1.2. Topological bands without magnetic fields: The quantum anomalous Hall effect

1.2.1. Preliminaries

We seek for models with the following properties:

* Lattice models of non-interacting fermions
* Band insulators
* Non-zero Chern number
* No magnetic field (!)

Bands with non-zero Chern number

$$\boxed{\text{Are there non-magnetic models with Chern band?}}$$

Definition: Chern Insulator

Chern Insulator* (CI^*) = $\begin{cases} \text{Lattice model} \\ \text{Band insulator} \\ \text{Chern bands} \end{cases}$

Prototype: Hofstadter model

Chern Insulator (CI) = $\begin{cases} \text{Lattice model} \\ \text{Band insulator} \\ \text{Chern bands} \\ \text{Non magnetic field} \end{cases}$

Prototype: Haldane model

$\boxed{\text{Are there Chern insulators ?}}$

Lattice models with two bands

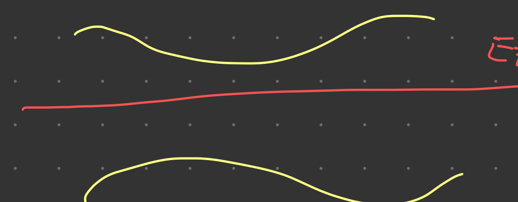
1. ✗ Most general two-band Hamiltonian on a lattice:

$$H = \bigoplus_{\vec{k} \in T^2} \tilde{H}(\vec{k}) \quad \text{with} \quad \tilde{H}(\vec{k}) = \varepsilon(\vec{k}) \mathbb{1} + \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

$\vec{\sigma} = \begin{pmatrix} \sigma^x \\ \sigma^y \\ \sigma^z \end{pmatrix}$ Pauli matrices

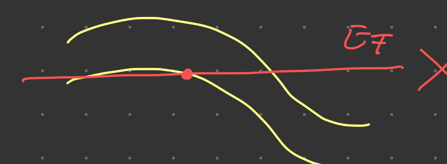
* T^2: Brillouin zone (= Torus)
* $\vec{d}(\vec{k}) : T^2 \longrightarrow \mathbb{R}^3$: real, vector-valued function on BZ

2. Spectrum: $E_\pm(\vec{k}) = \varepsilon(\vec{k}) \pm |\vec{d}(\vec{k})|$



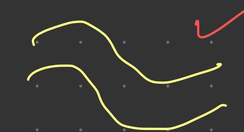
→ Band insulator iff

$$\min_{\vec{k} \in T^2} E_+(\vec{k}) > \max_{\vec{k} \in T^2} E_-(\vec{k})$$



→ Weaker condition:

$$\forall \vec{k} \in T^2 : \quad E_+(\vec{k}) - E_-(\vec{k}) = 2|\vec{d}(\vec{k})| > 0$$



→ Normalization possible:

$$\hat{d}(\vec{k}) := \frac{\vec{d}(\vec{k})}{|\vec{d}(\vec{k})|} \quad \text{such that} \quad T^2 \to \underbrace{S^2}_{\text{unit sphere in } \mathbb{R}^3}$$

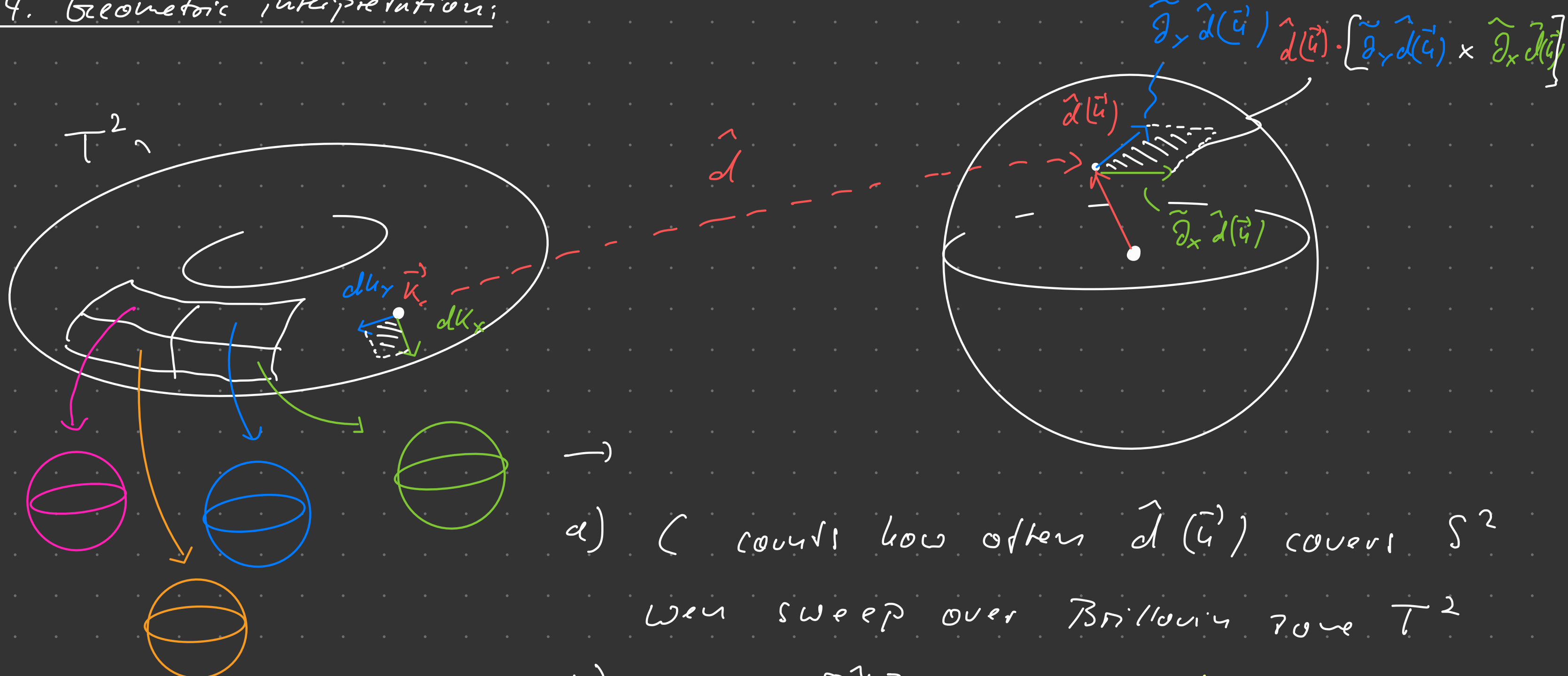
3. Chern number of the lower band:

$$\tilde{\partial}_y = \frac{\partial}{\partial k_y}$$

$$\boxed{C \doteq -\frac{1}{4\pi} \underbrace{\int_{T^2} \underbrace{\hat{d}(\vec{k}) \cdot \left[\tilde{\partial}_x \hat{d}\,\vec{k} \times \tilde{\partial}_y \hat{d}\,\vec{k} \right]}_{\text{(Oriented) Jacobian for surface integral}} d^2k}_{4\pi \cdot \mathbb{Z}}} \in \mathbb{Z}$$

Proof: Problemset 4

4. Geometric interpretation:



$\longrightarrow$

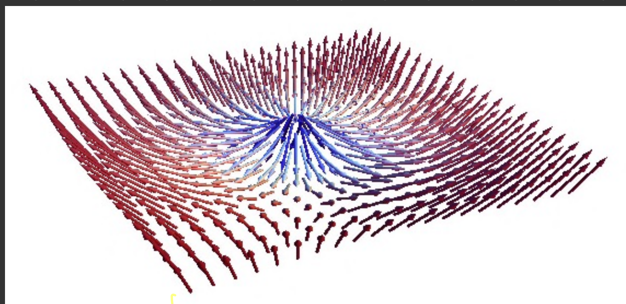
a) C counts how often $\hat{d}(\vec{u})$ covers S^2 when sweep over Brillouin zone T^2

b) $C = C[\hat{d}] \in \mathbb{Z}$ is a topological invariant

c) Hamiltonian H_a can be continuously deformed into H_b without closing the gap iff $\hat{d}_a$ can be continuously deformed into $\hat{d}_b$

c) C labels different topological phases

5. Skyrmion interpretation:



$\hat{d}(\vec{k}^*)$ = "Skyrmion"

$\Rightarrow$ see Script

Summary:

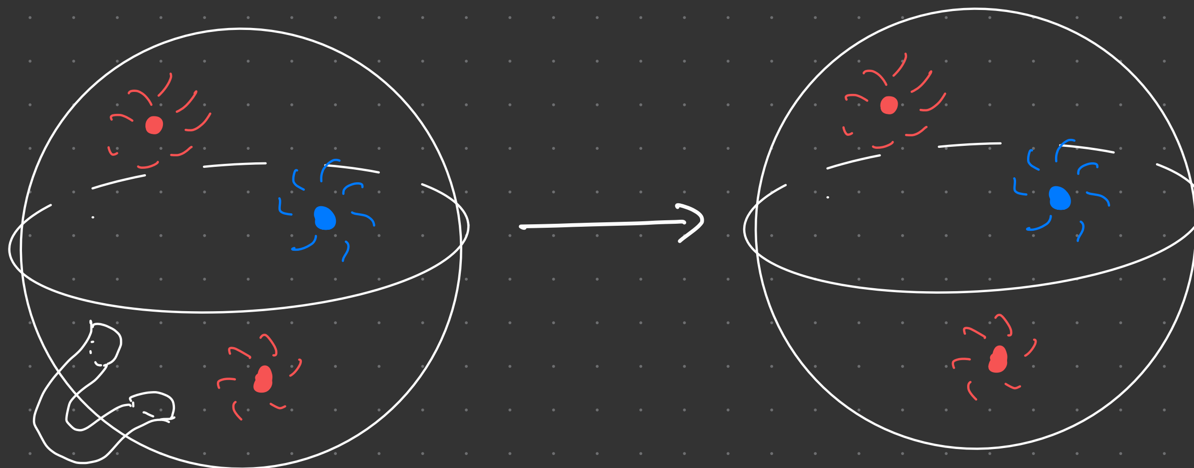
* Skyrmions are "twist" $\hat{d}$ and "live" on the BZ

* Positive (negative) Berry curvature indicates a finite (Anti-) Skyrmion density

* The Chern number is the number of Skyrmions minus the number of Antiskyrmions

Note:

T^2 - Torus $\qquad\qquad$ S^2 - Sphere



$\hat{d}: T^2 \longrightarrow S^2$ $\qquad\qquad$ $\hat{d}: S^2 \longrightarrow S^2$

$\pi_2(S^2) = \mathbb{Z}$

2nd Homotopy group

$\Rightarrow$ Torus (BZ) not important for definition of Chern number/ Skyrmion picture

Time-reversal Symmetry

Setting:

* Single-particle (SP) Hilbert space $\mathcal{H} = \text{span}\{|\psi_{i\alpha}\rangle\}_{i\alpha}$

  i: spatial (site) index, $i = 1, \dots, N$; α: internal DOF, $\alpha = 1, \dots, M$

$$H = \sum_{i\alpha, j\beta} H_{i\alpha, j\beta} |\psi_{i\alpha}\rangle\langle\psi_{j\beta}|$$

$NM \times NM$ matrix

* Many-body (MB) Hilbert space $\hat{\mathcal{H}} = \bigoplus_n \Lambda^n(\mathcal{H})$ (= fermionic Fock space)

  with MB Hamiltonian:

$$\hat{H} = \sum_{i\alpha, j\beta} c^\dagger_{i\alpha} H_{i\alpha, j\beta} c_{j\beta}$$

$c^\dagger_{i\alpha}$: fermionic creation operator for state $|\psi_{i\alpha}\rangle$

* Translation symmetry $\rightarrow$ $M \times M$ matrix: Bloch Hamiltonian

$$\hat{H} = \sum_{\vec{k}, \alpha, \beta} c^\dagger_{\vec{k}\alpha} \overbrace{H_{\alpha\beta}(\vec{k})} c_{\vec{k}\beta} \quad \text{with} \quad c_{\vec{k}\alpha} := \frac{1}{\sqrt{N}} \sum_i e^{i\vec{x}_i \cdot \vec{k}} c_{i\alpha}$$

$\vec{x}_i$: position of site i

SP Hamiltonian: $\quad H = \bigoplus_{\vec{k}} H(\vec{k})$

Spectrum

$$H(\vec{k}) = \sum_{n=1}^{M} \varepsilon_n(\vec{k}) \, |u_{n\vec{k}}\rangle\langle u_{n\vec{k}}|$$

Band index $\qquad$ Bloch wavefunction

$\to$ SP Hilbert space $\quad \mathcal{H} = \bigoplus_{\vec{k}} \mathcal{H}_{\vec{k}}$ $\quad$ with mode space

$$\mathcal{H}_{\vec{k}} = \mathrm{span}\{|u_{n\vec{k}}\rangle\}_n$$

$$\boxed{\mathcal{H} \cong T^2 \times \underbrace{\mathbb{C}^M}_{\mathcal{H}_{\vec{k}}}}$$

1. TRS: $t \mapsto -t$ is a $\mathbb{Z}_2$-symmetry and should act as

$$T x T^{-1} \overset{!}{=} x \quad \text{but} \quad T p T^{-1} \overset{!}{=} -p$$

$$\Rightarrow \quad T i\hbar T^{-1} = T [x,p] T^{-1} = -[x,p] = -i\hbar$$

$\Rightarrow$ T is antiunitary:

complex conjugation

$$T_u = U K$$

representation of TRS on the SP Hilbert space

unitary operator

$\rightarrow$ SP Hamiltonian H is time-reversal symmetric iff $[H, T_u] = 0$

2. <u>Consequences of antiunitarity:</u>

$$T_u^2 = UU^* = U(U^T)^{-1} \quad (1)$$

* UU^* is unitary
* $[H, UU^*] = 0$

$\rightarrow$ UU^* is a symmetry of H

→ Assume H does not have any additional unitary symmetries

= Hamiltonian irreducible → $UU^* = \lambda \mathbb{1}$

(1) → $U = \lambda U^T \Leftrightarrow U^T = U\lambda$

$\Rightarrow U = \lambda^2 U$

$\Rightarrow \lambda = \pm 1$

→ $$\boxed{T_u^2 = \pm \mathbb{1}}$$

3. Examples:

* Spinless particles: $$\boxed{T_0 := K \Rightarrow T_0^2 = + \mathbb{1}}$$

$T_u \sigma^i T_u^{-1} = -\sigma^i$ / $i = x, y, z$

* Spin-$\frac{1}{2}$ particles: $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$: angular momentum operator: $T_u \vec{S} T_u^{-1} \overset{!}{=} -\vec{S}$

→ Solution: $$\boxed{T_{\frac{1}{2}} := \sigma^y K \Rightarrow T_{\frac{1}{2}}^2 = -\mathbb{1}}$$

Γ Often: $T_{\frac{1}{2}} = -i\sigma^y K$ from Spin Rotation $\rightarrow$

4. Consequence of $T_u^2 = -\mathbb{1}$:

Kramer's theorem:

Every eigenenergy of a time-reversal symmetric Hamiltonian H with $T_u^2 = -1$ is at least two-fold degenerate.

Proof: $\Rightarrow$ Problemset 4

5. Action on Fock space:

a) $\exists$ Representation $\mathcal{T}_u$ of TRS on ferm. Fock space $\tilde{\mathcal{H}}$:

$$\boxed{\mathcal{T}_u\, i\, \mathcal{T}_u^{-1} := -i \quad \text{and} \quad \mathcal{T}_u\, c_{i\alpha}^{+}\, \mathcal{T}_u^{-1} := \sum_\beta \left(U_{\alpha\beta}^{+}\right)^{*} c_{i\beta}^{+}}$$

b)

$$T_u \hat{H} T_u^{-1} = \sum_{i\alpha', j\beta'} c^\dagger_{i\alpha'} \underbrace{\sum_{\alpha\beta} \left[U_{\alpha'\alpha} H^*_{i\alpha, j\beta} U^\dagger_{\beta\beta'} \right]}_{\overset{!}{=}} c_{j\beta'}$$

$$\overset{!}{=} \hat{H} = \sum_{i\alpha', j\beta'} c^\dagger_{i\alpha'} \underbrace{H_{i\alpha', j\beta'}}_{} c_{j\beta'}$$

$$\boxed{\begin{array}{l} \rightarrow [\hat{H}, T_u] = 0 \iff T_u H T_u^{-1} = H \\ \text{with } \underset{NM \times NM \text{ matrix}}{T_u = \bar{U} K} \text{ where } \bar{U} := \oplus_i U_i \text{ with } U_i = \underset{M \times M \text{ matrix}}{U} \end{array}}$$

c) Translation invariance:

$$T_u c_{\vec{u}\alpha} T_u^{-1} = \frac{1}{\sqrt{N}} \sum_i e^{-i x_i^{-1} \vec{k}} \sum_\beta U^\dagger_{\alpha\beta} c_{i\beta} = \sum_\beta U^\dagger_{\alpha\beta} c_{-\vec{u}\beta}$$

$\rightarrow$ T_u inverts momenta & mixes internal DOFS

$$T_u \hat H T_u^{-1} = \sum_{\vec u, \alpha', \beta'}{}' \; c^\dagger_{-\vec u \alpha'} \underbrace{\sum_{\alpha\beta}\left[U_{\alpha'\alpha} H^*_{\alpha\beta}(\vec u)\, U^\dagger_{\beta\beta'}\right]} c_{-\vec u \beta'}$$

$$\overset{!}{=} \hat H = \sum_{u, \alpha', \beta'}{}' \; c^\dagger_{-\vec u \alpha'} \overbrace{H_{\alpha'\beta'}(-\vec u)}\, c_{-\vec u, \beta'}$$

$\rightarrow$

$$\boxed{[\hat H, T_u] = 0 \iff \tilde T_u H(\vec u)\, \tilde T_u^{-1} = H(-\vec u) \quad \text{with } \tilde T_u = UK}$$

<u>In summary:</u>

$$\boxed{\begin{aligned}
[\hat H, T_u] = 0 &\iff T_u H T_u^{-1} = H \\
&\iff U H^* U^\dagger = H \\
&\iff \tilde T_u H(\vec u)\, \tilde T_u^{-1} = H(-\vec u) \\
&\iff U H^*(\vec u)\, U^\dagger = H(-\vec u)
\end{aligned}}$$

Furthermore:

$$\boxed{\begin{array}{ccccc} T_u^2 = +1 & \Leftrightarrow & \tilde{T}_u^2 = +\mathbb{1} & \Leftrightarrow & \mathcal{T}_u^2 = +\mathbb{1} \\ T_u^2 = -1 & \Leftrightarrow & \tilde{T}_u^2 = -\mathbb{1} & \Leftrightarrow & \mathcal{T}_u^2 = \underbrace{(-1)^{\hat{N}}}_{P} \end{array}}$$

P: fermion parity operator

$$\hat{N} = \sum_{i\alpha} c_{i\alpha}^{+} c_{i\alpha}$$

6. Consequence of TRS for the spectrum:

$$\underbrace{\overset{\tilde{T}_u}{\vee} H(\vec{u}) \overset{\tilde{T}_u^{-1}\tilde{T}_u}{\vee}}_{H(-\vec{u})} |u_{n\vec{u}}\rangle = \overset{\tilde{T}_u}{\vee} \varepsilon_n(\vec{u}) |u_{n\vec{u}}\rangle$$

$$H(-\vec{u}) \underbrace{U|u_{n\vec{u}}^*\rangle}_{|\psi\rangle} = \varepsilon_n(\vec{u}) \underbrace{U|u_{n\vec{u}}^*\rangle}_{|\psi\rangle}$$

$\Rightarrow$ Inversion-symmetric band structure

7. Consequence of TRS for the Chern number:

* $\nexists$ Two bands from pseudo-spin-$\frac{1}{2}$: $\tilde{T}_0 = K$

$$H^*(\vec{k}) = H(-\vec{k}) \iff \begin{aligned} \hat{d}_{x,z}(\vec{k}) &= \hat{d}_{x,z}(-\vec{k}) \\ \hat{d}_y(\vec{k}) &= -\hat{d}_y(-\vec{k}) \end{aligned}$$

* $\nexists$ Two bands from real spin-$\frac{1}{2}$: $\tilde{T}_{\frac{1}{2}} = \sigma^y K$

$$\sigma^y H^*(\vec{k})\sigma^y = H(-\vec{k}) \iff \hat{d}(\vec{k}) = -\hat{d}(-\vec{k})$$

Both cases $\to$

$$C = -\frac{1}{4\pi}\int_{-\pi}^{\pi} dk_x \int_{-\pi}^{\pi} dk_y \, \varepsilon_{ijk} \, \hat{d}_i(\vec{k}) \, \tilde{\partial}_x d_j(\vec{k}) \, \tilde{\partial}_y d_k(\vec{k}) = 0$$

→ Important:

$\boxed{\text{Systems with Chern bands must } \underline{\text{break}} \text{ time-reversal symmetry.}}$

Note: $\sigma \mapsto -\sigma$ $\qquad \vec{J} = \sigma \vec{E}$

$\qquad\qquad -\vec{J} \qquad +\vec{E}$

$\sigma = \sigma_a + \sigma_s = 0$