1.1.3 Berry connection and Berry holonomy

Setting:

* Continuous family of gapped Hamiltonians $H(\vec{\Gamma})$ with k parameters $\vec{\Gamma} = (\Gamma_1, \Gamma_2, \dots \Gamma_k)$ and n-fold degenerate ground state space $V(\vec{\Gamma}) \equiv V(H(\Gamma))$

In the following: $H(\vec{\Gamma})\,|\psi\rangle = 0 \quad |\psi\rangle \in V(\vec{\Gamma})$ for all Γ

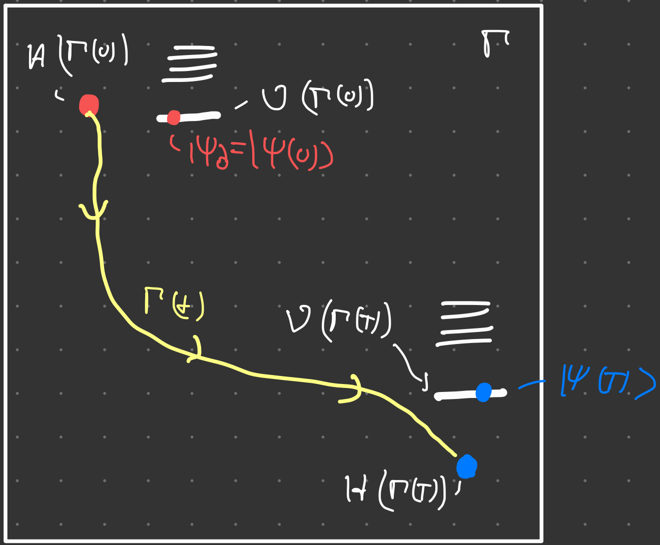
compared to the energy gap above $V(\vec{\Gamma})$

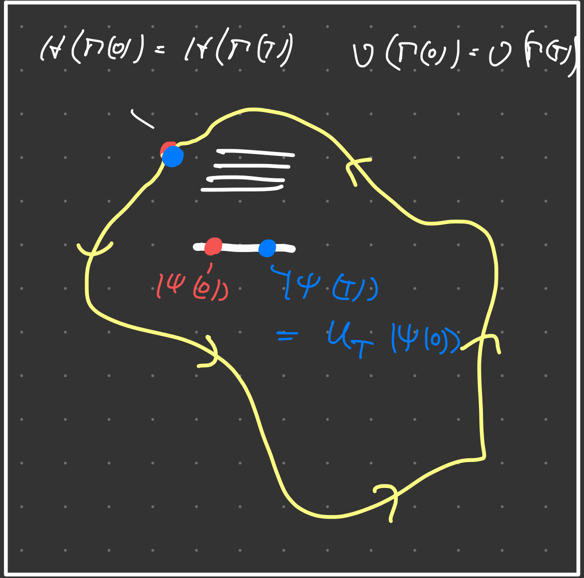
* Slow variation of parameters $\vec{\Gamma}(t)$ for $0 \le t \le T$
* Initial ground state $|\psi_0\rangle \in V(\vec{\Gamma}(0))$

Question: What happens with $|\psi_0\rangle$ as $H(\vec{\Gamma}(0))$ evolves to $H(\vec{\Gamma}(T))$?

Adiabatic theorem

A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum.





1. Pick a basis $|v_i(\vec{r})\rangle$ of $V(\vec{r})$ for every r $(i = 1, 2, \ldots, n)$

2. Time-dependent Schrödinger equation:

$$i\hbar\, \partial_t |\psi(t)\rangle = H(\vec{r}(t))\, |\psi(t)\rangle = 0$$

3. Adiabatic theorem $\rightarrow$ $|\psi(t)\rangle = \sum_i \psi_i(t)\, |v_i(\vec{r}(t))\rangle$

$\rightarrow$

$$\partial_t |\psi(t)\rangle = \left(\partial_t \psi_i(t)\right) |v_i(\vec{r}(t))\rangle + \psi_i(t) \left(\partial_{r_l} |v_i(\vec{r}(t))\rangle\right) \cdot \left(\partial_t r_l(t)\right)$$

$$= 0$$

4. Apply $\langle v_j(\vec{r}(t))|$ from the left:

$$\partial_t \psi_j(t) = -\psi_i(t)\, \underline{\langle v_j(\vec{r}(t))|\, \partial_{r_l} |v_i(\vec{r}(t))\rangle} \cdot \left(\partial_t r_l(t)\right)$$

5. Define the

$$\boxed{\text{Berry connection} \quad \left(\mathcal{A}_l(\vec{r})\right)_{ji} := -i \langle v_j(\vec{r})|\, \partial_{r_l} |v_i(\vec{r})\rangle \in u(n)}$$

6. Then

$$\partial_t \vec{\psi}(t) = -i(\partial_t r_l(t)) \cdot \mathcal{A}_l(\vec{r}(t))\, \vec{\psi}(t)$$

can be solved with the time-ordered ($\mathcal{T}$) or path-ordered ($\mathcal{P}$) exponential:

$$\vec{\psi}(T) = \mathcal{T} \exp\left\{-i \int_0^T \mathcal{A}_l(\vec{r}(t)) \cdot \partial_t r_l(t)\, dt\right\} \vec{\psi}_0$$

$$= \underbrace{\mathcal{P} \exp\left\{-i \int_\Gamma \vec{\mathcal{A}}(\vec{r})\, d\vec{r}\right\}}_{=: U_\Gamma} \vec{\psi}_0$$

7. Change local basis by unitary $\Omega(\vec{r}) \in U(n)$: $|v_i'(\vec{r})\rangle := \Omega_{ij}(r)\,|v_j(\vec{r})\rangle$

$\Rightarrow$

$$\mathcal{A}_l' = \Omega \mathcal{A}_l \Omega^\dagger - i \frac{\partial \Omega}{\partial r_l} \Omega^\dagger$$

$\overset{\circ}{\Rightarrow}$ $$U'_\Gamma = \Omega(\vec{\Gamma}(T))\, U_\Gamma\, \Omega^\dagger(\Gamma(0))$$

→ For an open path Γ, U_Γ is gauge dependent and does not contain physical information!

→ ∃ Closed loops Γ in parameter space $\left(\upsilon(\vec{\Gamma}(0)) = \upsilon(\vec{\Gamma}(T)) \right)$

8. The

$$\boxed{\text{Berry holonomy} \qquad U_\Gamma = \mathcal{P} \exp\left\{ -i \oint_\Gamma \vec{A}\, d\vec{\Gamma} \right\} \in U(n)}$$

is gauge covariant:

$$U'_\Gamma = \Omega(\vec{\Gamma}(0))\, U_\Gamma\, \Omega^\dagger(\Gamma(0))$$

[Γ gauge invariant: $U'_\Gamma = U_\Gamma$]

9.

$$\boxed{\text{Berry curvature} \quad \mathcal{F}_{lm} := \frac{\partial A_l}{\partial r_m} - \frac{\partial A_m}{\partial r_l} - i\,[A_l, A_m] \quad \in u(n)}$$

is gauge covariant as well:

$$\mathcal{F}'_{ij}(\vec{r}) = \Omega(\vec{r})\, \mathcal{F}_{ij}(\vec{r})\, \Omega^{+}(\vec{r})$$

Berry phase and Chern number

* Special case $n=1$: $\quad \mathcal{V}(\vec{r}) = \text{span}\{|v(\vec{r})\rangle\}$

Berry connection: $\quad A_l(\vec{r}) = -i\,\langle v(\vec{r})|\,\partial_{r_l}\,|v(\vec{r})\rangle \quad \in u(1) \simeq \mathbb{R}$

Berry holonomy: $\quad U_\Gamma = \exp\left\{-i \oint_\Gamma \vec{A}\, d\vec{r}\right\} \equiv e^{i\gamma(\Gamma)} \quad \in U(1)$

Berry curvature: $\quad \mathcal{F}_{lm} = \frac{\partial A_l}{\partial r_m} - \frac{\partial A_m}{\partial r_l} \quad \in u(1) \simeq \mathbb{R}$

Gauge transformations: $\Omega(\vec{r}) = e^{i\xi(\vec{r})} \longrightarrow$

$$\vec{A}' = \vec{A} + \nabla_{\vec{r}}\, \xi$$

$$U'_{\Gamma} = U_{\Gamma} \qquad \text{(gauge invariant)}$$

$$\mathcal{F}'_{lm} = \mathcal{F}_{lm} \qquad \text{(gauge invariant)}$$

Definition: Berry phase

For $n=1$, the exponent of the Berry holonomy is called Berry phase:

$$\gamma(\Gamma) = -\oint_{\Gamma} \vec{A}\, d\vec{r} = i \oint_{\Gamma} \langle v(\vec{r}) | \partial_{r_l} | v(\vec{r}) \rangle\, dr_l \qquad \in \mathbb{R}$$

Examples:

* Spin-$\frac{1}{2}$ in a variable magnetic field
* Focault pendulum
* Aharonov-Bohm effect

Gauge transformation:

$$\gamma'(\Gamma) = -\oint_\Gamma \vec{A}'\, d\vec{r} = -\oint_\Gamma (\vec{A} + \nabla_r \xi)\, d\vec{r} = \gamma(\Gamma) - \left[\xi(\Gamma(T)) - \xi(\Gamma(0))\right]$$

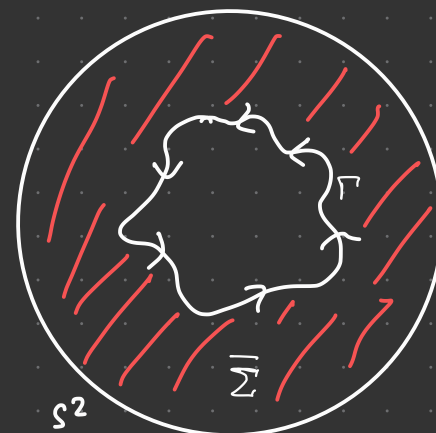
Continuity of the gauge transformation: $\Omega(\vec{r}(0)) = \Omega(\vec{r}(T))$

$\rightarrow$ (1) $\quad \xi(\vec{r}(T)) - \xi(\vec{r}(0)) = 2\pi m \quad$ for $\quad m \in \mathbb{Z}$

$\rightarrow$ $\gamma(\Gamma)$ is gauge invariant up to multiples of 2π

Computation of the Berry phase for $k=2$ on a compact manifold $\mathcal{M}$ (sphere, torus):

* Closed path Γ on sphere $\mathcal{M} = S^2$ and submanifolds with $\Sigma \cup \bar{\Sigma} = \mathcal{M}$ and $\partial\Sigma = \Gamma = \partial\bar{\Sigma}$:



Important:

In general it is <u>not</u> possible to choose a gauge that is continuous (= non-singular) everywhere on $\mathcal{M}$!

$\exists$ Continuous gauge $\vec{A}_1$ on Σ $\rightarrow$

(2) $$\oint_\Gamma \vec{A}_1 \, d\vec{\Gamma} \overset{\text{Stokes}}{=} \int_\Sigma \mathcal{F}_{lm} \, d\sigma^{lm}$$

$\exists$ Continuous gauge $\vec{A}_2$ on $\bar{\Sigma}$ $\rightarrow$

(3) $$\oint_\Gamma \vec{A}_2 \, d\vec{\Gamma} \overset{\text{Stokes}}{=} -\int_{\bar{\Sigma}} \mathcal{F}_{lm} \, d\sigma^{lm}$$

Combine (1), (2), (3):

$$\int_M \mathcal{F}_{lm} \, d\sigma^{lm} \overset{(2)+(3)}{=} \oint_\Gamma \vec{A}_1 d\vec{\Gamma} - \oint_\Gamma \vec{A}_2 d\vec{\Gamma} \overset{(1)}{=} 2\pi m$$

$m \in \mathbb{Z}$

$$\mathcal{A} = A_x dx + A_y dy + \dots$$

$$\mathcal{F} = d\mathcal{A} \qquad -dx \wedge dy$$

$$\mathcal{F} = \frac{\partial A_x}{\partial y} \overbrace{dy \wedge dx} + \frac{\partial A_y}{\partial x} dx \wedge dy + \dots$$

$$= \underbrace{\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right)}_{\mathcal{F}_{xy}} \underbrace{dx \wedge dy}_{d\sigma^{xy}}$$

$$\oint_\Gamma \mathcal{A} = \int_\Sigma d\mathcal{A} = \int_\Sigma \mathcal{F}$$

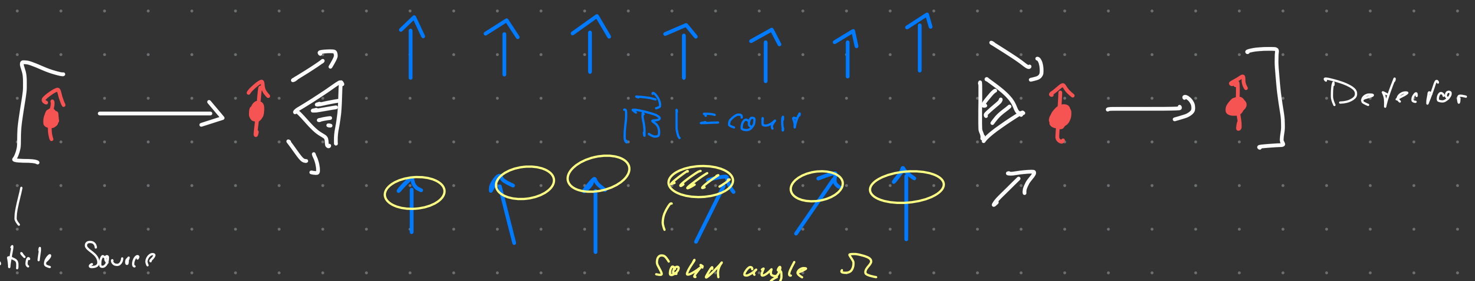
Definition: Chern number

For a compact, closed two-dimensional parameter space $\mathcal{M}$ with Berry curvature $\mathcal{F}$, the Chern number is an integer and defined as

$$C = \frac{1}{2\pi} \int_{\mathcal{M}} \mathcal{F}_{lm} \, d\sigma^{lm} \in \mathbb{Z}$$

Aside: Observation of the Berry phase

§ Spin-polarized particles:



P-Set 1 $\Rightarrow$ $e^{i\gamma(\Gamma)} = e^{i\Omega/2}$ $(0 \leq \Omega \leq 4\pi)$ $\rightarrow$ Interference $I = |1 + e^{i\gamma(\Gamma)}|^2$

Aside: Geometric interpretation of the Berry curvature