# Theoretische Physik IV: Statistische Mechanik, Exercise 13

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**1. Quantum Ising model with transverse field: Part 1 (Oral)**

We consider a system of Ising-coupled (quantum) spins on a lattice where each spin has $z$ nearest neighbors ($z$ is known as the coordination number of the lattice). A magnetic field of strength $\Omega$ is applied perpendicular to the preferential direction of the spins. The Hamiltonian is given by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z + \Omega \sum_i \hat{\sigma}_i^x \tag{1}$$

where $\hat{\sigma}_i^\alpha$ are the spin-1/2 Pauli matrices and $\langle i, j \rangle$ describes the sum over nearest neighbors. We consider a ferromagnetic coupling $J > 0$.

In this exercise we are exploring the physics of this model within a mean-field analysis. We define the mean-field $m \equiv \langle \hat{\sigma}^z \rangle$ as the average magnetization in $z$-direction.

(a) We can always write the spin operators as $\hat{\sigma}_i^z = m + \hat{\delta}_i^z$ where $\hat{\delta}^z$ contains the residual operator character and describes the deviation from the mean-field.

Transform the Hamiltonian into a sum of uncoupled spins by assuming that the deviations from the mean-field are small, such that we can neglect terms of second order in $\hat{\delta}^z$. Substitute all occurences of $\hat{\delta}^z$ with $\hat{\sigma}^z - m$ after making the approximation.

(b) Diagonalize the resulting single-spin Hamiltonian. Let $|\nearrow\rangle$ and $|\swarrow\rangle$ be the eigenstates.

(c) Compute the probability $p_\nearrow(T)$ to be in the state $|\nearrow\rangle$ at temperature $T$. Then, we can express the magnetization as

$$\begin{aligned} m = \langle \hat{\sigma}^z \rangle &= p_\nearrow(T)\ \langle \nearrow | \, \hat{\sigma}^z | \nearrow \rangle + p_\swarrow(T)\ \langle \swarrow | \, \hat{\sigma}^z | \swarrow \rangle \\ &= (2p_\nearrow(T) - 1) \, \langle \nearrow | \, \hat{\sigma}^z | \nearrow \rangle \, . \end{aligned} \tag{2}$$

Compute the right hand side of this equation (as a function of $m$). The resulting equation is called a *self-consistency equation*.

(d) Derive the phase diagram as a function of $\omega = \Omega / J$ and $t = k_B T / J$. To this end, derive an analytic expression for the critical temperature $t_c$ as a function of $\omega$.
<u>Hint:</u> While the self-consistency equation can not be solved analytically, you can get the idea of how to derive the phase boundary by inspecting the solutions graphically.

**2. Quantum Ising model with transverse field: Part 2 (Written)**

We consider the model from the first part at zero temperature. Intuitively, for vanishing field $\Omega/J \longrightarrow 0$, the system favors a configuration where all spins point in either positive or negative $z$-direction. On the other hand, for $\Omega/J \longrightarrow \infty$, the external field aligns all spins in $x$-direction.

Ignoring correlations between the spins, we can use these observations to devise a variational wave function for the system

$$|\Psi_\alpha\rangle = \prod_{i=1}^{N} |\swarrow_\alpha\rangle_i = \prod_{i=1}^{N} R_y(\alpha)\,|\downarrow\rangle_i\,. \tag{3}$$

Here, $R_y(\alpha)$ describes a rotation around the $y$-axis in spin-space. An explicit representation is given by

$$R_y(\alpha) = \mathrm{e}^{-i\frac{\alpha}{2}\sigma_y} = \mathbb{1}\cos\left(\frac{\alpha}{2}\right) - i\sigma_y \sin\left(\frac{\alpha}{2}\right). \tag{4}$$

(a) Calculate the energy per spin of the variational state $E(\alpha) = \langle\Psi_\alpha|\,\mathcal{H}\,|\Psi_\alpha\rangle/N$.

(b) Show that the variational ansatz yields the true ground state in the limits described above.

(c) Visualize the change in the energy landscape $E(\alpha)$ as $\Omega/J$ crosses the critical value for the phase transition.

**3. Absence of Bose-Einstein condensation in 2D (Oral)**

Determine the grand-canonical partition function $\mathcal{Z}(z, V, T)$ for the ideal Bose gas. By $z \equiv \mathrm{e}^{\beta\mu}$ we denote the *fugacity*. Use the partition function to calculate the mean density $n = n(z, T) = \langle N\rangle/V$ of the gas for $d = 2, 3$ dimensions.

Show that the ideal Bose gas does not condense in two dimensions at any $T > 0$.

<u>Remark 1:</u> A powerful generalization of this result is known as Mermin-Wagner theorem. It states that a continous symmetry cannot be spontaneously broken at finite temperature in $d \le 2$ dimensions. A Bose-Einstein condensate has a broken $U(1)$ symmetry due to the overall phase of the wave function — and is therefore forbidden in $d \le 2$ dimensions.

<u>Remark 2:</u> The above result is only valid for a uniform system. A two dimensional Bose gas which is harmonically trapped condenses at a finite critical temperature.