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Problem 1: Edge modes of the Su-Schrieffer-Heeger chain

> **Learning objective**
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> In the lecture we claimed (and numerically demonstrated) that the ground space degeneracy of the open-boundary SSH chain is induced by exponentially localized *edge modes* which are present everywhere in the topological phase—even in the presence of sublattice-symmetric disorder. Here you substantiate this claim analytically.

The many-body Hamiltonian of the SSH chain reads for open boundary conditions

$$\hat{H}_{\text{SSH}} = t \sum_{i=1}^{L} (a_i^\dagger b_i + b_i^\dagger a_i) + w \sum_{i=1}^{L-1} (b_i^\dagger a_{i+1} + a_{i+1}^\dagger b_i)\,, \tag{1}$$

with real and homogeneous hopping strengths $t > 0$ and $w > 0$; $a_i^{(\dagger)}$ and $b_i^{(\dagger)}$ are fermionic creation- and annihilation operators.

a) Show that the operators

$$\tilde{a}_l = \mathcal{N} \sum_{i=1}^{L} (-x)^{i-1}\, a_i \quad \text{and} \quad \tilde{b}_r = \mathcal{N} \sum_{i=1}^{L} (-x)^{i-1}\, b_{L-i+1} \tag{2}$$

with $x = t/w$ describe two fermionic modes and determine the normalizing factor $\mathcal{N}$.

b) Prove that the ground space of $\hat{H}_{\text{SSH}}$ is four-fold degenerate in the thermodyamic limit if the system is in the topological phase ($x < 1$) by showing that

$$\left[\tilde{a}_l, \hat{H}_{\text{SSH}}\right] = \mathcal{O}\left(x^L\right) \quad \text{and} \quad \left[\tilde{b}_r, \hat{H}_{\text{SSH}}\right] = \mathcal{O}\left(x^L\right)\,, \tag{3}$$

and demonstrate that the energy splitting of the edge modes vanishes exponentially with the system size.

Your final goal is to demonstrate that these results are robust to disorder that breaks translational invariance, particle-hole and time-reversal symmetry (but not sublattice symmetry!). To this end, consider the generalized SSH chain Hamiltonian from the lecture

$$\hat{H}'_{\text{SSH}} = \sum_{i=1}^{L} (t_i\, a_i^\dagger b_i + t_i^*\, b_i^\dagger a_i) + \sum_{i=1}^{L-1} (w_i\, a_{i+1}^\dagger b_i + w_i^*\, b_i^\dagger a_{i+1})\,, \tag{4}$$

with site-dependent couplings $t_i, w_i \in \mathbb{C}$.

We define the local ratio $x_i = t_i/w_i$ ($i = 1, \dots, L-1$) and assume that the moduli $|x_i|$ are i.i.d. (= independent and identically distributed) random variables with probability density function $P(x)$ for $x \in [0, \infty)$.

c) For a given realization of couplings $\{x_i\}$, we generalize the edge mode operators as follows:

$$\tilde{a}_l = \mathcal{N} \sum_{i=1}^{L} X_i\, a_i \quad \text{and} \quad \tilde{b}_r = \mathcal{N} \sum_{i=1}^{L} X_i^*\, b_{L-i+1} \tag{5}$$

with $X_i := \prod_{j<i}(-x_j)$ and $X_1 := 1$. Verify that all algebraic statements from subtask a) remain valid, i.e., these constitute two fermion modes.

d) Focus only on the left edge mode $\tilde{a}_l$ and show that

$$\left[\tilde{a}_l, \hat{H}'_{\text{SSH}}\right] = \mathcal{N} X_L t_L\, b_L\,. \tag{6}$$

Derive a condition on $P(x)$ such that

$$X_L \in \mathcal{O}(e^{-\lambda L}) \quad \text{which means} \quad |X_L| < e^{-\lambda(L-1)} \quad \text{for} \quad L \to \infty \tag{7}$$

with some $\lambda > 0$.

(*Hint:* Strictly speaking, the limit $L \to \infty$ is to be taken in a stochastic sense: Use the (strong) *law of large numbers* to convert X_L into an integral over $P(x)$ in the limit $L \to \infty$ (this limit is a so called "almost surely" convergence).)

e) For the sake of concreteness, assume that the moduli $|x_i| \in \mathcal{U}(\delta, \tilde{x})$ are *uniformly distributed* random variables in the interval $[\delta, \tilde{x}]$; $0 < \delta \ll 1$ is a regularization of no physical importance. The upper cutoff $\tilde{x} > \delta$ is the parameter of the model.

Use your result in d) to show that for $\tilde{x} < 1$ the ground space of $\hat{H}'_{\text{SSH}}$ is four-fold degenerate in the thermodynamic limit—despite the disorder in the hoppings.

Problem 2: The Zak phase

Learning objective

In the lecture we have shown that the two quantum phases of the SSH chain can be characterized by the winding number of the Bloch vector around the origin in the d_x-d_y-plane. Here you show that the two quantum phases can also be distinguished by the Berry phase collected over the Brillouin zone. This phase is known as the *Zak phase*[a] and has already been measured in experiments[b].

[a] J. Zak, *Berry's phase for energy bands in solids*, PRL **62**, 2747 (1989)
[b] M. Atala et al., *Direct measurement of the Zak phase in topological Bloch bands*, Nature Physics **9**, 795 (2013)

The Bloch Hamiltonian of the SSH chain is given by

$$H(k) = \vec{d}(k) \cdot \vec{\sigma} \quad \text{with} \quad \vec{d}(k) = \begin{pmatrix} t + w\, \cos k \\ w\, \sin k \\ 0 \end{pmatrix} \tag{8}$$

for $k \in [0, 2\pi)$ and $t, w > 0$.

a) Diagonalize the Bloch Hamiltonian and compute the eigenstates $|u_\pm(k)\rangle$.

b) Compute the integral of the Berry phase over the Brillouin zone

$$\varphi_{\text{Zak}} = \int_0^{2\pi} i \langle u_\pm(k) | \partial_k u_\pm(k) \rangle \, \mathrm{d}k \tag{9}$$

and show that $\varphi_{\text{Zak}} = \pi \mod 2\pi$ in the topological phase ($t < w$) and $\varphi_{\text{Zak}} = 0 \mod 2\pi$ in the trivial phase ($t > w$).

c) Let $|u'_\pm(k)\rangle := e^{i\varphi_k} |u_\pm(k)\rangle$ be a continuous gauge transformation and compute the effect on the Zak phase φ_{Zak}.

Problem 3: The Bogoliubov-de Gennes Hamiltonian and particle-hole "symmetry"

Learning objective

For the topological classification of the Majorana chain we used the "intrinsic" particle-hole symmetry of the Bogoliubov-de Gennes Hamiltonian. In the lecture, it was claimed that this is not a real symmetry (in the sense that some operator commutes with the Hamiltonian) but rather a tautological constraint on the BdG Hamiltonian that arises from the algebra of fermion operators. Here you show this claim in detail.

We are interested in a generic quadratic fermion Hamiltonian

$$\hat{H} = \sum_{i,j=1}^{L} H_{ij}\, c_i^\dagger c_j + \frac{1}{2}\left(\Delta_{ij}\, c_i^\dagger c_j^\dagger + \Delta_{ij}^*\, c_j c_i\right) \tag{10}$$

with mean-field pairing terms parametrized by $\Delta_{ij} \in \mathbb{C}$ and hopping Hamiltonian $H_{ij} \in \mathbb{C}$.

a) Show that $H^\dagger = H$ and w.l.o.g. $\Delta^T = -\Delta$.

b) Introduce the Nambu spinor

$$\mathbf{\Psi} = \left(c_1 \, \dots \, c_L \, c_1^\dagger \, \dots \, c_L^\dagger\right)^T \tag{11}$$

and show that the Hamiltonian can be written in the form

$$\hat{H} = \frac{1}{2} \mathbf{\Psi}^\dagger \, H_{\text{BdG}} \, \mathbf{\Psi} + \text{const.} \tag{12}$$

with *Bogoliubov-de Gennes Hamiltonian*

$$H_{\text{BdG}} = \begin{pmatrix} H & \Delta \\ -\Delta^* & -H^* \end{pmatrix} . \tag{13}$$

c) Show that H_{BdG} features a particle-hole "symmetry" as required for the tenfold way classification.

Note that this reality constraint on H_{BdG} does not impose any constraints on $\hat{H}$ but follows simply from the algebraic properties of the fermion operators; it is, in this sense, "intrinsic" or "tautological".

Problem 4: From the Majorana chain to the transverse-field Ising model

Learning objective

The transverse-field Ising model is a one-dimensional spin-$\frac{1}{2}$ model of interacting spins with a quantum phase transition that exemplifies the notion of spontaneous symmetry breaking. By contrast, the Majorna chain is a quadratic fermion model that can be solved exactly and features a topological phase transition without symmetry breaking. Remarkably, there is a mathematically exact mapping between fermionic and spin-$\frac{1}{2}$ systems known as *Jordan-Wigner transformation* that relates these two models. The point of this task is then (1) to solve the transverse-field Ising model exactly by mapping it to the Majorana chain and (2) to understand how the *topological* phase transition of the Majorana chain translates to the *spontaneous symmetry breaking* phase transition of the transverse-field Ising model.

In the lecture we introduced the mean-field Hamiltonian of a one-dimensional p-wave superconductor,

$$\hat{H}_{\text{MC}} = -\frac{\mu}{2}\sum_{i=1}^{L}(i\gamma_{2i-1}\gamma_{2i}) + w\sum_{i=1}^{L-1}(i\gamma_{2i}\gamma_{2i+1}) \qquad \text{(OBC)} \tag{14}$$

commonly referred to as *Majorana chain*; here, $w = \Delta$ is the hopping amplitude/superconducting gap parameter and μ the chemical potential; the γ_n are $2L$ Majorana operators. Because the Hamiltonian is quadratic in fermion operators, we had not trouble computing the spectrum in the Bogoliubov-de Gennes representation.

The goal is to show that the model can be mapped exactly onto the *transvere-field Ising model (TIM)* (with open boundary conditions), given by the spin-$\frac{1}{2}$ Hamiltonian

$$H_{\text{TIM}} = -J\sum_{i=1}^{L-1}\sigma_i^x\sigma_{i+1}^x + h\sum_{i=1}^{L}\sigma_i^z \qquad \text{(OBC)} \tag{15}$$

which we introduced in the first lecture of this course as an example of spontaneous symmetry breaking; here, $J > 0$ is the ferromagnetic coupling and $h \in \mathbb{R}$ the transverse field.

a) Consider the Hilbert space $\mathcal{H}_{\text{Spin}} = (\mathbb{C}^2)^{\otimes L}$ of a spin-$\frac{1}{2}$ system with L spins and Pauli matrices σ_i^α for $\alpha = x, y, z$ and $i = 1, \ldots, L$. Show that the operators

$$\gamma_{2i-1} := \left[\prod_{j<i}\sigma_j^z\right]\sigma_i^x \quad \text{and} \quad \gamma_{2i} := \left[\prod_{j<i}\sigma_j^z\right]\sigma_i^y \tag{16}$$

satisfy the algebraic relations of Majorana fermions and therefore establish a Fock space representation $\mathcal{H}_{\text{Fock}} \simeq \mathcal{H}_{\text{Spin}}$ via $c_i = \frac{1}{2}(\gamma_{2i-1} + i\gamma_{2i})$.

This transformation is known as *Jordan-Wigner transformation*. Note that the transformation Eq. (16) is highly non-local; the non-local product of σ^z-operators is sometimes referred to as *Jordan-Wigner string* which can be troublesome for the simulation of fermionic systems on quantum computers (because qubits = spin-$\frac{1}{2}$).

b) Apply the Jordan-Wigner transformation to the Majorana chain Eq. (14) with open boundary conditions and show that it results in the transverse-field Ising model Eq. (15). How do the parameters of the two models relate? Conclude from this where the gap of the transverse-field Ising model closes and the symmetry-breaking phase transition occurs.

c) Demonstrate how the ground state(s) of the Majorana chain at the fixed points (trivial: $w = 0$ and $\mu > 0$; topological: $w > 0$ and $\mu = 0$) map to ground state(s) of the transverse-field Ising model.

What happens to the long-range correlations $\lim_{|i-j|\to\infty}\langle\sigma_i^x\sigma_j^x\rangle$ of the transverse-field Ising model in the symmetry-broken phase under Jordan-Wigner transformation?

What is the fermionic counterpart of the global spin-flip symmetry

$$Z = \prod_i \sigma_i^z \quad \text{with} \quad [Z, H_{\text{TIM}}] = 0 \tag{17}$$

of the transverse-field Ising model?