Theoretische Physik IV: Statistische Mechanik, Exercise 11

Prof. Dr. Hans Peter Büchler WS 2013/14, 7. January 2014

1. **Ideal Paramagnet Part 1 (Written)**
   For a microcanonical ensemble, we consider a perfectly isolated large system, made out of N independent magnetic moments of total energy E. Through the application of a homogeneous magnetic field H, each moment m_i will take one of two values, $m_i = \pm m$. The Hamiltonian $\mathcal{H}$ for the system is

$$\mathcal{H} = -\sum_{i=1}^{N} H\, m_i = -H\, M = -H\, n\, m\,, \tag{1}$$

   where $n = n_+ - n_-$ denotes the difference of positive and negative moments and $M = n\, m$ the magnetization.

   (a) The phase space for this system is discrete, why? $\Omega(M)$ is the number of states for a given magnetization, explicitly calculate and show, that it takes on the form
   $$\Omega(n) = \frac{N!}{\left[\frac{1}{2}(N+n)\right]!\,\left[\frac{1}{2}(N-n)\right]!}\,.$$

   (b) Compute the entropy $S(E,H)$ for the system with $S(E,H) = k_\mathrm{B}\log[\Omega\,(E,\,H)]$, neglect terms of order $\mathcal{O}(\log N)$ and smaller.

   (c) Determine the temperature $\frac{1}{T} = \frac{\partial S}{\partial E}\Big|_H$ and afterwards solve it for $E = E(T,H)$.

   (d) Calculate the magnetization $M = T\,\frac{\partial S}{\partial H}\Big|_E$ and explicitly demonstrate, that in the high temperature limit ($k_\mathrm{B}T \gg H\,m$) the Curie behavior $M = N\frac{Hm^2}{k_\mathrm{B}T}$ can be obtained. In addition, compute the isothermal magnetic susceptibility $\chi_T = \frac{\partial M}{\partial H}\Big|_T$.

2. **Entropy of a Simple System (Oral)**
   Consider a simple magnetic lattice system with N spin-1 atoms. Each atom can be in one of three spin states, namely $S_z = \pm 1, 0$. The respective number of atoms in each of those spin states is denoted by $n_{\pm 1}$, n_0. No magnetic field shall be present, hence all states are degenerate.

   (a) Determine the total entropy of the system as a function of $n_{\pm 1}$ and n_0. For sake of convenience, use Stirling's formula.

   (b) Find the configuration (n_{-1}, n_0, n_1) which maximizes the entropy. How can the entropy be understood?

   (c) Calculate the entropy for the maximizing configuration.

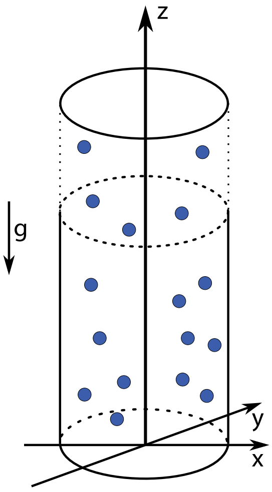

Figure 1: Classical ideal gas constrained to a cylinder.

3. Classical Gas in Homogeneous Field (Oral)

Consider a classical ideal gas in a homogeneous gravitational field

$$\mathcal{H} = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + mgz_i \right) , \tag{2}$$

which is constrained in a cylinder with high walls ($z_{max} = L \to \infty$), see figure 1. Here, the gravitational field points along the z-direction of the cylinder. The system is considered to be in the canonical ensemble.

(a) Calculate the probability $P(z)\,dz$ for finding a particle within $[z, z + dz]$, for $z > 0$.

(b) Compute the mean kinetic energy for a particle.

(c) Determine the mean potential energy for a particle.

4. Ideal Paramagnet Part 2 (Oral)

Here, we again consider an ideal paramagnet, as investigated in problem 1 from above. In this scenario, the analysis takes place in a canonical ensemble. Calculate the following quantities for the canonical ensemble:

(a) Internal energy $E(T, H, N)$

(b) Entropy $S(T, H, N)$

(c) Magnetization $M(T, H, N)$ and isothermal magnetic susceptibility χ_T . Compare these two quantities with the microcanonical scenario of problem 1.