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You can find detailed information about the lecture, tutorials and the general organization of this course on the website

`https://ilias3.uni-stuttgart.de/goto_Uni_Stuttgart_crs_2346781.html`.

Please keep yourself up to date concerning this course.

The solutions will be discussed in the online tutorial via `Webex` two weeks later.

Problem 1: The integer quantum Hall effect in the symmetric gauge

Learning objective

In this task you revisit the Hamiltonian of a charged particle in a magnetic field. In contrast to the lecture, you will work in the *symmetric gauge* to construct an alternative set of basis functions for Landau levels. The benefit of this approach is that the basis wavefunctions can be naturally expressed in complex coordinates, a feature that becomes handy especially in the context of the fractional quantum Hall effect.

In the lecture it was shown that the Hamiltonian of a charged particle in a magnetic field can be written as a harmonic oscillator,

$$H = \frac{1}{2m}(\underbrace{\boldsymbol{p} + e\boldsymbol{A}}_{\boldsymbol{\pi}})^2 = \frac{\boldsymbol{\pi}^2}{2m} = \hbar\omega_B\left(a^\dagger a + \frac{1}{2}\right) \tag{1}$$

with the cyclotron frequency $\omega_B = \frac{eB}{m}$ and the ladder operators defined by the *kinetic* momentum operators π_i,

$$a = \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y) \tag{2}$$

with $\left[a, a^\dagger\right] = 1$.

So far, these results are gauge invariant (we did not yet fix a gauge). The result in Eq. (1) already determines the spectrum of the problem but leaves open the question for eigenstates and the degeneracy of the eigenenergies (the Landau levels). The goal of this exercise is to answer these questions.

a) Define the additional "momentum operators"

$$\tilde{\boldsymbol{\pi}} := \boldsymbol{p} - e\boldsymbol{A} \tag{3}$$

and show that $[\tilde{\pi}_x, \tilde{\pi}_y] = ie\hbar B$. Is $\tilde{\boldsymbol{\pi}}$ gauge invariant?

b) Compute all commutators $[\pi_i, \tilde{\pi}_j]$ of the momentum $\tilde{\boldsymbol{\pi}}$ and the kinetic momentum $\boldsymbol{\pi}$. What does the result tell you about the possibility to construct good quantum numbers to label eigenstates?

c) Show that by fixing the *symmetric gauge*

$$\boldsymbol{A} := -\frac{1}{2}\boldsymbol{r} \times \boldsymbol{B} = -\frac{yB}{2}\boldsymbol{e}_x + \frac{xB}{2}\boldsymbol{e}_y \tag{4}$$

all commutators $[\pi_i, \tilde{\pi}_j]$ vanish.

Then define the additional ladder operators

$$b := \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x + i\tilde{\pi}_y) \quad \text{and} \quad b^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x - i\tilde{\pi}_y) \tag{5}$$

and show that $\left[b, b^\dagger\right] = 1$ and $[a, b] = 0$.

d) Write down the eigenstates of Hamiltonian Eq. (1) and label them with appropriate quantum numbers $n \in \mathbb{N}_0$ for $a^\dagger a$ and $m \in \mathbb{N}_0$ for $b^\dagger b$. Which states are energetically degenerate?

Now that you formally derived a full basis set for each Landau level $n = 0, 1, 2, \ldots$, you can construct these states in real space. To do so, it is convenient to switch to complex coordinates:

$$z := x - iy \quad \text{and} \quad \bar{z} := x + iy \tag{6}$$

(The unconventional sign makes the functions below holomorphic instead of antiholomorphic.)

With the complex (Wirtinger) derivatives

$$\partial := \frac{1}{2}(\partial_x + i\partial_y) \quad \text{and} \quad \bar{\partial} := \frac{1}{2}(\partial_x - i\partial_y) \tag{7}$$

it follows $\partial z = \bar{\partial}\bar{z} = 1$ and $\partial\bar{z} = \bar{\partial}z = 0$. A function of complex variables is then holomorphic (= satisfies the Cauchy-Riemann equations) if and only if $\bar{\partial}f = 0$, i.e., $f = f(z)$.

e) Use $p_i = -i\hbar\partial_i$ and the symmetric gauge for $\boldsymbol{A}$ to express the ladder operators a, $a^\dagger$, b, $b^\dagger$ in terms of ∂, $\bar{\partial}$, z, and $\bar{z}$.

f) Show that the wavefunctions in the lowest Landau level (LLL) $n = 0$ take the form

$$\langle z, \bar{z}|\Psi_{\text{LLL}}\rangle = \Psi_{\text{LLL}}(z, \bar{z}) = f(z)e^{-z\bar{z}/4l_B^2} \tag{8}$$

with an arbitrary holomorphic function $f(z)$; $l_B = \sqrt{\frac{\hbar}{eB}}$ denotes the magnetic length.

g) Derive now the real space representation $\langle z, \bar{z}|n = 0, m = 0\rangle$ for the LLL state with $m = 0$ and show that in general

$$\langle z, \bar{z}|n = 0, m\rangle = \Psi_{\text{LLL},m}(z, \bar{z}) \propto \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4l_B^2}\,. \tag{9}$$

How does this relate to the result in Eq. (8)? (Recall your course on complex analysis.)

h) Show that in the LLL, m labels angular momentum eigenstates for rotations in the x-y-plane.

(*Hint:* Write the generator of rotations in terms of ∂ and $\bar{\partial}$ and use Eq. (9).)

Why is a single quantum number sufficient to specify the angular momentum eigenstates?

Now that you have the real space wavefunctions at hand (at least for the LLL; but at this point it should be clear how to construct the wavefunctions for higher Landau levels), you can derive the degeneracy of the Landau levels.

i) Show that the extension of the LLL wavefunctions Eq. (9) is given by the radius $r = \sqrt{2m}\, l_B$ and use this result to compute the number of independent states in the LLL in a disc shaped region of radius R; show that there is one state per quantum of flux in the LLL. Compare this result to the one derived in the lecture in the Landau gauge.

Problem 2: Berry phase of a spin in a magnetic field

Learning objective

In the lecture we introduced on very general grounds the concepts of Berry connection, Berry curvature, Berry phase and the Chern number. The simplest model to observe these concepts in action is a single spin-$\frac{1}{2}$ in a slowly varying magnetic field. In this task, you will study this example in detail.

Consider a spin-$\frac{1}{2}$ with Hilbert space $\mathcal{H} = \mathbb{C}^2$ in a external magnetic field $\boldsymbol{B}$ with Hamiltonian

$$H = -\boldsymbol{B}\cdot\boldsymbol{\sigma} + B \quad \text{with} \quad B = |\boldsymbol{B}| \tag{10}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$ the vector of Pauli matrices. We consider $\boldsymbol{B} = \boldsymbol{\Gamma}$ as tunable parameters so that the parameter space is isomorphic to $\mathbb{R}^3$ ("the space of magnetic fields at the position of the spin") and parametrize $\boldsymbol{B}$ with spherical coordinates,

$$\boldsymbol{B} = B \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} . \tag{11}$$

a) Write the Hamiltonian Eq. (10) as a 2×2-matrix that parametrically depends on θ, ϕ, and B. Compute its spectrum and show that the eigenstates can be written as

$$|n_+(\boldsymbol{B})\rangle = \begin{pmatrix} e^{-i\phi}\cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \quad \text{and} \quad |n_-(\boldsymbol{B})\rangle = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} . \tag{12}$$

For which parameters $\boldsymbol{B}$ is the ground state $|n_+(\boldsymbol{B})\rangle$ well-defined?

b) Consider only parameters for which the ground state $|n_+(\boldsymbol{B})\rangle$ is well-defined and compute the Berry connection $\mathcal{A}_i^{[+]} = -i\langle n_+(\boldsymbol{B})|\,\partial_i\,|n_+(\boldsymbol{B})\rangle$ in spherical coordinates $i \in \{B, \theta, \phi\}$. Then show that the only non-vanishing term of the Berry curvature is

$$\mathcal{F}_{\theta\phi}^{[+]} = -\frac{\sin(\theta)}{2} . \tag{13}$$

c) Make the gauge transformation $|\tilde{n}_+(\boldsymbol{B})\rangle := e^{i\phi}\,|n_+(\boldsymbol{B})\rangle$. For which parameters is this gauge well-defined? Compute again the Berry connection and the Berry curvature and compare your results with b).

d) Show that the Berry curvature for the ground state manifold reads in *cartesian coordinates* R_i with labels $i, j \in \{x, y, z\}$

$$\mathcal{F}_{ij}^{[+]}(\boldsymbol{B}) = \varepsilon_{ijk}\, g \frac{B_k}{B^3} \quad \text{with} \quad g = -\frac{1}{2} . \tag{14}$$

This is the field strength of a *magnetic monopole* at $\boldsymbol{B} = \boldsymbol{0}$ with charge g (in parameter space, not in real space!).

e) Compute the Berry phase γ_Γ for a closed path $\Gamma : t \mapsto \boldsymbol{B}(t)$ that avoids the origin, $\boldsymbol{B}(t) \neq \boldsymbol{0}$ for all $t \in [0, T]$, and show that it can be written as

$$\exp\left(i\gamma_\Gamma\right) = \exp\left(i\frac{\Omega_\Gamma}{2}\right) \tag{15}$$

where $0 \leq \Omega_\Gamma \leq 4\pi$ is the *solid angle* that is traced out by the path.

(*Hint:* Use that $\mathcal{F}^{[+]}_{ij}\mathrm{d}\sigma^{ij} = g/B^3\,\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{\sigma}$ with the surface element $\mathrm{d}\boldsymbol{\sigma}$.)

f) Finally, compute the Chern number C for the Berry curvature in Eq. (14). What follows in this particular case for $g = -1/2$? In the lecture we showed that $C \in \mathbb{Z}$. What follows for the magnetic monopole charge g in general?

Note: The last result is mathematically equivalent to the famous *Dirac quantization condition* for magnetic monopoles.