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Problem 1: The Chern number as Skyrmion number

Learning objective

For the special case of 2-band Bloch Hamiltonians, the expression for the Chern number can be interpreted as a winding number that counts how often the sphere S^2 is wrapped when traversing the Brillouin zone. This number counts topological twists in the Bloch vector field on the Brillouin zone; these twists are known as *(Anti-)Skyrmions*. Here you derive the expression for the Chern number that is crucial for this interpretation.

Consider a translation invariant system in two dimensions with Bloch Hamiltonian

$$H(\boldsymbol{k}) = \vec{d}(\boldsymbol{k}) \cdot \vec{\sigma} \,. \tag{1}$$

Denote the ground and excited states as $|u_{\boldsymbol{k}}\rangle$ and $|v_{\boldsymbol{k}}\rangle$ with eigenenergies $-d(\boldsymbol{k}) = -|\vec{d}(\boldsymbol{k})|$ and $+d(\boldsymbol{k})$, respectively; the gap is therefore $2d(\boldsymbol{k})$.

The Chern number for Bloch bands was derived in the lecture and reads in this case

$$C = \frac{i}{2\pi} \int_{T^2} \left\{ \left\langle \tilde{\partial}_y u_{\boldsymbol{k}} \middle| \tilde{\partial}_x u_{\boldsymbol{k}} \right\rangle - \left\langle \tilde{\partial}_x u_{\boldsymbol{k}} \middle| \tilde{\partial}_y u_{\boldsymbol{k}} \right\rangle \right\} \mathrm{d}^2 k \tag{2}$$

with $\tilde{\partial}_i = \partial_{k_i}$.

Your goal is to massage this expression into a new form with a straightforward topological interpretation.

a) As a preliminary step, show that

$$\varepsilon_{ijk}\, \hat{d}_i (\tilde{\partial}_x \hat{d}_j)(\tilde{\partial}_y \hat{d}_k) = \varepsilon_{ijk}\, \frac{d_i (\tilde{\partial}_x d_j)(\tilde{\partial}_y d_k)}{d^3} \tag{3}$$

with $\hat{d} = \vec{d}/|\vec{d}|$.

b) Now show that

$$\left\langle v_{\boldsymbol{k}} \middle| \tilde{\partial}_x u_{\boldsymbol{k}} \right\rangle = \frac{\langle v_{\boldsymbol{k}}| \left[\tilde{\partial}_x H\right] |u_{\boldsymbol{k}}\rangle}{-2d(\boldsymbol{k})} \tag{4}$$

and use this and the completeness of the Bloch basis $\{|v_{\boldsymbol{k}}\rangle, |u_{\boldsymbol{k}}\rangle\}$ to derive the expression

$$C = -\frac{1}{4\pi} \int_{T^2} \mathrm{Im} \left[\langle u_{\boldsymbol{k}}| \sigma^i |v_{\boldsymbol{k}}\rangle \langle v_{\boldsymbol{k}}| \sigma^j |u_{\boldsymbol{k}}\rangle \right] \frac{\tilde{\partial}_y d_i \tilde{\partial}_x d_j}{d^2} \,\mathrm{d}^2 k \,. \tag{5}$$

(*Hint:* You may also use results from Problem 1 on Problem Set 2.)

c) Show that $\langle u_{\boldsymbol{k}}| \sigma^k |u_{\boldsymbol{k}}\rangle = -\hat{d}_k$ and with this

$$\mathrm{Im}\left[\langle u_{\boldsymbol{k}}| \sigma^i |v_{\boldsymbol{k}}\rangle \langle v_{\boldsymbol{k}}| \sigma^j |u_{\boldsymbol{k}}\rangle\right] = -\varepsilon_{ijk}\hat{d}_k \tag{6}$$

so that with a) finally

$$C = -\frac{1}{4\pi}\int_{T^2} \hat{d}\cdot\left(\tilde{\partial}_x\hat{d}\times\tilde{\partial}_y\hat{d}\right)\,\mathrm{d}^2k\,. \tag{7}$$

Identify the Berry curvature $\mathcal{F}_{xy}(\boldsymbol{k})$.

This is the expression used in the lecture to motivate the interpretation of Chern number and Berry curvature in terms of Skyrmion number and -density, respectively.

Problem 2: The Berry curvature of Dirac Hamiltonians

Learning objective

In the lecture we identified Dirac Hamiltonians as useful tools to study *changes* in Chern numbers. Here you derive a simple expression for the integral of the Berry curvature of a general Dirac Hamiltonian over its (non-compact) momentum space $\mathbb{R}^2$.

Consider the general Dirac Hamiltonian

$$H_D(\boldsymbol{k}) = \sum_{i,j=1}^{2} k_i h_{ij}\sigma^j + h_z\sigma^z \tag{8}$$

with 2×2 matrix h and "mass" h_z.

Show that the integral of the Berry curvature (of the lower band) over $\mathbb{R}^2$ yields

$$C = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathcal{F}_{xy}\,\mathrm{d}^2k = -\frac{\mathrm{sign}[h_z]\,\mathrm{sign}[\det(h)]}{2}\,. \tag{9}$$

(*Hint:* Use the result derived in Problem 1 and make the linear substitution $\boldsymbol{k}' = h^T\boldsymbol{k}$.)

Problem 3: Kramers theorem

Learning objective

Kramers theorem guarantees the degeneracy of eigenenergies for time-reversal invariant Hamiltonians of *half-integer total spin*. This has consequences in many situations, e.g., in atomic physics where it explains the degeneracy of energy levels for states with an odd number of fermions in the absence of a magnetic field. The theorem was mentioned in the lecture but not proven; here you complete this task.

Mathematically, Kramers theorem states that every eigenenergy of a time-reversal invariant ($[H, T_U] = 0$) Hamiltonian H with a (anti-unitary) time-reversal symmetry $T_U = U\mathcal{K}$ that squares to minus one, $T_U^2 = -\mathbb{1}$, is at least two-fold degenerate.

Prove this!