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## Problem 1: System of three interacting spin-1/2 particles (Written)

**Learning objective**

You will begin this exercise sheet be going through an instructive example of an few particle spin system. The task is to show that using the theorem of addition of angular momentum, it is possible to analytically determine the ground state of three spins coupled antiferromagnetically; for a larger number of spins, this task is highly non-trivial and still poses unsolved problems in theoretical physics.

Let us consider a system composed of three spin-1/2 particles.

a) What is the dimension of the Hilbert space?

The total spin operator can be defined as $\mathbf{S} = \sum_{i=1}^{3} \mathbf{S}^{(i)}$ and its $z$ projection as $S_z = \sum_{i=1}^{3} S_z^{(i)}$. What are the eigenvalues and eigenstates of $\mathbf{S}^2$ and $S_z$?

b) The Hamiltonian of the system is

$$H = J \sum_{i=1}^{3} \mathbf{S}^{(i)} \cdot \mathbf{S}^{(i+1)}, \qquad J > 0.$$

Here we assume a periodic system (for $i = 3$ take $i + 1 = 1$). Calculate the eigenstates and eigenenergies of this Hamiltonian.

**Tip**: Rewrite H as a function of $S^2$ and $S^{(i)2}$.

## Problem 2: Dynamics in a harmonic potential (Oral)

**Learning objective**

You will now study how a Gaussian wave packet moves inside a harmonic trap. By doing so, you will use both the Schrödinger and Heisenberg picture to extract dynamical properties of a quantum system and build a bridge between the classical and quantum world.

The quantum harmonic oscillator is given by the Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = \hbar\omega(a^\dagger a + 1/2) \tag{1}$$

(see exercise sheet 1). Its eigenstates are given by

$$\psi_n(x) = \sqrt{\frac{1}{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{-1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{2\hbar}}.$$

$H_n$ denotes the n-th Hermite polynomial and is defined as

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} = \frac{\mathrm{d}^n}{\mathrm{d}t^n} e^{2xt-t^2}\Big|_{t=0}.$$

a) Assume you prepare the system in the state $|\Psi_\mu\rangle$, defined by

$$\langle x|\Psi_\mu\rangle \equiv \Psi_\mu(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{-1/4} e^{-\frac{m\omega(x-\mu)^2}{2\hbar}}.$$

Calculate the time evolution of $|\Psi_\mu(t)\rangle$ by decomposing $|\Psi_\mu\rangle$ into the eigenstates of $H$ and show that you can write $|\Psi_\mu(t)\rangle$ as $\left|\Psi_{\mu(t)}\right\rangle$ up to a global phase.

b) Use the previous results to determine the expectation value $\langle x(t)\rangle \equiv \langle\Psi_\mu(t)|\hat{x}\,|\Psi_\mu(t)\rangle$. Compare this result to the classical motion of a particle in a harmonic potential with initial position $x(0) = \mu$ and initial momentum $p(0) = 0$. This result is generalised by Ehrenfest's theorem.

c) Instead of evolving $|\Psi_\mu\rangle$ in time, which is called the Schrödinger picture, we can apply the time evolution onto operators. For this consider

$$\begin{aligned}\langle\psi_n(t)|A|\psi_m(t)\rangle &= \langle\psi_n|e^{iHt/\hbar}Ae^{-iHt/\hbar}|\psi_m\rangle \\ &\equiv A_{nm}(t).\end{aligned}$$

Taking $\psi_n$, $\psi_m$ to be basis elements of the Hilbert space we can define the time evolution of the operator $A$ as

$$A(t) \equiv e^{iHt/\hbar}Ae^{-iHt/\hbar}.$$

Use this to derive the equation of motion for $A(t)$, i.e., determine $\partial_t A(t)$.

d) Solve the equation of motion for the ladder operators $a$ and $a^\dagger$ and then for the position and momentum operator $\hat{x}$ and $\hat{p}$.

e) Finally, re-evaluate $\langle x(t)\rangle$, using your solution for $\hat{x}(t)$.