Theoretische Physik IV: Statistische Mechanik, Exercise 14

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1. Black body radiation (Oral)

The goal of this exercise is to derive Planck's law of black body radiation. We consider a gas of photons at thermal equilibrium. For simplicity, we consider a box of volume $V = L^3$ and periodic boundary conditions. Due to the boundary conditions, there are discrete energy levels (modes). Each mode is labeled by a set of quantum numbers $\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{z}$ with $\boldsymbol{z} = (z_x, z_y, z_z) \in \mathbb{Z}^3$ and a polarization $\sigma \in \{\pm 1\}$. The energy of the mode $(\boldsymbol{k}, \sigma)$ is given by $E_{\boldsymbol{k},\sigma} = \hbar\omega_{\boldsymbol{k}} = \hbar c|\boldsymbol{k}| = \hbar ck$.

(a) Each mode can be occupied by $n_{\boldsymbol{k},\sigma} = 0, 1, 2, \ldots$ photons. Determine the partition function $Z_{\boldsymbol{k},\sigma}(\beta, V)$ of a single mode and the total partition function $Z(\beta, V)$.

(b) In the following, we assume that the modes are 'dense' such that we can go over to a continuum description. In $\boldsymbol{k}$-space there are exactly two modes (polarization) per 'volume' $(2\pi)^3/V$. Show that there are $N(k)\mathrm{d}k \equiv \frac{V}{\pi^2}k^2\mathrm{d}k$ states in a spherical shell of radius k and thickness $\mathrm{d}k$. The quantity $N(k)$ is called the density of states in k-space. Use the relation $D(k)\mathrm{d}k = D(\omega)\mathrm{d}\omega$ to transform to the (frequency) density of states $D(\omega)$, i.e. the number of states in the frequency interval $[\omega, \omega + \mathrm{d}\omega]$.

(c) Show that we can use the density of states to write a sum $\sum_{\boldsymbol{k},\sigma} f(k)$ in the continuum limit as $\int \mathrm{d}\omega\, D(\omega) f(\omega)$.

(d) Write the energy density $u \equiv U/V = -(\partial_\beta \ln Z)/V$ in the form $u = \int \mathrm{d}\omega\, u(\omega)$. The expression for the spectral energy density $u(\omega)$ is known as Planck's law.

(e) Find the Stefan-Boltzmann law $u = \sigma T^4$ by integrating the spectral energy density over the frequency. Determine the value of σ.

Remark: The following exercises are intended as a repetition and preparation for the exam.

2. Rubber band (Oral)

In its relaxed state, a rubber band is of length L_0 over a temperature range $[T_i, T_f]$. The force F needed to expand the band to length $L > L_0$ has been experimentally determined to be

$$F = bT\left(\frac{L}{L_0} - \frac{L_0^2}{L^2}\right), \qquad b > 0. \tag{1}$$

Furthermore, the heat capacity at constant length $c_L \equiv T\,(\partial S/\partial T)_L$ is independent of the temperature for $L = L_0$ within the given temperature range, i.e. $c_L(T, L_0) = c_L(T_0, L_0)$.

(a) Determine the internal energy and the entropy of the rubber band as a function of T and L close to the point (T_0, L_0) with $T_0 \in [T_i, T_f]$. To this end, derive the relation $\mathrm{d}U = c_L \mathrm{d}T$ and show that $c_L(T, L) = c_L(T_0, L_0)$.

(b) Determine the amount of work needed to expand the rubber band isothermally in a reversible way (slowly) from length L_0 to $L_1 > L_0$ at temperature $T = T_0$. What is the entropy change of the rubber band during this process?

(c) Starting from state (T_0, L_1) the rubber band is now released without contact to the heat bath (and without extracting any energy from the system). Which temperature does the band have in the relaxed state? What is the entropy change? Is this process reversible?

3. Thermodynamic relations (Oral)

We consider a material whose internal energy is independent of the volume, that is $(\partial U/\partial V)_T = 0$. Prove the following statements:

(a) The specific heat at constant volume c_V is a function of T only.

(b) The volume V is a function of P/T only.

(c) The difference between the specific heat at constant pressure and at constant volume $c_P - c_V$ is a function of P/T only.

4. Collection of harmonic oscillators (Oral)

We consider a system of N uncoupled classical harmonic oscillators. Each oscillator is described by a Hamiltonian $H(p, q) = p^2/2m + m\omega^2 q^2/2$.

(a) Calculate the canonical partition function $Z_N(T)$.

(b) Determine the free energy, the internal energy and the specific heat c_V.