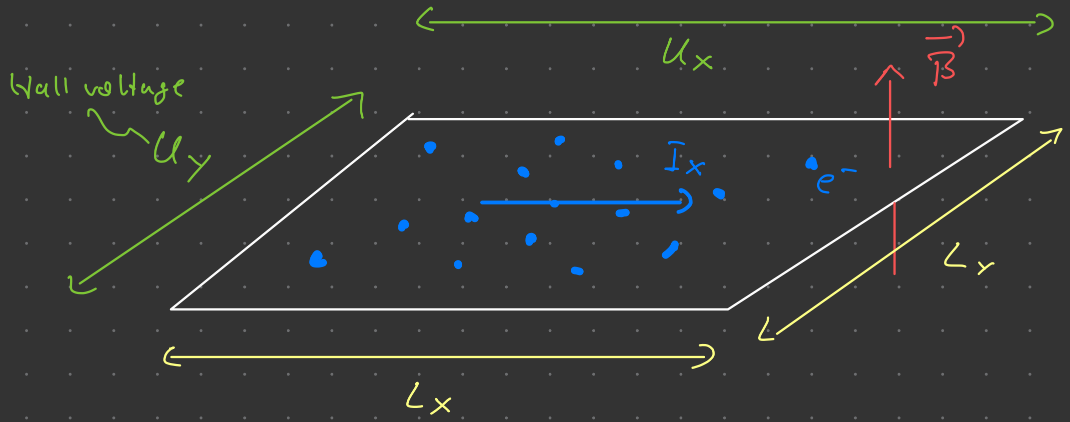
1. Topological Phases of non-interacting fermions

1.1. The integer quantum Hall effect

1.1.1. From the classical to the quantum Hall effect

$\lozenge$ 2D electron gas (2DEG) in perpendicular magnetic field $\vec{B} = B\,\vec{e}_z$:



Drude model:

$$m \frac{d\vec{v}}{dt} = -e\vec{E} - e\,\vec{v} \times \vec{B} - \frac{m}{\tau}\vec{v}$$

τ: scattering time

With current density $\vec{j} = -\underbrace{n}_{\text{electron density}} e \vec{v}$ and $\frac{d\vec{v}}{dt} = 0$ $\Rightarrow$

$$\underbrace{\vec{j} = \sigma \cdot \vec{E}}_{\text{Ohm's law}}$$

$$\underbrace{\sigma}_{\text{conductivity tensor}} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{yx} & \sigma_{yy} \end{pmatrix} = \frac{\sigma_0}{1+\omega_B^2\tau^2} \begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix}$$

with

$$\boxed{\omega_B = \frac{eB}{m} \quad \text{the cyclotron frequency}}$$

and $\sigma_0 = ne^2\tau/m$ the DC conductivity

$\Rightarrow$ <u>Resistivity tensor:</u>

$$\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix} = \sigma^{-1} = \frac{1}{\sigma_0} \begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix}$$

Note:

$$R_{xy} = \frac{U_y}{I_x} = \frac{\cancel{L_y} E_y}{\cancel{L_y} J_x} = \frac{E_y}{J_x} = -\rho_{xy}$$

(Hall resistance

$$R_{xx} = \frac{U_x}{I_x} = \frac{L_x E_x}{L_y J_x} = \frac{L_x}{L_y} \rho_{xx}$$

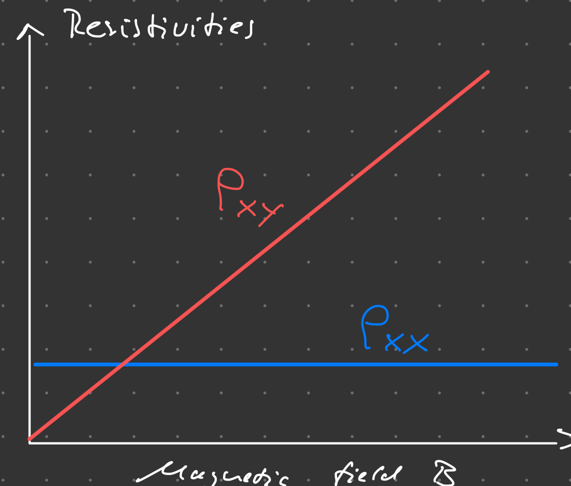
→ In particular:

$$\rho_{xy} = \frac{\omega_B \tau}{\sigma_0} = \frac{B}{ne}$$

independent of τ

$$\rho_{xx} = \frac{m}{ne^2 \tau}$$

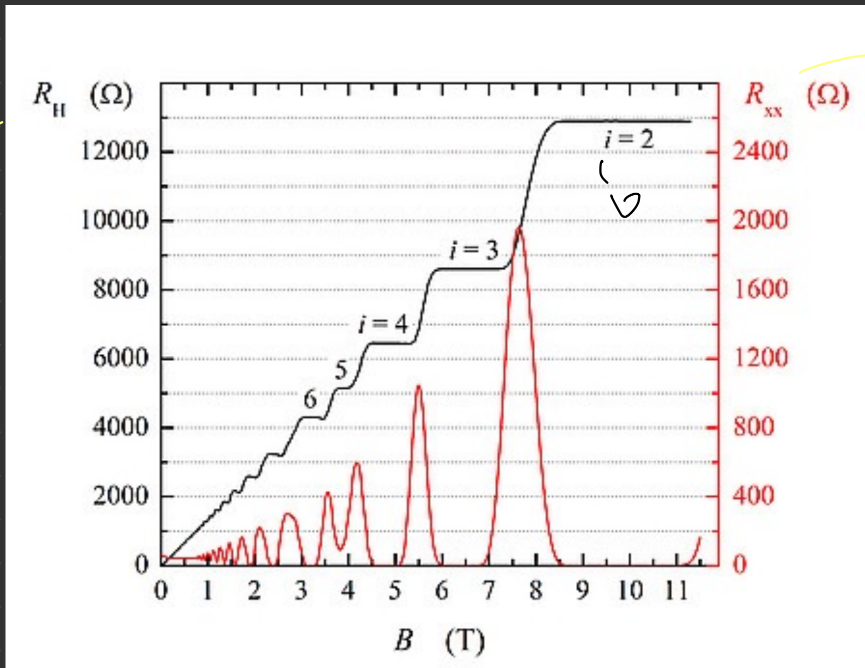
→ Classical prediction:



Experimentally valid for high temperatures & weak magnetic fields ($\hbar\omega_B \ll k_B T$).

But <u>not</u> for low temperatures & strong magnetic fields ($\hbar\omega_B \gg k_B T$):

$R_{xy} = \rho_{xy}$



$\frac{L_x}{L_y} \cdot \rho_{xx}$

→ <u>Quantized plateaus</u> for Hall resistivity:

$$\rho_{xy} = \underbrace{\frac{2\pi\hbar}{e^2}}_{R_K} \cdot \frac{1}{\nu} \qquad \nu = \{1, 2, 3, \dots\}$$

"quantum of resistivity"

($R_K = 25.8\ k\Omega$)

Important: The exact quantization of the (macroscopic) Hall response in disordered samples is a remarkable and unexpected feature that demands for an explanation.

1.1.2. Landau levels

Important: The integer quantum Hall effect can be understood in the context of non-interacting fermions. Thus we will focus on single-particle wavefunctions in the following.

Single-particle Hamiltonian:

$$H = \frac{1}{2m}\Big(\underbrace{\overset{\text{canonical momentum}}{\vec{p}} + e\vec{A}}_{\vec{\pi}\,:\ \text{kinetic momentum}}\Big)^2$$

gauge potential with $\nabla \times \vec{A} = B\vec{e}_z$

Canonical quantization: $\quad [x_i, p_j] = i\hbar\,\delta_{ij}$

$\rightarrow [\pi_x, \pi_y] \hat{=} -ie\hbar B$

With

$$a = \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \quad \text{and} \quad a^+ = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y)$$

we find $[a, a^+] = 1$ and

$$\boxed{H \hat{=} \hbar\omega_B\left(a^+a + \frac{1}{2}\right)}$$

$\rightarrow$ Discrete spectrum $\quad \underbrace{E_n = \hbar\omega_B\left(n + \frac{1}{2}\right)}_{\text{Landau levels}}$ with $n = 0, 1, 2, 3, \ldots$

$\rightarrow$ Eigenstates? Degeneracy?

Landau gauge

Gauge choice: $\vec{A} = x B \vec{e}_y$

→ Hamiltonian: $$\mathcal{H} = \frac{1}{2m}\left[p_x^2 + (p_y + eBx)^2\right]$$

→ Ansatz: $$\psi_k(x,y) = e^{iky} f_k(x)$$

→ Shifted harmonic oscillator:

$$\mathcal{H} \hat{=} \frac{1}{2m} p_x^2 + \frac{m\omega_B^2}{2}\left(x + k l_B^2\right)^2$$

with the magnetic length

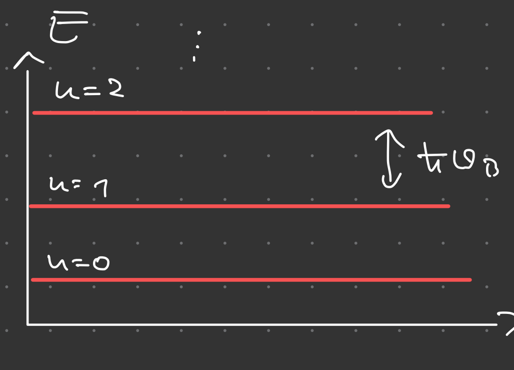
$$\boxed{l_B = \sqrt{\frac{\hbar}{eB}}}$$

$$\psi_{n,k}(x,y) = \mathcal{N} \cdot \underbrace{e^{iky}}_{\text{Plane wave in y-direction}} \underbrace{H_n(x + k l_B^2)\, e^{-\frac{(x+kl_B^2)^2}{2 l_B^2}}}_{\text{Harmonic oscillator in x-direction}}$$

with $n = 0, 1, 2, \ldots$ (Landau levels)

and $k = \frac{2\pi}{L_y}\mathbb{Z}$ the y-momentum

→ <u>Spectrum</u>: $E_n = \hbar\omega_B \left(n + \frac{1}{2}\right)$



<u>Degeneracy</u>: $0 \le x \le L_x$ → Allowed k: $-\frac{L_x}{l_B^2} \le k \le 0$

→ Number of states in each Landau level: $L_x L_y$: Area of the sample

$$N = \frac{L_x / l_B^2 - 0}{2\pi / L_y} = \frac{L_x L_y}{2\pi l_B^2} = \frac{AB}{\Phi_0} = \frac{\Phi}{\Phi_0}$$

$\Phi_0 = \frac{2\pi\hbar}{e}$: quantum of flux

→ Extensive degeneracy