Theoretische Physik IV: Statistische Mechanik, Weihnachtsaufgabe

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Why does an airplane fly?
(Adapted from
http://www.mathpages.com/home/kmath258/kmath258.htm)

1. Preparation (Oral)

During this exercise, we will study the basic principle for lift forces and consequently why an airplane can fly. The analysis is based on an incompressible fluid described by a potential flow ϕ. For simplicity, the wing is considered infinitely long and the problem reduces to a two-dimensional situation.

The potential flow satisfies the Laplace equation

$$\partial_x^2\phi + \partial_y^2\phi = 0 \tag{1}$$

with the flow field $\mathbf{v} = \nabla\phi$. In the following, we use the strength of analytical functions to solve the problem of the flow around an airplane wing. We introduce the complex variable $z = x + iy$ and consider an analytical function $f(z) = \phi(x, y) + i\psi(x, y)$ with ϕ and ψ real valued.

(a) Show that the condition that $f(z)$ is an analytical function leads to the Laplace equation for ϕ and ψ.

(b) Show that $\nabla\psi$ is orthogonal to $\nabla\phi$.

(c) Show that we can interpret the real part of an analytical function as the potential flow of an incompressible fluid, while the imaginary part gives rise to the streamline function: $\psi(x, y) = \mathit{const}$ describes the trajectory a particle takes during the flow.

The goal is therefore to find suitable analytical functions $f(z) = \phi + i\psi$, where the shape of an obstacle is described by $\psi = \mathit{const}$. Then, the flow of an incompressible fluid around this obstacle is given by the real part of the analytical function. To warm up, we study the flow around a cylinder first.

2. Potential flow around a cylinder (Oral)

We start with an interesting analytical map of the complex plane onto the complex plane (so called Joukowsky map). This mapping is defined by the function, see Fig. 1

$$z = f(w) = w + \sqrt{w^2 - 1} \qquad\qquad w = f^{-1}(z) = \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{2}$$

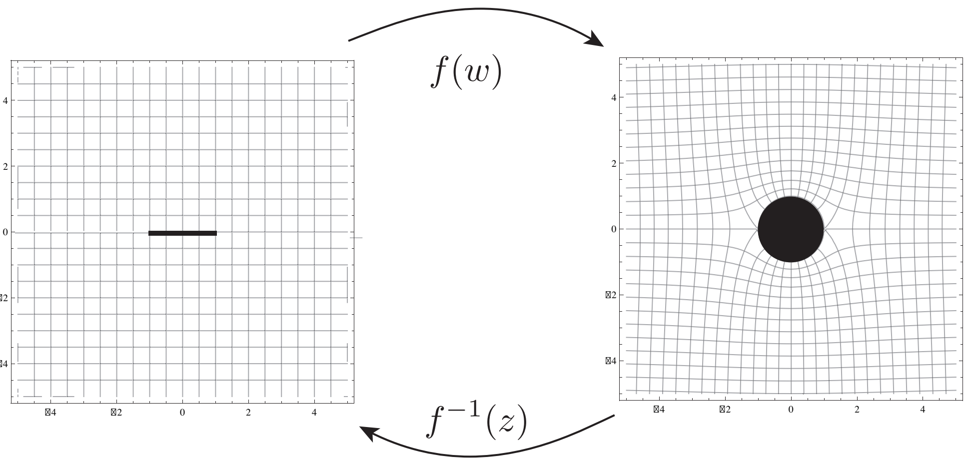

Figure 1: Analytical mapping of a thin plate onto a cylinder

It has the special property, that the flat piece (solid black) for $-1 \leq x \leq 1$ and $y = 0$ is mapped onto a circle with radius 1. This mapping allows us to read of the potential flow around the cylinder:

(a) Show that the surface of the circle satisfy $\mathrm{Im} f^{-1}(z) = 0$, i.e., the streamline function matches the shape of the cylinder.

(b) Derive the potential flow $\phi(x, y)$ around the sphere.

(c) What is the velocity $\mathbf{v}_0$ of the fluid far a way from the cylinder.

(d) Express the potential in polar coordinates and derive the velocity field $\mathbf{v}$ in polar coordinates. How does the flow on the surface of the cylinder look like?

3. Potential flow around a cylinder with finite vorticity (Written)

An additional solution for the flow of a fluid around a cylinder is given by the analytical function $h(z) = i\Gamma \ln(x + iy)/2\pi$, which exhibits a finite vorticity.

(a) Draw the potential lines and streamline for this flow.

(b) Express the flow potential and the velocity field in polar coordinates?

(c) Show that the vorticity $\mathbf{\Gamma} = \oint_\gamma ds\, \mathbf{n} \wedge \mathbf{v}$ reduces to $\mathbf{\Gamma} = \Gamma \mathbf{e_z}$ with $\mathbf{e}_z$ the unit vector along the cylinder and $\mathbf{n}$ a unit vector orthogonal onto the surface.

(d)* The square root in $f(z)$ as well as the logarithm in $h(z)$ requires the definition of a branch cut. Therefore, $h(z)$ is not globally defined but is only local an analytical function. The correct positioning of the branch cut places an important role for the illustration of the streamlines. E.g., the seemingly discontinuities at the front of the wing in Fig. 3(b) is just a consequence of the branch cut and an artifact of the choice for the positioning of the branch. Discuss the role of the branch cut and an optimal choice. Point out how one can enforce Mathematica to use a different branch cut in plotting the streamlines.

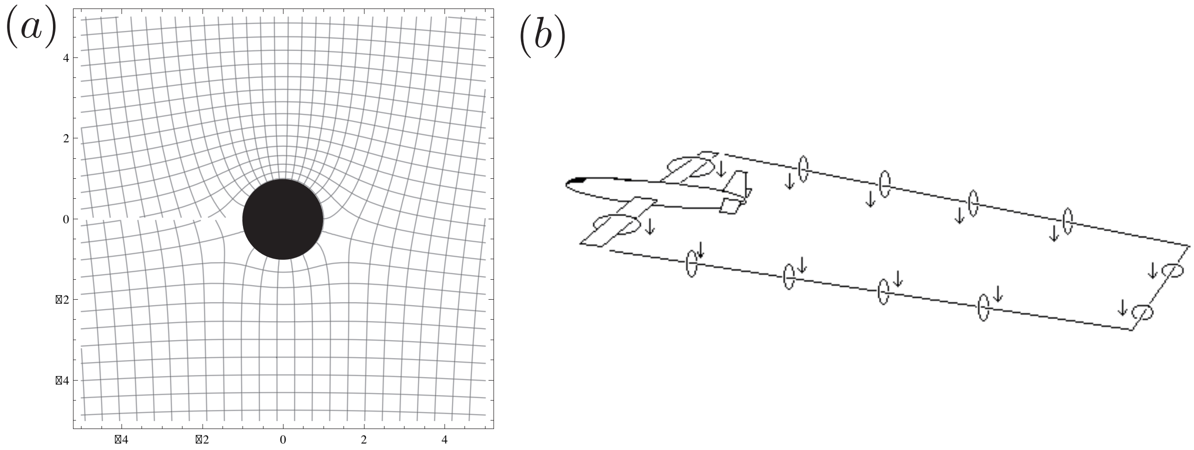

Figure 2: (a) Analytical mapping of a thin plate onto a cylinder (b) Illustration of the vorticity ring trailing the airplane. (Figure taken from http://www.mathpages.com/home/kmath258/kmath258.htm)

4. **A Cylinder with a finite lift force (Written)**

Now we combine the two solutions to obtain the flow around a cylinder with a finite vorticity, see Fig 2(a). The flow is defined by the analytical function $f^{-1}(x+iy) + h(x+iy)$. The lift force on the cylinder can be calculated by the pressure on the surface of the cylinder

$$\mathbf{F}_{\text{Lift}} = \oint_{\gamma} ds\, \mathbf{n}\, p \tag{3}$$

with $\mathbf{n}$ the vector normal on the surface γ of the obstacle. In turn, the pressure is derived from Bernoullis law.

$$p = -\frac{\rho \mathbf{v}^2}{2} + const. \tag{4}$$

(a) Use the above results for the flow field on the surface of the cylinder to derive the lift force. Show that the lift force can be written in the form

$$\mathbf{F}_{\text{Lift}} = \rho \mathbf{v}_0 \wedge \mathbf{\Gamma}. \tag{5}$$

We conclude, that the lift force appears is a general consequence of a finite vorticity around the obstacle. This phenomena explains the lift and drift of rotating balls, which we can observe in tennis, soccer, etc.,

5. **A model of an airplane wing (Written)**

Remarkably, the above mapping $f^{-1}(z)$ not only allows us to solve trivially the flow around a cylinder, but it also allows us to derive the flow around an object resembling the shape of an airplane wing. Remember, that the mapping $f^{-1}(z)$ mapps a unit circle onto the flat piece in Fig. 1. However, if we translate the circle and scale its radius before the mapping, then the shape of the circle is mapped onto

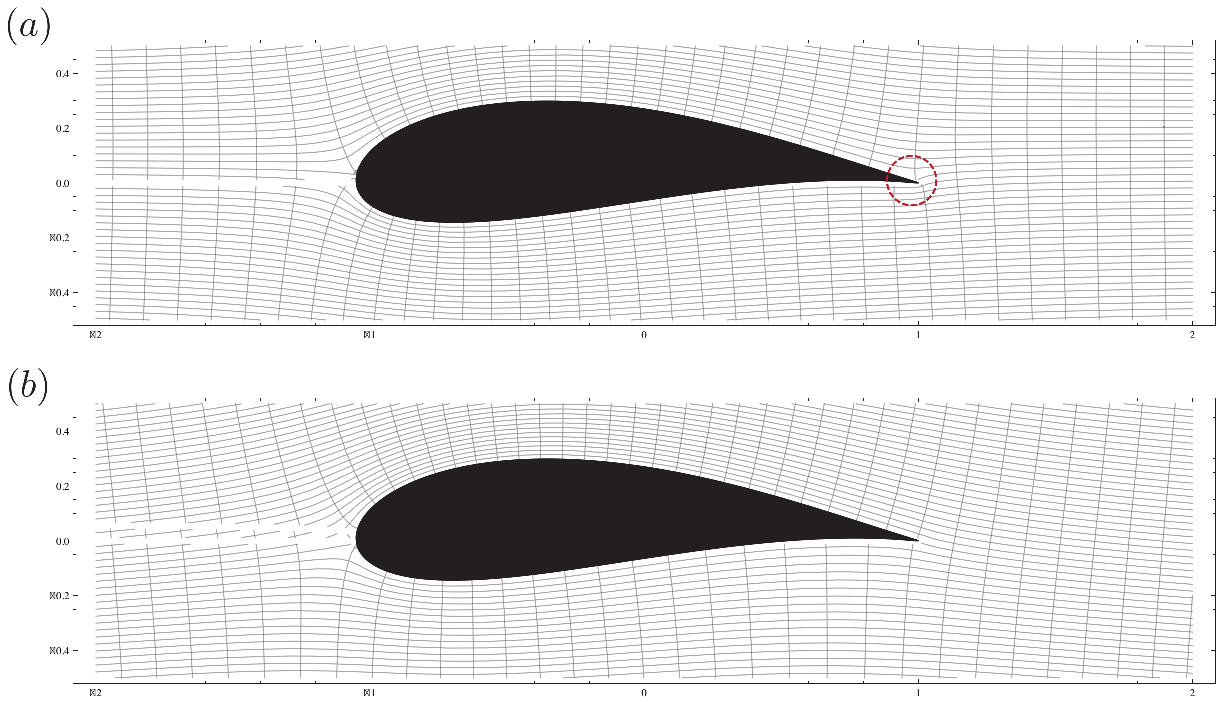

Figure 3: Flow around an airplane wing: (a) without vorticity with a highly unstable region (red circle). (b) A flow with a finite vorticity in turn becomes stable due to the disappearance of sharp edges in the flow pattern. It is this vorticity which provides the lift of the airplane.

a shape resembling a realistic airplane wing, see Fig. 3. The scaling and translation is achieved by the mapping

$$g(z) = r\,[z - (a + ib)] \tag{6}$$

(a) Show that the unit circle is mapped onto a shape shown in Fig. 3 under the mapping $f^{-1}[g(z)]$. Play around to derive an optimal set of parameters. (Hint: The values $a = 0.15$, $b = -0.1$, and $r = 1.2$ is a good starting point.)

(b) How does the shape look like for $b = 0$ and $a = 0$?

(c) Demonstrate that the potential flow around the above obstacle in absence of a vorticity is now given by the analytical function

$$f(g^{-1}(f^{-1}(x + iy))). \tag{7}$$

(d) Demonstrate, that the lift force also for this shape reduces to

$$\mathbf{F}_{\text{Lift}} = \rho \mathbf{v}_0 \wedge \mathbf{\Gamma} \tag{8}$$

Therefore, in absence of a finite vorticity, the airplane wing does not experience a lift force. (Hint: the analytical mapping provides a orthogonal coordinate transformation)

(e) Show, that the flow fields reduces to the shape as shown in Fig. 3(a) by plotting the streamlines in mathematica.

It turns out, that in absence of a vorticity, the flow is highly unstable due to the curvature at the end of the wing (red circle). This flow decays via the formation of a finite vorticity. Then, the flow with a finite vorticity is stable and does not exhibit such sharp edges, see Fig. 3(b).

(f) Derive the potential flow for this situation from the solution of the flow around a cylinder with a vorticity.

(g) Find by try and error a somehow optimal value for the vorticity, where the flow at the end of the wing becomes smooth. Which vorticity is required?

Then, we obtain a finite lift force by the general formula in 4(b). This explains why a airplane can fly: the shape of the wing induces a stable flow with a finite vorticity. It is this finite vorticity, which is responsible for the lift of an airplane via the magnus force. Note, that vorticity can not disappear, but can only be annihilated by the opposite vorticity. Therefore, two strings of finite vorticity are trailing the airplane and disconnect from the wing at the tip, see Fig. 2(b); it is the latter vorticity, which is visible, when an airplane flies through smoke.