# 1.4. Topological Insulators in 1D: The Su-Schrieffer-Heeger Chain

## 1.4.1. Preliminaries: Sublattice Symmetry

1. * <u>Unitary symmetries $\mathcal{U}$:</u>

$$\mathcal{U}\, i\, \mathcal{U}^{-1} = i \quad \text{and} \quad \mathcal{U} c_i \mathcal{U}^{-1} = \sum_j U^{\dagger}_{ij} c_j$$

$$[\hat{H}, \mathcal{U}] = 0 \quad \Leftrightarrow \quad U^{\dagger} H U = H \quad \Leftrightarrow \quad [H, U] = 0$$

* <u>Time-reversal symmetry</u> $\mathcal{T}_U$:

$$\mathcal{T}_U\, i\, \mathcal{T}_U^{-1} = -i \quad \text{and} \quad \mathcal{T}_U c_i \mathcal{T}_U^{-1} = \sum_j U^{\dagger}_{ij} c_j$$

$$[\hat{H}, \mathcal{T}_U] = 0 \quad \Leftrightarrow \quad U^{\dagger} H^{*} U = H \quad \Leftrightarrow \quad [H, UK] = 0$$

2. * $\underline{\mathcal{U} \quad c_i \leftrightarrow c_i^\dagger \quad ?}$

$$\mathcal{C}_U \, i \, \mathcal{C}_U^{-1} = i \quad \text{and} \quad \mathcal{C}_U c_i \mathcal{C}_U^{-1} = \sum_j U_{ij}^{*\dagger} c_j^\dagger$$

$$[\hat{H}, \mathcal{C}_U] = 0 \iff U^\dagger H^* U = -H \iff \{H, U\mathcal{K}\} = 0$$

→ Particle-hole symmetry $\mathcal{C}_U$

⊜ Next lecture

* $\underline{\mathcal{T}_U \quad c_i \leftrightarrow c_i^\dagger \quad ?}$

$$S_U \, i \, S_U^{-1} = -i \quad \text{and} \quad S c_i S_U^{-1} = \sum_j U_{ij}^{*\dagger} c_j^\dagger$$

$$[\hat{H}, S_U] = 0 \iff U^\dagger H U = -H \iff \{H, U\} = 0$$

→ Chiral- or Sublattice symmetry $S_U$

⊕ today

Note:

$$\{H, U\} = 0 \implies [H, U^2] = 0 \implies U^2 = e^{i\varphi}\mathbb{1}$$

$\rightarrow$ Redefine $\tilde{U} = e^{-i\frac{\varphi}{2}} U \rightarrow \tilde{U}^2 = +\mathbb{1} \rightarrow$ o.l.o.g. $U^2 = +\mathbb{1}$

---

## 3. Why "sublattice symmetry"?

a) $\exists$ SP Hamiltonian $H$: $U^\dagger H U = -H$

$\rightarrow$ Spectrum $\sigma(H) = \sigma(-H)$

$\rightarrow$ Spectrum symmetric about $E = 0$

b) $\exists$ $H$ is $2L \times 2L$ matrix $\rightarrow$

$\exists$ Unitary $M$: $\quad M H M^\dagger = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$ diagonal matrix

$$\rightarrow (QM) H (QM)^\dagger = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \quad \text{with} \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}$$

c)

$$\hat{H} = \sum_{ij} c_i^\dagger H_{ij} c_j \underset{!}{=} \sum_{ij} \tilde{c}_i^\dagger \tilde{H}_{ij} \tilde{c}_j$$

$[\hat{H}, S_y] = 0$

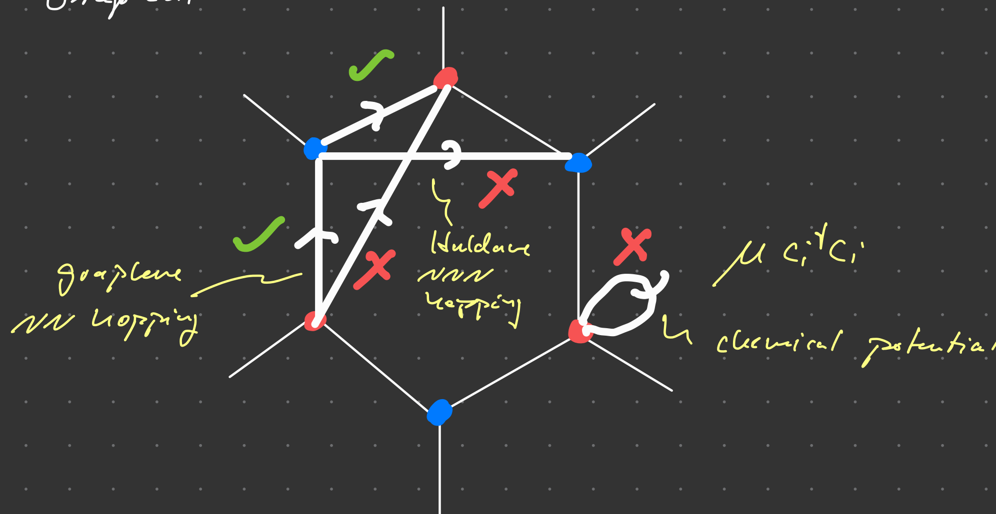
with block off-diagonal Hamiltonian

$$\tilde{H} = \begin{array}{cc} \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} & \begin{array}{l} \} A \\ \} B \end{array} \\ \begin{array}{cc} \underbrace{\phantom{h}}_{A} & \underbrace{\phantom{h}}_{B} \end{array} & \end{array}$$

Sublattices $A$ + $B$

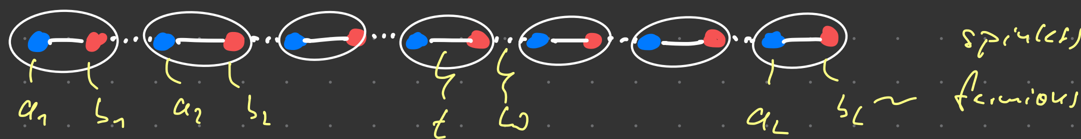
d) $\tilde{H}$ couples only modes between sublattices $A$ + $B$:

e) Example: Graphene



## 1.4.2. The Su-Schrieffer-Heeger chain

1. ∃ 1D lattice with $L$ unit cells with $2L$ fermionic sites:




→ <u>SSH Hamiltonian:</u>

$$\boxed{\hat{H}_{SSH} = \underbrace{t}_{\in \mathbb{R}} \sum_{i=1}^{L} \left(a_i^\dagger b_i + b_i^\dagger a_i\right) + w \sum_{i=1}^{L'} \left(b_i^\dagger a_{i+1} + a_{i+1}^\dagger b_i\right)}$$

$L' = \begin{cases} L-1 & \text{OBC} \\ L & \text{PBC} \end{cases}$

2. <u>Symmetries:</u>

* Particle-number symmetry (PNS)
* Translation symmetry (TS)

* Time-reversal symmetry (TRS):

$\mathcal{T} i \mathcal{T}^{-1} := -i \quad , \quad \mathcal{T} a_i \mathcal{T}^{-1} = a_i \quad \text{and} \quad \mathcal{T} b_i \mathcal{T}^{-1} := b_i$

$\rightarrow [H_{SSH}, \mathcal{T}] \overset{!}{=} 0$

* Particle-hole symmetry (PHS):

$\mathcal{C} i \mathcal{C}^{-1} = i \quad \text{and} \quad \mathcal{C} a_i \mathcal{C}^{-1} = a_i^\dagger \quad \text{and} \quad \mathcal{C} b_i \mathcal{C}^{-1} = -b_i^\dagger$

$\rightarrow [H_{SSH}, \mathcal{C}] \overset{!}{=} 0$

* Sublattice symmetry (SLS):

$S i S^{-1} = -i \quad \text{and} \quad S a_i S^{-1} = a_i^\dagger \quad \text{and} \quad S b_i S^{-1} = -b_i^\dagger$

$\rightarrow [H_{SSH}, S] \overset{!}{=} 0$

$$S = \prod_i (a_i^\dagger - a_i)(b_i^\dagger + b_i) \circ K$$

3. ☇ "Generic" SSH chain:

$$\hat{H}'_{SSH} = \sum_i \Big( \underbrace{t_i}_{\in \mathbb{C}}\, a_i^\dagger b_i + t_i^*\, b_i^\dagger a_i \Big) + \sum_{i=1}^{L'} \Big( \underbrace{\omega_i}_{\in \mathbb{C}}\, b_i^\dagger a_{i+1} + \omega_i^*\, a_{i+1}^\dagger b_i \Big)$$

$\xrightarrow{\circ}$ Symmetries left: PN + SLS

$\rightarrow$ $\boxed{\text{The "natural" symmetry of the SSH chain is SLS}}$

## 1.4.3. Diagonalization

1. ☇ $\hat{H}_{SSH}$ with PBC and Fourier transform $\xrightarrow{\circ}$

$$\hat{H}_{SSH} = \sum_{\underbrace{k \in BZ}_{\text{Circle } S^1}} \left( \tilde{a}_k^\dagger \;\; \tilde{b}_k^\dagger \right) \underbrace{\begin{pmatrix} 0 & t + \omega e^{-ik} \\ t + \omega e^{ik} & 0 \end{pmatrix}}_{h(k)} \begin{pmatrix} \tilde{a}_k \\ \tilde{b}_k \end{pmatrix}$$

## 2. Bloch Hamiltonian:

$$H(k) = (t + \omega \cos k)\,\sigma^x + \omega \sin(k)\,\sigma^y = \vec{d}(k) \cdot \vec{\sigma}$$

Bloch vector :

$$\boxed{\vec{d}(k) = \begin{pmatrix} t + \omega \cos k \\ \omega \sin k \\ 0 \end{pmatrix}}$$

## 3. Band Structure:

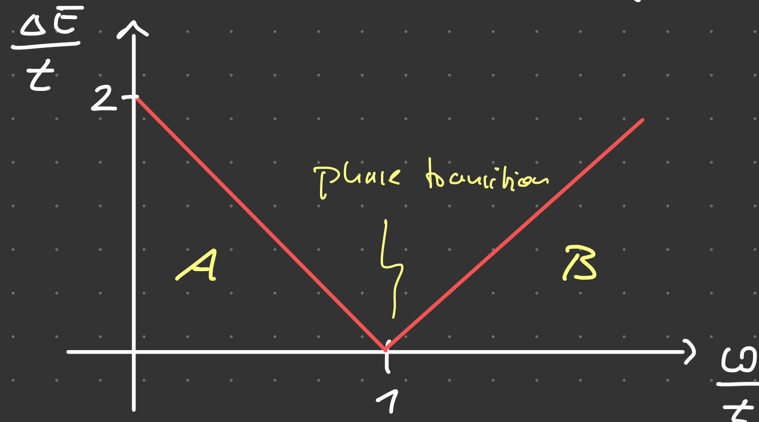
$$E_\pm(k) = \pm \left| \vec{d}(k) \right| = \pm \sqrt{t^2 + \omega^2 + 2 t \omega \cos(k)}$$

## 4. Phase diagram:

Band gap :

$$\Delta E \overset{\omega}{=} \big|\, |t| - |\omega| \,\big|$$

Let $t, \omega > 0$ → Gapless point $\omega = t$, gapped insulator for $\omega \neq t$




→ Unique ground state in A and B (↔ no symmetry breaking)

$$\boxed{\text{Can we use SLS to define a topological invariant to distinguish the two phases?}}$$

## 1.1.4. A new topological invariant

1. <u>Observation:</u> PNS does not contain $H(k)$

2. <u>∄ SLS:</u>

$$[\vec{H}_{SSH}, S] = 0 \iff U^{\dagger} H U = -H \iff \sigma^{z} H(k) \sigma^{z} = -H(k)$$

3. <u>Constrained Bloch vector:</u>

$$\boxed{d_z(k) \stackrel{!}{=} 0 \qquad \forall k \in BZ}$$

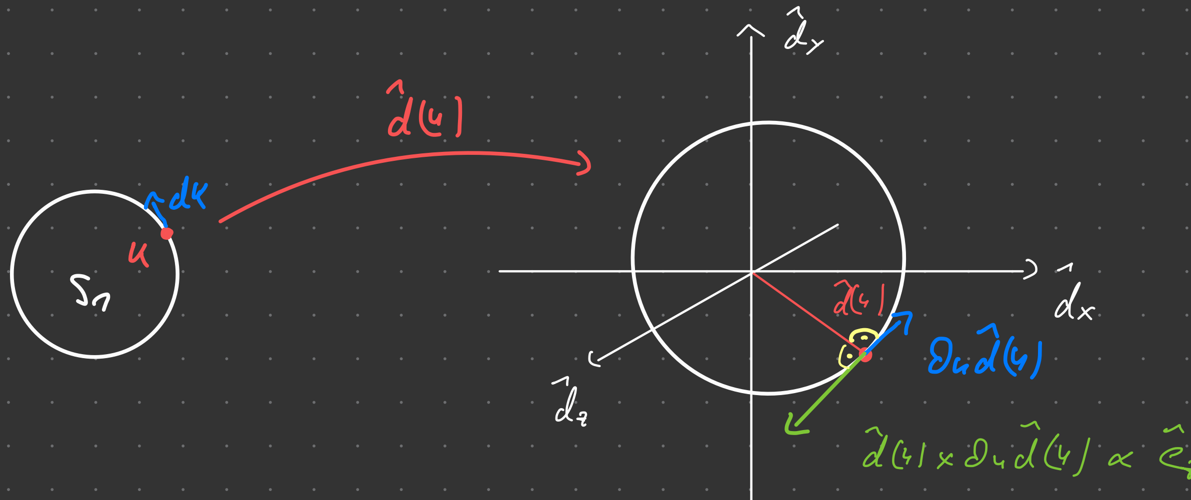
$\rightarrow$ $\vec{d}(k)$ cannot leave to x-y-plane

4. ∄ Gapped phases $\rightarrow$ Normalization possible:

$$\hat{d}(k) = \frac{\vec{d}(k)}{|\vec{d}(k)|}$$

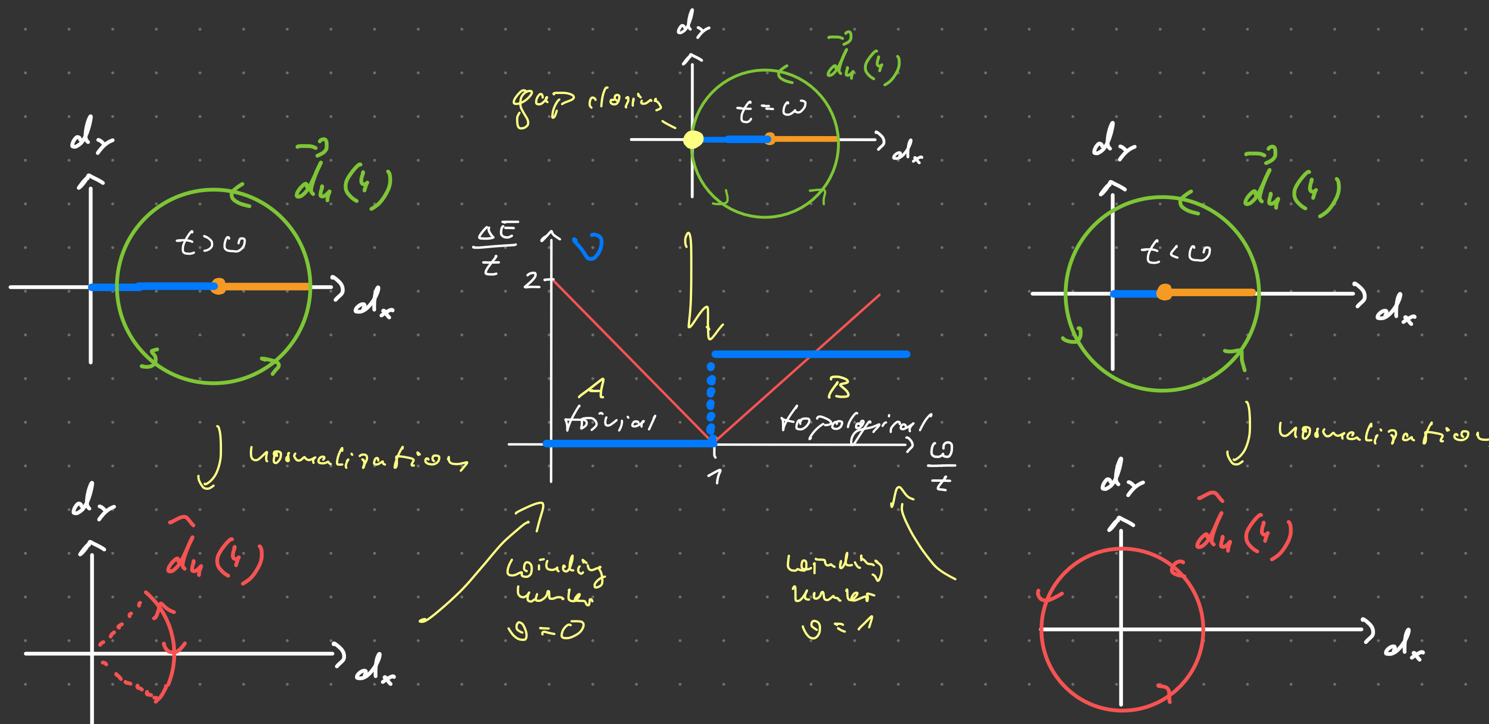
# 5. Winding number:

$$\boxed{\nu[\hat{d}] = \frac{1}{2\pi} \int_{S^1} \hat{e}_z \cdot \left[ \hat{d}(k) \times \partial_k \hat{d}(k) \right] dk \quad \in \mathbb{Z}}$$



# 6. Two phases:

$$\boxed{\nu = \begin{cases} 0 & \text{for} \quad t > \omega \quad \text{(Phase A)} \\ 1 & \text{for} \quad t < \omega \quad \text{(Phase B)} \end{cases}}$$



## 7. Mathematical side note

Homotopy group $\pi_p(X)$ — topological space

$p = 1, 2, 3, \ldots$

$\alpha: S^p \to X$

| | 1D | 2D | 1D + SLS |
|---|---|---|---|
| $p$ | $1 \; (BZ = S^1)$ | $2 \; (BZ = T^2 \to S^2)$ | $1 \; (BZ = S^1)$ |
| $X$ | $X = S^2$ | $X = S^2$ | $X = S^1$ |
| $\pi_p(X)$ | $\pi_1(S^2) = 0$ | $\pi_2(S^2) = \mathbb{Z}$ | $\pi_1(S^1) = \mathbb{Z}$ |
| | → no Chern number in 1D | (Chern number) | (Winding number $\nu$) |

8. Note: Zak phase! $\forall$ lower band $|u(k)\rangle$

$$\varphi_{Zak} = \int_{S^1} \underbrace{i\langle u(k)|\partial_k u(k)\rangle}_{\text{Berry connection}} dk$$

Berry phase collected over 1D BZ

$$\Delta\varphi_{Zak} = \left(\varphi_{Zak}^{top} - \varphi_{Zak}^{triv}\right) \bmod 2\pi = \pi$$

$\Rightarrow$ Problem 6

---

## 1.4.5. Breaking the Symmetry

1. Connect phases A and B without closing the gap but with breaking SLS?
2. Add staggered chemical potential:

$$\hat{H}'_{SSH} = \hat{H}_{SSH} + \underbrace{\mu \sum_{j=1}^{L} (a_j^\dagger a_j - b_j^\dagger b_j)}_{\hat{H}_\mu} \rightarrow [\hat{H}_\mu, S] \neq 0$$

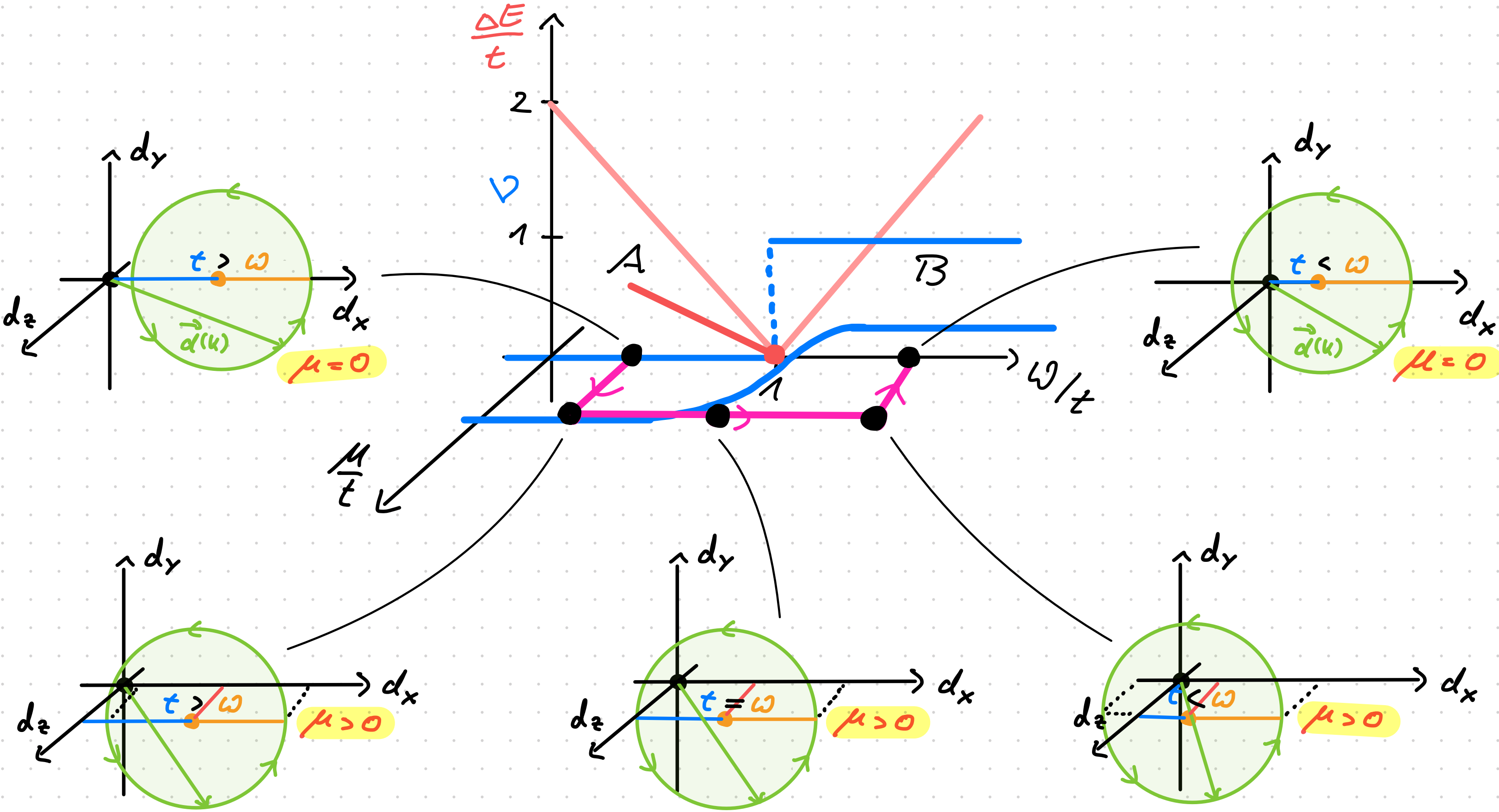
3. Bloch vector:

$$\vec{d}(k) = \begin{pmatrix} t + \omega \cos k \\ \omega \cos k \\ \mu \end{pmatrix}$$

→ Spectrum: $\pm E_\pm(k) = |\vec{d}(\vec{k})| = \sqrt{\mu^2 + t^2 + \omega^2 + t\omega \cos k} \geq |\mu|$

→ Gapped for all $\omega, t$ (in particular $\omega = t$) if $\mu \neq 0$

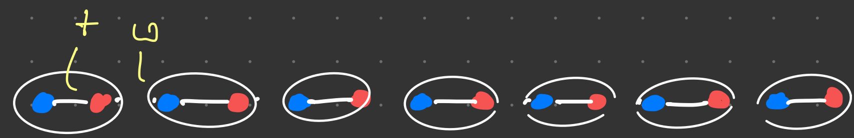
4. Connect phases without closing the gap:

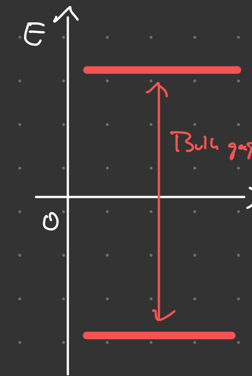


## 1.4.6. Edge modes

1. Open chain of length $L$:

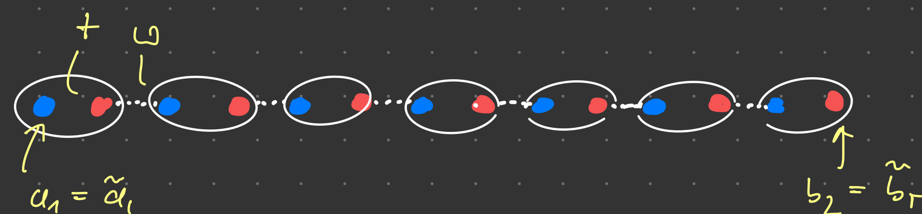
* Trivial phase: $t > 0$ and $w = 0$

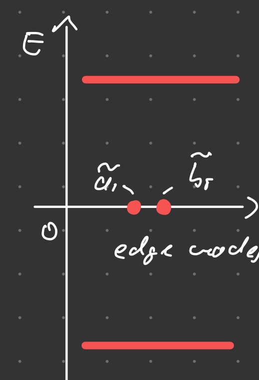



→ Spectrum:

* Topological phase: $t = 0$ and $w > 0$




→ Edge modes $\tilde{a}_1$ and $\tilde{b}_r$ commute with $\hat{H}_{SSH}$

→ 4-fold degenerate ground space

$|n_1, n_r\rangle$ $\quad$ $\tilde{a}_1^\dagger \tilde{a}_1 |n_1, n_r\rangle = n_1 |n_1, n_r\rangle$ etc.

2. Edge modes persist for $t>0$ as long as $t<\omega$:

$$\tilde a_l \approx N \sum_{i=1}^{L} \left(-\frac{t}{\omega}\right)^{i-1} a_i \quad \text{and} \quad \tilde b_r \approx N \sum_{i=1}^{L} \left(-\frac{t}{\omega}\right)^{i-1} b_{L-i+1}$$

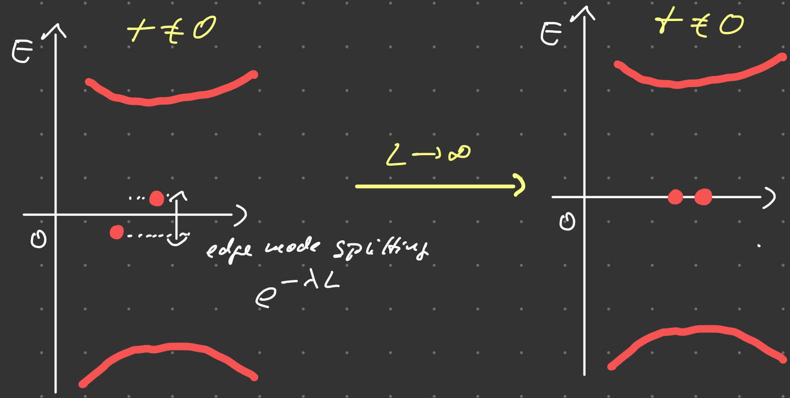
$\rightarrow$ exponentially localized on edges

Show: ($\Rightarrow$ Problemset 6)

* $\{\underset{\tilde b_r}{\tilde a_l}, \underset{\tilde b_r}{\tilde a_l}\} = 0$, $\{\underset{\tilde b_r}{\tilde a_l}, \underset{\tilde b_r}{\tilde a_l^\dagger}\} = 1$, $\{\tilde a_l^{(\dagger)}, \tilde b_r^{(\dagger)}\} = 0$

* $$[\underset{\tilde b_r}{\tilde a_l}, \hat H_{SSH}] = \underbrace{\mathcal{O}\left(\left(\frac{t}{\omega}\right)^L\right)}_{\text{vanishes exponentially } L\to\infty \text{ if } \frac{t}{\omega}<1 \text{ (top. phase)}}$$

→ Finite-size Scaling of SP spectrum:



## 3. Degeneracy of edge modes is robust against SLS-preserving disorder:

a) Clean system:

$t = 1-\omega$

b) SLS-preserving disorder:

$t \to t_i$ , $\omega \to \omega_i$

$\langle t_i \rangle = t$ $\langle \omega_i \rangle = \omega$ , Variance 20% of the mean

c) SLS-breaking disorder:

$\mu_i^a a_i^\dagger a_i + \mu_i^b b_i^\dagger b_i$

$\langle \mu_i^x \rangle = 0$, Variance $\langle \mu_i^x \rangle = 0.1$